A Generalization of Uniform Boundedness Principle

Zeinab Bandpey and Bhamini M. P. Nayar
Department of Mathematics, Morgan State University,
Baltimore, MD 21251, USA.
E-mail: Zeinab.Bandpey@morgan.edu

ABSTRACT. In 1998, several generalizations of uniform boundedness principle (U.B.P) was given for functions from topological spaces to topological spaces by Joseph, Kwack, and Nayar. Recently, using the definition of $\alpha$-closure operator and $\alpha$-set, we introduced three classes of continuous functions, namely $\alpha u$-continuous, strongly $u$-continuous and semi-$\alpha u$-continuous functions. In this study the U.B.P. is extended using these generalized continuous functions.

1. Introduction

In a study of generalized continuous functions [3] the following concepts were introduced. They are strongly $u$-continuous functions, $au$-continuous functions and semi-$\alpha au$-continuous functions. A function $g : X \to Y$ is $\alpha u$-continuous (strongly $u$-continuous, semi-\$\alpha u$-continuous) at $x \in X$, if for each $\alpha$-set ($\alpha$-set, open set) $W$ which contains a closed neighborhood of $g(x)$, there exists some $\alpha$-set (open set, $\alpha$-set) $V$ which contains a closed neighborhood of $x$ which satisfies the condition $g(clV) \subseteq clW$. If $g$ is $\alpha u$-continuous (strongly $u$-continuous, semi-$\alpha u$-continuous) at each $x \in X$, we say $g : X \to Y$ is $\alpha u$-continuous (strongly $u$-continuous, semi-$\alpha u$-continuous) on $X$.

In [17] generalizations of uniform boundedness principle were given. The functions considered there were from a class of Baire topological spaces to a countable union of its subspaces satisfying some generalized compactness properties. In [12] other generalizations of the uniform boundedness principle were given for functions from topological spaces to topological spaces as applications of $u$-continuous functions introduced there. The

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topological spaces considered there were Urysohn closed spaces and Quasi Urysohn Closed spaces. The definitions of these concepts are given in the section for preliminaries. In [3] we defined and studied the above mentioned three classes functions as generalizations of u-continuous function. Further, these classes of functions are used to give several characterizations of compact and extremally disconnected spaces in [4]. The present article is a study of another application of these classes of functions to give generalizations of uniform boundedness principle in topological spaces.

In classical analysis uniform boundedness principle is stated as follow; [19] let \( \mathcal{F} \) be a family of continuous real-valued functions on a complete metric space \( X \) such that for each \( x \in X \), there is some constant \( M_x \) such that \( |f(x)| \leq M_x \) for all \( f \in \mathcal{F} \). Then there is some constant \( M \) and a non-empty open set \( U \) in \( X \) such that \( |f(x)| \leq M \) for each \( x \in U \) and each \( f \in \mathcal{F} \). To provide a generalization of this principle we introduce a class of spaces called \( \alpha \)-Quasi Urysohn Closed (\( \alpha \)-QUC) space, a subclass of the well-known class of QUC spaces. We provide a second category type property for this class of spaces. We further used the class of function introduced in [3] to identify collection of subsets of \( \alpha \)-QUC space \( X \) on which a family of functions from \( X \) to an arbitrary space \( Y \) is uniformly bounded. These concepts are defined in the section of preliminaries.

2. Preliminaries

The concept of \( \alpha \)-sets are defined by Njastad in 1965 [16]. Sets for which \( A \subseteq int(cl(intA)) \), where \( intA \) and \( clA \) represents the interior and closure respectively of \( A \). The complement of each \( \alpha \)-set is called \( \alpha \)-closed set. One can easily see that every open set is an \( \alpha \)-set, but every \( \alpha \)-set need not be open as is demonstrated in [16].

