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SOLUTION OF SPLIT COMMON NULL POINT PROBLEM USING YOSIDA APPROXIMATION ITERATION METHOD

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ABSTRACT. In this paper, we solve split common null point problem using Yosida approximation operators of monotone mappings in Hilbert spaces. Two weak convergence theorems to solve split common null point problems are proved with three supporting lemmas. The results given in this paper are new and different from those, which solved split common null point problems using resolvents of multi-valued monotone mappings.

1. Introduction

In an outstanding work [7], Censor, Gibali and Reich introduced and studied a new type of variational inequality problem known as split variational inequality problem (SVIP), in which the solution of one variational inequality problem under the image of a bounded linear operator is also a solution of another variational inequality problem, that is

$$Find\ x^* \in H_1, \text{ such that } \langle f_1(x^*), x - x^* \rangle \geq 0,\ for\ all\ x \in P, \tag{1.1}$$

and such that

$$Ax^* = y^* \in H_2, \text{ such that } \langle f_2(y^*), y - y^* \rangle \geq 0,\ for\ all\ y \in Q, \tag{1.2}$$

where P, Q are closed and convex subset of Hilbert spaces H_1 and H_2, respectively; $A : H_1 \to H_2$ is abounded linear operator; $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are single-valued mappings.

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Moudafi [13] introduced the split monotone variational inclusion problem (SMVIP), which is a natural generalization of SVIP.

$$\mathit{Find}\ x^* \in H_1,\ \text{such that } 0 \in f_1(x^*) + V_1(x^*), \tag{1.3}$$

and such that

$$Ax^* = y^* \in H_2,\ \text{and } 0 \in f_2(y^*) + V_2(y^*), \tag{1.4}$$

where $V_1 : H_1 \to 2^{H_1}, V_2 : H_2 \to 2^{H_2}$ are multi-valued mappings on Hilbert spaces H_1 and H_2 respectively; $A : H_1 \to H_2$ is a bounded linear operator; $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are single-valued mappings.

Moudafi studied the weak convergence of following iterative method for SMVIP(1.3)-(1.4). For $x_0 \in H_1$ be arbitrary, compute the iterative sequence $\{x_n\}$ generated by:

$$x_{n+1} = U[x_n + \gamma A^*(G - I)Ax_n],\ \ n \in \mathbb{N}. \tag{1.5}$$

where $\gamma \in (0, \frac{1}{l})$ with l is the spectral radius of operator AA^*, A^* is the adjoint of A, $U = R_\lambda^{V_1}(I - \lambda f_1)$ and $G = R_\lambda^{V_2}(I - \lambda f_2)$ such that $R_\lambda^{V_1}, R_\lambda^{V_2}$ are resolvents of $V_1$and V_2, respectively.

If we replace $f_1 = f_2 = 0$, then SMVIP(1.3)-(1.4) reduces to the following split common null point problem (SCNPP):

$$\mathit{Find}\ x^* \in H_1,\ \text{such that } 0 \in V_1(x^*), \tag{1.6}$$

and such that

$$Ax^* = y^* \in H_2,\ \text{and } 0 \in V_2(y^*), \tag{1.7}$$

Recently, Byrne et al. [5] studied the weak and strong convergence of the following iterative scheme for SCNPP(1.6)-(1.7):

$$x_{n+1} = R_\lambda^{V_1}[x_n + \gamma A^*(R_\lambda^{V_2} - I)Ax_n)], \tag{1.8}$$

Kazmi et al. [10] introduced and analyzed the iterative methods to estimate a common solution of SCNPP (1.6)-(1.7) and a fixed point problem for nonexpasive mapping in real Hilbert spaces.

We observed that to show the convergence of their proposed algorithms Moudafi, Byrne and Kazmi et al. were focused on the resolvents of multi-valued monotone mappings and the properties of averaged mappings which required the pre-estimation of spectral radius of the operator A^*A.

It is known to us that the monotone mappings on Hilbert spaces can be regularized into a uni-valued Lipschitz continuous monotone operators through a process known as Yosida approximation. Such operators have been studies broadly due to its significant role in convex analysis, partial differential equations, variational inclusions etc. The Yosida approximation operators are used to estimate the solutions of variational inclusion or system of variational inclusion, multi-valued differential equations and elliptic boundary value problems. The existence of multi-valued stochastic differential equations with maximal monotone mappings have been proved through the Yosida approximation method [14]. We notice that for being firmly nonexpansive, the resolvent of maximal monotone mappings are given much attention in comparison to the Yosida approximation operator, to solve various variational inclusions, split variational inclusion problems and split common null point problem etc. Many authors have studied and generalized split common null point problems in different directions and used the resolvents of multi-valued monotone mappings and their properties to show the convergence of proposed algorithms see for example [1, 2, 3, 6, 8, 10, 15, 11, 12, 16] and references therein.