The collection of \( \alpha \)-sets of a space form a topology and it is denoted by \( \tau_\alpha \). Let \( X \) be a topological space. For \( A \subset X \), \( \Sigma(A) \) represents the collection of open sets containing \( A \). The family \( \Gamma(A) \) is the collection of closed neighborhoods of \( A \) and let \( \Lambda(A) = \bigcup_{\Gamma(A)} \Sigma(A) \), union of open sets containing closed neighborhood of the set. (if \( A = \{x\} \) the notation \( \Sigma(x) \), \( \Gamma(x) \), or \( \Lambda(x) \) will be used). We shall denote \( \alpha\Lambda(A) \) and \( \alpha\Lambda(x) \) as collection of \( \alpha \)-sets containing a closed neighborhood of \( A \) and \( x \) respectively. The concept of \( \theta \)-closure of a set was introduced by Veličko [18]. For a topological space \( X \) and for a filterbase \( \Omega \) on \( X \), the adherence of \( \Omega \) is denoted by \( ad\Omega \) and \( ad\Omega = \cap_{\tau_\Omega} clF \).

In [10] the concept of \( u \)-closure of set was introduced by Joseph and was used to study, among others, compact spaces as well as Urysohn-closed spaces. Let \( A \subseteq X \). The \( u \)-closure of \( A \), denoted as \( cl_u(A) = \{x : clV \cap A \neq \emptyset, V \in \Lambda(x)\} \). A set \( A \) is \( u \)-closed if \( A = cl_uA \). This concept was further used in the study of compact and extremally disconnected spaces [4]. A filterbase \( \Omega \) on a space \( X \) is said to \( u \)-converge to \( x \in X \) if for each \( W \in \Lambda(x) \) there exists \( F \in \Omega \) such that \( F \subset clW \). A filter base \( \Omega \) on a space \( X \) \( u \)-converges to \( x \in X \) if for each \( \alpha \)-set \( W \) which contains a closed neighborhood of \( x \), there exists \( F \in \Omega \) such that \( F \subset clW \).

A point \( x \in X \) is in \( u \)-closure of \( A \), denoted as \( cl_{au}A \) [3], if \( cl_uV \cap A \neq \emptyset \) for every \( \alpha \)-set \( V \) containing a closed neighborhood of \( x \). i.e. \( cl_{au}A = \{x \in X : clV \cap \Lambda \neq \emptyset, V \in \tau_\alpha, \Lambda \text{ contains a closed neighborhood of } x\} \). A is called \( au \)-closed if \( cl_{au}A = A \). So, \( au \)-closure of a set \( A \) is denoted as \( cl_{au}A \) and defined as \( cl_{au}A = \cap_{\Lambda(A)} clV \).

Since every open set is an \( \alpha \)-set, the \( au \)-closure of a set is contained in the \( u \)-closure of the set. Let \( \Omega \) be a filter base on \( X \). A point \( x \) is an \( au \)-adherent point of a filter base \( \Omega \), denoted as \( ad_{au}\Omega \) if \( x \) belongs to the \( au \)-closure of every \( F \) in \( \Omega \) and \( au \)-adherence of \( \Omega \) denoted by \( ad_{au}\Omega = \cap_{\tau_\Omega} cl_{au}F \).
A space \( X \) is Urysohn if each singleton is \( u \)-closed [12]. A Urysohn filterbase on a space \( X \) is an open filterbase \( \Omega \) such that whenever \( x \in X - ad\Omega \) some \( V \in \Sigma(x) \) and \( F \in \Omega \) satisfies \( clV \cap clF = \emptyset \). A Urysohn space is Urysohn-closed if each Urysohn filterbase on the space has non-empty adherence. An arbitrary space is Quasi Urysohn-closed (QUC) if each Urysohn filterbase on the space has non-empty intersection. Herrington [10] has defined a space to be Quasi Urysohn-closed if each Urysohn filterbase on the space has non-empty adherence. An arbitrary space is Quasi Urysohn-closed (QUC) if each filterbase on the space has non-empty \( u \)-adherence.

A function \( g : X \to Y \) is \( u \)-continuous at \( x \in X \) [12] if for each \( W \in \Lambda(g(x)) \) there exists a \( V \in \Lambda(x) \) such that \( g(clV) \subseteq clW \) and \( g \) is \( u \)-continuous on \( X \) if \( g \) is \( u \)-continuous at each \( x \in X \). A filterbase \( \Omega \) on a space \( X \) is said to \( u \)-converge to \( x \in X \) [12] (\( \Omega \to_u x \)) if for each \( W \in \Lambda(x) \) some \( F \in \Omega \) satisfies \( F \subseteq clW \). [12] For a filter base \( \Omega \) on \( X \), \( u \)-adherence of \( \Omega \) is denoted by \( ad_u \Omega \) and is defined as \( \bigcap \Omega cl_u F \). The concept of \( a \)-set is defined by Njastad in 1965 [14]. Sets for which \( A \subseteq int(cl(intA)) \) are \( a \)-sets, where \( intA \) represents the interior of \( A \). One can easily see that every open set is an \( a \)-set, but an \( a \)-set need not be open as is demonstrated in [14].

If \( X \) is a space, let \( S(x) = \{ V - \{ x \}, V \in \Sigma(x) \} \). For spaces \( X \) and \( Y \), a set valued function \( F : X \to 2^Y \) is a function from \( X \) to \( 2^Y - \{ \emptyset \} \), where \( 2^Y \) is the power set of \( Y \). A set valued function \( F \subseteq X \times 2^Y \) is subclosed (strongly subclosed)\( [u \)-strongly subclosed\] if \( adF(S(x)) \subseteq F(x) \) \( (ad_u F(x) \subseteq F(x)) \) \( [ad_u F(x) \subseteq F(x)] \), for each \( x \in X \) for which \( S(x) \) is a filterbase on \( X \). The value function \( F \) is closed (strongly-closed)\( [u \)-strongly closed\] if \( F(x) \) is closed (\( \theta \)-closed) \( [u \)-closed\] in \( Y \) for each \( x \in X \). [17]

A subset \( A \) of a space \( X \) is Quasi Urysohn-closed (QUC) relative to the space \( X \) if each filterbase \( \Omega \) on \( A \) satisfies \( A \cap ad_u \Omega \neq \emptyset \).

### 3. Generalizations of Uniform Boundedness Principle

Uniform boundedness principle is defined as follow as stated in the introduction: [19] let \( \mathcal{F} \) be a family of continuous real-valued functions on a complete metric space \( X \) such that for each \( x \in X \), there is some constant \( M_x \) such that \( |f(x)| \leq M_x \) for all \( f \in \mathcal{F} \). Then there is some constant \( M \) and a non-empty open set \( U \) in \( X \) such that \( |f(x)| \leq M \) for each \( x \in U \) and each \( f \in \mathcal{F} \).

Generalizations of uniform boundedness principle are presented here. As stated earlier, in [17] uniform boundedness principle was generalized for functions from Baire spaces satisfying certain properties. A topological space is a Baire space if the intersection of each sequence of dense open subsets of the space is dense in the space [12]. A subset \( Q \) of a topological space is of second category in \( X \) if \( Q \) is not the union of a countable number of nowhere dense subsets of \( X \). It is clear that a topological space is a Baire space iff each non-empty open subset of the space is of second category in \( X \) [12].

Henceforth, we adopt the terminology and notations used in [12].

**Definition 3.1.** A family \( \mathcal{F} \) of functions from a space \( X \) to a space \( Y \) is uniformly bounded on \( A \subseteq X \) by \( \Delta \subseteq 2^Y \) if there is a \( C \in \Delta \) such that \( g(A) \subseteq C \) for each \( g \in \mathcal{F} \) [12]. Let \( \mathcal{U}(\mathcal{F}, X, Y, \Delta) = \{ A \subseteq X : \mathcal{F} \text{ uniformly bounded on } A \text{ by } \Delta \} \).

Using the above terminology, Uniform Boundedness Principle is stated as follows[12]: let \( X \) be a complete metric space, let \( \mathbb{R} \) be the Euclidean line, let \( \mathcal{F} \) be the family of real-valued continuous functions on \( X \) and let \( \Delta \) be...
non-empty countable family of compact subsets of the reals. If \( \{B \mid [F, X, R, \Delta] \} \) contains the collection of singletons then \( \{B \mid [F, X, R, \Delta] \} \) has a non-empty open subset as an element.