Motivated by the work of Censor, Moudafi, Byrne et al., in this paper, we propose and investigate two iterative algorithms using Yosida approximation of multi-valued monotone mappings to solve SCNPP (1.6)-(1.7) such that the pre-calculation of spectral radius of the operator A^*A is not needed. We establish weak convergence results to solve SCNPP (1.6)-(1.7) with new three supporting lemmas. Our method for solving SCNPP (1.6)-(1.7)) is new and different from others which have been proved earlier using resolvents of multi-valued monotone mappings.

2. Preliminaries

Throughout this paper, we always assume that H, H_1 and H_2 are real Hilbert spaces, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$.

Definition 2.1 Let $V : H \to 2^H$ be a multi-valued mapping, then V is called monotone, if

$$\langle u - v, x - y \rangle \geq 0, \; for\ all\ u \in V(x)\ and\ v \in V(y).$$

V is called maximal, if the $graph(V)$ is not properly contained in the graph of any other monotone mapping. The resolvent and Yosida approximation of V are defined as

$$R_\lambda^V(x) = (I + \lambda V)^{-1}(x) \;\; and \;\; J_\lambda^V(x) = \frac{1}{\lambda}(I - R_\lambda^V)(x), \; for\ all\ x \in H\ and\ \lambda > 0.$$

It is well known that R_λ^V and J_λ^V are uni-valued if V is maximal monotone.

Definition 2.2 Let $T : H \to H$ be a mapping, then T is called nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in H.$$

T is called firmly nonexpansive, if

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2, \forall x, y \in H.$$

T is called α-inverse strongly monotone, there exits a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2, \forall x, y \in H.$$

Remark 2.3 It can be easily seen that T is α-inverse strongly monotone, if αT is firmly nonexpansive for $\alpha > 0$.

Lemma 2.4 [4] Let H be a Hilbert space. A mapping $T : H \to H$ is α-inverse strongly monotone if and only if $I - \alpha T$ is firmly nonexpansive.

Lemma 2.5 If x, y and z are positive real numbers, then the following inequality holds

$$x^2 + \frac{y^2}{z} \geq \frac{(x+y)^2}{1+z}.$$

Theorem 2.6 [9] Let H be a Hilbert space and $T : H \to H$ be a firmly nonexpansive mapping such that $Fix(T) = \{x \in H : Tx = x\} \neq \phi$. Define

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n. \end{cases}$$

If α_n is a sequence in $[0, 2]$ such that

$$\sum_{n=1}^{\infty} \alpha_n(2-\alpha_n) = \infty,$$

Then the sequence $\{x_n\}$ converges weakly to an element $x \in Fix(T)$ and $x = \lim_{n\to\infty} P_{Fix(T)} x_n$.

Theorem 2.7(Krasnosel'skill-Mann Theorem) Let $T : H \to H$ be averaged and assume that $Fix(T) \neq \phi$. Then, for any starting point x_0, the sequence $\{T^n x_0\}$ converges weakly to a fixed point of T.

3. Main Results

We denote the solution set of SCNPP (1.6)-(1.7), by

$$\Gamma = \{x \in H_1 : 0 \in V_1(x) \text{ and } Ax \in H_2 \text{ such that } 0 \in V_2(Ax)\}.$$

Lemma 3.1 Let V be multi-valued maximal monotone mapping and R_λ^V and J_λ^V be resolvent and Yosida approximation of V, respectively. Then the following are equivalent

(i) $x^* \in H$ such that $0 \in V(x^*)$,

(ii) $R_\lambda^V x^* = x^*$,

(iii) $J_\lambda^V x^* = 0$.

Proof. Let V be a maximal monotone operator and $\lambda > 0$. Then it can be easily seen that

$$0 \in V(x^*) \Leftrightarrow x^* = R_\lambda^V x^* \Leftrightarrow 0 = x^* - R_\lambda^V x^* \Leftrightarrow 0 = \frac{1}{\lambda}(x^* - R_\lambda^V x^*) \Leftrightarrow 0 = J_\lambda^V x^*.$$

□

To elaborate Lemma 3.1, we have the following example.