The following definitions are needed before we present our generalizations of uniform boundedness principle.

**Definitions 3.2.** [4]

a) A relation \( R \subseteq X \times Y \) is \( au \)-strongly subclosed if \( ad_{au} R(\Omega) \subseteq R(x) \) for each \( x \in X \) and each filterbase \( \Omega \) on \( X - \{x\} \) with \( \Omega \to x \).

A function \( f : X \to Y \), \( f \subseteq X \times Y \) has \( au \)-strongly subclosed inverse if \( f^{-1} \subseteq f(X) \times X \) is \( au \)-strongly subclosed.

b) A relation \( R \subseteq X \times Y \) is \( u \)-strongly subclosed if \( ad_{u} R(\Omega) \subseteq R(x) \) for each \( x \in X \) and each filterbase \( \Omega \) on \( X - \{x\} \) with \( \Omega \to x \).

A function \( f \) from \( X \) to \( Y \), \( f \subseteq X \times Y \) has \( u \)-strongly subclosed inverse if \( f^{-1} \subseteq f(X) \times X \) is \( u \)-strongly subclosed.

A \( u \)-strongly subclosed relation is an \( au \)-strongly subclosed relation as well.

**Definitions 3.3.**

a) A set \( A \) is said to be \( au \)-dense in \( X \), if \( cl_{au} A = X \).

b) A set \( A \) is \( au \)-nowhere dense in \( X \), if \( \text{int}(cl_{au} A) = \emptyset \), where a point \( x \in A \) is an \( a \)-interior point of \( A \), if there exists an \( a \)-set containing \( x \) contained in \( A \).

c) A subset \( A \) is of \( au \)-first category in \( X \) if \( A = \cup_{F \in \mathcal{A}} F \), where \( \mathcal{A} \) is a countable family of subsets of \( X \) such that \( \text{int}_{a}(cl_{au}(F)) = \emptyset \) for \( F \in \mathcal{A} \).

d) A subset \( A \) is of \( au \)-second category in \( X \) if it is not of \( au \)-first category, i.e. \( A \) is of \( au \)-second category if \( A \) is not the union of countable family \( \mathcal{A} \) of subsets of \( X \) such that \( \text{int}_{a}(cl_{au}(F)) = \emptyset \) for \( F \in \mathcal{A} \).

e) A space is \( au \)-Baire if \( cl_{au} W \) is of the \( au \)-second category for each non-empty \( a \)-subset of \( X \).

**Result 3.4.**

If \( A \) is an open \( au \)-dense subset of \( X \), then \( X - A \) is \( au \)-nowhere dense set.

**Proof.** Let \( A \) be an open \( au \)-dense, then \( cl_{au} A = X \). Therefore \( \text{int}(cl_{au} A) = \text{int}(X) = X \). Therefore, \( \text{int}(cl_{au}(X - A)) = cl(X - (X - cl_{au}(X - A))) = cl(cl_{au}(X - A)) = X - \text{int}(cl_{au} A) = X - X = \emptyset \).

**Definition 3.5.** A subset \( A \) of a space \( X \) is \( a \)-Quasi Urysohn Closed (\( a \)-QUC) relative to the space \( X \) if each filterbase \( \Omega \) on \( A \) satisfies \( A \cap ad_{au} \Omega \neq \emptyset \).

Note \( ad_{au} \Omega \subseteq ad_{a} \Omega \), therefore every \( a \)-QUC space is a QUC space but not every \( cl_{a} A \) for \( A \subset X \) is \( cl_{au} A \), so an \( a \)-QUC space might not be a QUC space.

**Theorem 3.6.**

Let \( X \) and \( Y \) be spaces and let \( g : X \to Y \) have a \( au \)-subclosed graph. Then \( g^{-1}(A) \) is \( au \)-closed in \( X \) for each \( a \)-QUC relative to \( Y \) subset \( A \).
Proof. For \( v \in cl_{au}(g^{-1}(A) - \{v\}) \), there is a filterbase \( \Omega \) on \( g^{-1}(A) - \{v\} \) such that \( \Omega \to_{au} v[3] \). Then \( g(\Omega) \) is a filterbase on \( A \) and since \( \emptyset \neq A \cap ad_{au}g(\Omega) \) as \( A \) is a-QUC relative to \( Y \). Also, \( A \cap ad_{au}g(\Omega) \subset \{g(v)\} \) since \( g \) has \( au \)-subclosed graph. It follows that \( \emptyset \neq A \cap ad_{au}g(\Omega) \subset \{g(v)\} \). Hence \( v \in g^{-1}(A) \). Thus \( g^{-1}(A) \) is \( au \)-closed.

Theorem 3.7.
Let \( X \) and \( Y \) be spaces and let \( A \) be a-QUC relative to \( Y \). If \( g : X \to Y \) has a \( au \)-subclosed graph and \( g(X) \subset A \) then \( g \) is \( au \)-continuous.

Proof. Let \( v \in X \). If \( V = \{v\} \), and \( clV = \{v\} \) for some \( v \)-set \( V \) containing a closed neighborhood of \( v \), then \( g \) is \( au \)-continuous at \( v \). Otherwise, \( \Omega = \{clV - \{v\} : V \text{ is an } \alpha \text{-set containing a closed neighborhood of } v\} \) is a filterbase on \( X \), and \( ad_{au}g(\Omega) \subset \{g(v)\} \) since \( g \) has \( au \)-subclosed graph. Let \( W \) be an \( \alpha \)-set containing closed neighborhood of \( g(v) \). Then there exists a \( F \in \Omega \) such that \( g(F) \subset clW \). Since \( g(X) \subset A \), an a-QUC relative to \( Y \), \( \{g(F) - clW\} \subset A \). If not then \( \gamma = \{g(F) - clW : F \in \Omega\} \) is a filterbase on \( A \), \( g(v) \notin ad_{au} \gamma \subset ad_{au}g(\Omega) \), which is a contradiction since we found a filterbase with empty \( au \)-adherence in an a-QUC set \( A \).

Lemma 3.8. If \( \{A_\beta; \beta \in \} \) is a collection of \( au \)-closed subsets of a space \( X \), then \( \bigcap_{\beta \in \} A_\beta \) is an \( au \)-closed set.

Proof. Since each \( A_\beta \) is \( au \)-closed for each \( \beta \in \), if \( x \in X - A_\beta \), there exists an \( \alpha \)-set \( U \) containing a closed neighborhood of \( x \) such that \( cl_{\alpha}U \cap A_\beta = \emptyset \). Hence there exists an \( \alpha \)-set containing a closed neighborhood of \( x \) such that \( cl_{\alpha}U \subset X - A_\beta \). This is true for each \( A_\beta \), therefore \( cl_{\alpha}U \subset \bigcup_{\beta \in \} X - A_\beta = X - \bigcap_{\beta \in \} A_\beta \). Hence \( cl_{\alpha}U \cap \bigcap_{\beta \in \} A_\beta \) = \( \emptyset \).

Theorem 3.9. Let \( X \) be a \( au \)-Baire space, Let \( Y \) be a space, Let \( \Delta \) be a non-empty countable family of a-QUC relative to \( Y \) subsets and \( \mathcal{F} \) be a family of functions from \( X \) to \( Y \) with \( au \)-subclosed graphs such that \( \mathcal{U}([\mathcal{F}, X, Y, \Delta]) \) contains the collection of singletons of \( X \). Then

1. There is a non-empty \( \alpha \)-set \( V \) of \( X \) such that \( cl_{au}V \in \mathcal{I}([\mathcal{F}, X, Y, \Delta]) \).
2. For each \( g \in \mathcal{F} \) the restriction of \( g \) to \( cl_{au}V \) of part (1) is \( au \)-continuous.
3. Each \( g \in \mathcal{F} \) is \( au \)-continuous at each point of \( V \) of part (1).
4. There is a \( W \) an \( \alpha \)-set in \( X \) such that each \( g \in \mathcal{F} \) is \( au \)-continuous at \( x \in W \) and \( cl_{au}W = X \).

Proof.