Example 3.2 Let $V_1, V_2 : \mathbb{R} \to \mathbb{R}$ are two maximal monotone mappings defined as $V_1(x) = (2x+3)$, and $V_2(x) =$

$-2(1-x)$. Then for $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{4}$, the resolvents and Yosida approximation operators of V_1 and V_2, are given by

$$R_{\lambda_1}^{V_1} = [I+\lambda_1 V_1]^{-1} = \frac{x-3}{3},$$
$$R_{\lambda_2}^{V_2} = [I+\lambda_1 V_2]^{-1} = \frac{2x+1}{3},$$
$$J_{\lambda_1}^{V_1} = \frac{1}{\lambda_1}[I - R_{\lambda_1}^{V_1}] = \frac{2x+3}{3},$$
$$J_{\lambda_2}^{V_2} = \frac{1}{\lambda_1}[I - R_{\lambda_1}^{V_2}] = \frac{4x-4}{3}.$$

We can easily see that

$$0 \in V_1(\frac{-3}{2}) \Leftrightarrow R_{\lambda_1}^{V_1}(\frac{-3}{2}) = \frac{-3}{2} \Leftrightarrow J_{\lambda_1}^{V_1}(\frac{-3}{2}) = 0.$$

and

$$0 \in V_2(1) \Leftrightarrow R_{\lambda_2}^{V_2}(1) = 1 \Leftrightarrow J_{\lambda_2}^{V_2}(1) = 0.$$

□

Keeping in mind the Lemma 3.1, we can write

$$\Gamma = \{x \in H_1 : J_{\lambda_1}^{V_1} x = 0 \text{ and } Ax \in H_2 \text{ such that } J_{\lambda_2}^{V_2} Ax = 0\}. \tag{3.1}$$

Now, we are prove the supporting lemmas which are used in our main results.

Lemma 3.3 Let $V_1 : H_1 \to 2^{H_1}$, $V_2 : H_2 \to 2^{H_2}$ be two multi-valued maximal monotone mappings and $A : H_1 \to H_2$ be a bounded linear operator. Then the operator $I - \theta L$ is firmly nonexpansive, where $L = J_{\lambda_1}^{V_1} + A^*(J_{\lambda_2}^{V_2})A$, $J_{\lambda_1}^{V_1}, J_{\lambda_2}^{V_2}$ are Yosida approximation operators of V_1 and V_2, respectively, $\lambda_1, \lambda_2 > 0$ and $\theta = \frac{\min\{\lambda_1,\lambda_2\}}{1+\|A\|^2}$.

Proof. Let $x, y \in H_1$, since $J_{\lambda_1}^{V_1}, J_{\lambda_2}^{V_2}$ are λ_1 and λ_2-inverse strongly monotone, respectively. Then, using Lemma 2.5, we have

$$\begin{aligned}
\langle x-y, Lx-Ly\rangle &= \langle x-y, J_{\lambda_1}^{V_1}x + A^*(J_{\lambda_2}^{V_2})Ax - J_{\lambda_1}^{V_1}y + A^*(J_{\lambda_2}^{V_2})Ay\rangle \\
&= \langle x-y, J_{\lambda_1}^{V_1}x - J_{\lambda_1}^{V_1}y\rangle + \langle Ax-Ay, (J_{\lambda_2}^{V_2})Ax - (J_{\lambda_2}^{V_2})Ay\rangle \\
&\geq \lambda_1\|J_{\lambda_1}^{V_1}x - J_{\lambda_1}^{V_1}y\|^2 + \lambda_2\|(J_{\lambda_2}^{V_2})Ax - (J_{\lambda_2}^{V_2})Ay\|^2 \\
&\geq \min\{\lambda_1,\lambda_2\}\Big\{\|J_{\lambda_1}^{V_1}x - J_{\lambda_1}^{V_1}y\|^2 + \frac{\|A^*(J_{\lambda_2}^{V_2})Ax - A^*(J_{\lambda_2}^{V_2})Ay\|^2}{\|A^*\|^2}\Big\} \\
&\geq \min\{\lambda_1,\lambda_2\}\Big\{\frac{(\|J_{\lambda_1}^{V_1}x - J_{\lambda_1}^{V_1}y\| + \|A^*(J_{\lambda_2}^{V_2})Ax - A^*(J_{\lambda_2}^{V_2})Ay\|)^2}{1+\|A^*\|^2}\Big\} \\
&\geq \frac{\min\{\lambda_1,\lambda_2\}}{1+\|A\|^2}\|Lx-Ly\|^2 \\
&= \theta\|Lx-Ly\|^2
\end{aligned}$$

that is, L is θ-inverse strongly monotone. Therefore by Lemma 2.4, $I - \theta L$ is firmly nonexpansive. □

Lemma 3.4 Let $\Gamma = \{x \in H_1 : J_{\lambda_1}^{V_1} x = 0 \text{ and } Ax \in H_2 \text{ such that } J_{\lambda_2}^{V_2} Ax = 0\} \neq \phi$, then $Fix(I - \theta L) = \Gamma$, where $L = J_{\lambda_1}^{V_1} + A^*(J_{\lambda_2}^{V_2})A$, and $\theta = \frac{\min\{\lambda_1,\lambda_2\}}{1+\|A\|^2}$.