1. For each \( C \in \Delta \) let \( I(C) = \cap g^{-1}(C) \). Then \( I(C) \) is \( au \)-closed, in view of Theorem 3.6 and Lemma 3.8. If \( x \in X \) there is a \( C(x) \in \Delta \) such that \( g(x) \in C(x) \) for all \( g \in \mathcal{F} \). Hence \( \{I(C); C \in \Delta \} \) is a cover of \( X \) by a countable family of \( au \)-closed sets. Since \( X \) is of \( au \)-second category it is not contained in the union of a countable family \( A \) of subsets of space such that \( int_{au}(cl_{au}(F)) = \emptyset \) for \( F \in A \). Therefore there exists a \( C_0 \in \Delta \) such that \( int_{au}(I(C_0)) \neq \emptyset \).

Let \( V = int_{au}(I(C_0)) \), so \( cl_{au}V \in I(C_0) \). Hence for each \( g \in \mathcal{F} \), \( g(cl_{au}V) \subset g(I(C_0)) \subset g(g^{-1}(C_0)) \subset C_0 \). Hence \( cl_{au}V \in \mathcal{I}([\mathcal{F}, X, Y, \Delta]) \).

2. Let \( V \) be a \( \alpha \)-set described in part (1), and let \( g \in \mathcal{F} \). Since \( C_0 \) is a-QUC relative to \( Y \), and the restriction of \( g \) to \( cl_{au}V \) has \( au \)-subclosed graph, by the theorem 3.7, \( g|_{clV} \) is \( au \)-continuous.
(3) Let V be an $\alpha$-set of the type guaranteed in part (1). Let $x \in V$, and let $g \in F$ and let W be an $\alpha$-set containing a closed neighborhood of $g(x)$, since restriction of g to $cl_{au} V$ is au-continuous, there is an $\alpha$-set $Q$ containing a closed neighborhood of $x$ such that $g|_{cl_{au} V}(cl(Q \cap cl_{au} V)) \subset cl(W)$. Since $Q$ is an $\alpha$-set and V is an $\alpha$-set, then $Q \cap V$ is an $\alpha$-set containing a closed neighborhood of $x$, and $cl(Q \cap V) \subseteq cl(Q \cap cl_{au} V)$ therefore $g(cl(Q \cap V)) \subseteq g(cl(Q \cap cl_{au} V)) \subset cl(W)$. Therefore $g(cl(Q \cap V)) \subset cl(W)$. Hence $g$ is au-continuous.

(4) Let $W$ be union of all $\alpha$-sets $V$ such that each $g \in F$ is au-continuous at each $x \in V$. Then each $g \in F$ is au-continuous in each $x \in W$. If $A$ is a non-empty $\alpha$-subset of the au-Baire space $X$, then $cl_{au} A$ is of au-second category and $F_{cl_{au} V} F$ satisfies the conditions on $F$ in the hypothesis (relative to $X$). Hence from part (1) and (2) there is a non-empty $\alpha$-set B of $cl_{au} V$ with each $h \in F_{cl_{au} V} F$ au-continuous at each point of the non-empty subset $A \cap H$ where $B = H \cap cl_{au} V$ and $H$ is a set in $X$. It follows as in the proof of part (3) that each $g \in F$ is a u-continuous at each $x \in A \cap H$. Hence $A \cap H \subset W$, $A \cap W \neq \emptyset$, and $cl_{au} W = X$.

The following results lead us to the final Main Theorem.

**Theorem 3.10.** The following statements are equivalent for the space $X$ and $A \subset X$.

1. The relation $int_A(cl_{au}(A)) = \emptyset$ holds.
2. If $V$ is a nonempty $\alpha$-subset of $X$ and $W$ is an $\alpha$-set containing a closed neighborhood of $V$, there is an $x \in V$ and $Q$, an $\alpha$-set containing closed neighborhood of $x$ such that $Q \subset W$ and $A \cap cl Q = \emptyset$.
3. For each nonempty $\alpha$-set $V$ in $X$ there is a nonempty $\alpha$-set $Q \subset V$ such that $Q \cap cl_{au}(A) = \emptyset$.

**Proof.**

1) $\Rightarrow$ 2) If (1) holds and $V$ is a nonempty $\alpha$-subset of $X$, then $V - cl_{au}(A) \neq \emptyset$. Let $x \in V - cl_{au}(A)$. Then there exists an $\alpha$-set $P$ containing a closed neighborhood of $x$ such that $A \cap cl P = \emptyset$. If $W$ is an $\alpha$-set containing a closed neighborhood of $V$ and if $Q$ is an $\alpha$-set containing a closed neighborhood of $x \in V$, then $Q \subset W$, and $A \cap cl Q = \emptyset$.

2) $\Rightarrow$ (3) Suppose (2) holds and let $V$ be a nonempty $\alpha$-set in $X$. From (2) there is an $x \in V$ and $\alpha$-set $Q$ containing a closed neighborhood of $x$ satisfying $A \cap cl Q = \emptyset$. Let $T$ be an $\alpha$-set containing $x$ with $Q \subset aA(T)$. Then $X - cl_{au} V \in aA(A)$. Since $Q \subset V$, $cl Q \subset cl V$, therefore $X - cl V \subset X - cl Q$, so $A \subset X - cl Q$. Also from (2), $A \cap cl Q = \emptyset$, since $cl T \subset Q$, then $A \cap cl T = \emptyset$. Therefore $A \subset X - cl Q \subset X - cl T$, and $cl_{au} A \subset cl_{au}(X - cl T)$. Consequently, $V \cap T \cap cl_{au}(A) \subset T \cap cl(X - cl T) = \emptyset$. So $V \cap T = T$ is the desired $\alpha$-set satisfying (3).

3) $\Rightarrow$ (1) Assume (3) holds, so for each $\alpha$-set $V$ in $X$ there is a nonempty $\alpha$-set $Q \subset V$ such that $Q \cap cl_{au} A = \emptyset$, therefore $int_A(cl_{au}(A)) = \emptyset$.

**Theorem 3.11.** If $X$ is a space $A \subset X$ satisfies $int_A(cl_{au}(A)) = \emptyset$, then for each nonempty $\alpha$-set $V \subset X$ there is a nonempty $\alpha$-set $Q \subset V$ such that $A \cap cl_{au} Q = \emptyset$.

**Proof.** If $int_A(cl_{au}(A)) = \emptyset$, and $V$ is a nonempty $\alpha$-set, subset of $X$, there is an $\alpha$-set $W$ containing a closed neighborhood of $A$ such that $V - cl W \neq \emptyset$. Then $Q = V - cl W$ satisfies $Q \subset V$ and $A \cap cl_{au}(Q) = \emptyset$. 


Theorem 3.12. If $X$ is an $a$-QUC space and $V \subset X$ is a nonempty $a$-set, then $cl_{au}(V)$ is not contained in the union of countable family, $\Omega$, of subsets of $X$ such that $int_{a}(cl_{au}(F)) = \emptyset$ for all $F \in \Omega$.

Proof. Let $\Omega = \{F(n) : n \in N\}$ such that $int_{a}(cl_{au}(F(n))) = \emptyset$ for each $F(n) \in \Omega$. Using Theorem 3.11, a decreasing sequence $\{V(n) : n \in N\}$ of nonempty $a$-sets is constructed inductively such that $V(n + 1) \subseteq V(n) \subseteq \ldots \subseteq V(1) \subseteq V$ and $F(n) \cap cl_{au}(V(n)) = \emptyset$ for each $n$. Therefore, $cl_{au}V \subset (\bigcup_{i=1}^{n} F(i)) = \emptyset$. Since $X$ is $a$-QUC, there is an $x \in \bigcap_{i} cl_{au}(V(n))$. So, $cl_{au}V \neq \bigcup_{i} F(n)$ and $int_{a}(cl_{au}(F(n))) = \emptyset$. Since, $x \in cl_{au}V - \bigcup_{i} F(n)$.

Corollary 3.13. Every nonempty $a$-subset of an $a$-QUC space is of $au$-second category.

Proof.

If $X$ is $a$-QUC and $V \subset X$ is a nonempty $a$-set, then $cl_{au}(V)$ is not contained in the union of countable family of $au$-closed subsets with empty $a$-interiors.