Proof. It can be easily seen that $\Gamma \subseteq Fix(I-\theta L)$. To show that $Fix(I-\theta L) \subset \Gamma$, let $u \in Fix(I-\theta L)$ and $p \in \Gamma$. Thus, $Lu = 0$ and by (3.1), we have $Lp = 0$. Now, using the same steps as in Lemma 3.3, we have

$$\langle u-p, Lu-Lp\rangle \geq \min\{\lambda_1,\lambda_2\}\Big\{\|J_{\lambda_1}^{V_1}u - J_{\lambda_1}^{V_1}p\|^2 + \|J_{\lambda_2}^{V_2}Au - J_{\lambda_2}^{V_2}Ap\|^2\Big\}$$
$$0 = \min\{\lambda_1,\lambda_2\}\Big\{\|J_{\lambda_1}^{V_1}u\|^2 + \|J_{\lambda_2}^{V_2}Au\|^2\Big\} \geq 0,$$

this implies that $J_{\lambda_1}^{V_1}u = 0$ and $J_{\lambda_2}^{V_2}Au = 0$. Thus $u \in \Gamma$. This completes the proof. □

Theorem 3.5 Let H_1 and H_2 be Hilbert spaces. Suppose $V_1 : H_1 \to 2^{H_1}$, $V_2 : H_2 \to 2^{H_2}$ be two multi-valued maximal monotone mappings and $J_{\lambda_1}^{V_1}$, $J_{\lambda_2}^{V_2}$ be Yosida approximation operators of V_1 and V_2, respectively. Define

$$\begin{cases} x_0 \in H_1, \\ x_{n+1} = x_n - \tau_n[J_{\lambda_1}^{V_1} + A^*(J_{\lambda_2}^{V_2})A]x_n, \end{cases}$$

where τ_n is a sequence in $[0, \frac{2\lambda}{1+\|A\|^2}]$, $\lambda = \min\{\lambda_1,\lambda_2\}$, such that

$$\sum_{n=1}^{\infty} \tau_n\Big(2 - \Big(\frac{1+\|A\|^2}{\lambda}\Big)\tau_n\Big) = \infty.$$

Then $\{x_n\}$ generated in the above iterative process converges weakly to $z \in \Gamma$ and $z = \lim_{n\to\infty} P_\Gamma x_n$.

Proof. Let $L = J_{\lambda_1}^{V_1} + A^*(J_{\lambda_2}^{V_2})A$, then by Lemma 3.3, $I - \theta L$ is firmly nonexpansive, where $\theta = \frac{\min\{\lambda_1,\lambda_2\}}{1+\|A\|^2}$. Furthermore

$$x_{n+1} = (I - \tau_n L)x_n = (1 - \frac{\tau_n}{\theta})x_n + \frac{\tau_n}{\theta}(I-\theta L)x_n.$$

Applying Theorem 2.6, Lemma 3.3 and Lemma 3.4, we obtain that $x_n \rightharpoonup z = Fix(I-\theta L) = \Gamma$ and $z = \lim_{n\to\infty} P_\Gamma(x_n)$. □

Theorem 3.6 Let H_1 and H_2 be two real Hilbert spaces; $A : H_1 \to H_2$ be a bounded linear operator and A^* be the adjoint of A. Let $V_1 : H_1 \to 2^{H_1}$ and $V_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Then for any arbitrary point x_0, the sequence $\{(I-\theta L)x_0\}$ converges weakly to a fixed point of $(I-\theta L)$, which is a solution of SCNPP (1.6)-(1.7), where $L = J_{\lambda_1}^{V_1} + A^*(J_{\lambda_2}^{V_2})A$.

Proof. From Lemma 3.3, the operator $\{(I-\theta L)\}$ is firmly nonexpansive and hence averaged, therefore Theorem 2.7 implies that the sequence generated by $\{(I-\theta L)x_0\}$ for an arbitrary $x_0 \in H_1$ converges weakly to fixed point of $\{(I-\theta L)\}$ which is by Lemma 3.4, is a solution of SCNPP (1.6)-(1.7). □

4. Conclusion

In this paper, we present two weak convergence theorems to solve split common null point problem. We present two iterative algorithms based on the Yosida approximation operators of corresponding monotone mappings in Hilbert spaces such that the convergence of proposed algorithm do not required the estimation of spectral radius of the operator AA^*. We also give three new supporting lemmas to prove our main results. Use of Yosida approximation operators to solve SCNPP make our results new and different from those found in the literature.

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