Corollary 3.14. If $X$ is $a$-QUC and $W$ is an $a$-set containing a closed neighborhood of $A$ for some nonempty subset $A$, then $clW$ is not contained in the union of countable family, $\Omega$, of subsets of $X$ such that $int_{a}(cl_{au}(F)) = \emptyset$ for $F \in \Omega$.

Proof. Let $V$ be an $a$-set containing a closed neighborhood of $x$, for $x \in A$ such that $cl_{au}V \subset clW$. In view of 3.12, $cl_{au}V$ is not contained in the union of countable family of $au$-closed subsets with empty $a$-interiors so neither could $clW$.

Theorem 3.15. Let $X$ be a $a$-QUC space, let $Y$ be a space, let $\Delta$ be a nonempty countable family of $a$-QUC relative to $Y$ subsets. Let $\mathcal{B}$ be a family of functions from $X$ to $Y$ with $au$-subclosed graphs such that $\bigcup \mathcal{B}[\mathcal{F}, X, R, \Delta]$ contains the collection of singletons of $X$. Then

1. There is a nonempty $a$-set $V \subset X$ such that $cl_{au}(V) \in \bigcup \mathcal{B}[\mathcal{F}, X, R, \Delta]$.
2. For each $g \in \mathcal{F}$ the restriction of $g$ to $cl_{au}(V)$ for $V$ as stated in part $(1)$ is $au$-continuous.
3. For each $V$ satisfying $(1)$, each $g \in \mathcal{F}$ is $au$-continuous at each point $x$ of $V$ with a closed neighborhood of $x$ is contained in $V$.

Proof. 

(1) It can be proved using Theorems 3.6, 3.7, 3.8, 3.9 part (1) and Theorem 3.12 and corollary 3.13.

(2) Let $V$ be a $a$-set described in part (1), and let $g \in \mathcal{F}$. Since $\Delta_0$ is $a$-QUC relative to $Y$, and the restriction of $g$ to $cl_{au}V$ has $au$-subclosed graph, by the theorem 3.7, $g|_{cl_{au}V}$ is $au$-continuous.

(3) Let $g \in \mathcal{F}$ and $W \in a\Lambda(x)$ satisfying $(1)$, let $Q \in a\Lambda(g(x))$. From (2) there exists an $a$-set $A$ in $X$ such that $A \cap cl_{au}W \in a\Lambda(x)$ in $cl_{au}W$, and $g(cl(A \cap cl_{au}W)) \subset clQ$. Choose $B$, an $a$-set containing $x$ in $X$ such that $B \cap cl_{au}W \subset cl(B \cap cl_{au}W) \subset A \cap cl_{au}W$. Let $P$ be an open set containing $x$ in $X$ such that $clP \subset W$. Then $P \cap B \cap W$ is an $a$-set containing $x$ in $X$ and $clP \cap cl(B \cap W) \subset clP \cap cl(B \cap W) \subset W \cap A$. Hence $W \cap A \in a\Lambda(x)$ in $X$ and $g(cl(A \cap W) \subset g(cl(A \cap cl_{au}W)) \subset clQ$.

4. Conclusion. We conclude by pointing out the following;
There have been studies on uniform boundedness principle in different topological spaces using different types of functions. In classical analysis uniform boundedness principle is defined for real-valued functions on complete metric spaces [19]. In [17] generalizations of uniform boundedness principle were given from a class of Baire topological spaces to a countable union of its subspaces satisfying some generalized compactness properties. In [12] other generalizations of the uniform boundedness principle were given for functions from topological spaces to topological spaces as applications of u-continuous functions introduced there. The topological spaces considered there were Urysohn Closed Spaces and Quasi Urysohn Closed Spaces. What makes the results in this paper unique is that the generalization of uniform boundedness principle given here is not for real-valued functions nor for continuous functions but for classes of functions which generalizes continuous functions and functions which satisfy certain graph conditions. An approach like the one presented here opens a path for new investigative avenues while seeking classes of spaces which form uniform bounds for more general classes of functions under more general conditions than usually seen in classical approaches.

REFERENCES


