International Journal of Advances in Mathematics
Volume 2019, Number 5, Pages 22-34, 2019
eISSN 2456-6098

Adjoint Peters and Pidduk polynomials

Gabriella Bretti [1], Pierpaolo Natalini * and Paolo Emilio Ricci[2]

[1] IAC-CNR (National Research Council)
Via dei Taurini, 19, 00185 - Roma, Italia E-mail: g.bretti@iac.cnr.it

* Università degli Studi Roma Tre, Dipartimento di Matematica e Fisica,
Largo San Leonardo Murialdo, 1, 00146 - Roma, Italia E-mail: natalini@mat.uniroma3.it

[2] International Telematic University UniNettuno,
Corso Vittorio Emanuele II, 39, 00186 - Roma, Italia E-mail: paoloemilioricci@gmail.com

ABSTRACT. In recent papers, new sets of Sheffer and Brenke polynomials based on higher order Bell numbers have been studied, and several integer sequences related to them have been introduced. In this article other types of Sheffer polynomials are considered, by introducing the adjoint Peters and adjoint Pidduk polynomials.

1. Introduction

In recent articles [8, 30], we have studied new sets of Sheffer [33] and Brenke [7] polynomials related to higher order Bell numbers [16]. Furthermore, several integer sequences [34] associated with the considered polynomial sets, both of exponential [2, 3] and logarithmic type [8], have been introduced.
It is worth to note that exponential and logarithmic polynomials have been recently studied even in the multivariate case [21, 22, 23].

* Corresponding Author.
Received March 15, 2019; revised August 15, 2019; accepted August 19, 2019.
2010 Mathematics Subject Classification: 11B83, 12E10, 33C99.
Key words and phrases: Special polynomials, Sheffer polynomials, Peters polynomials, Pidduk polynomials, Differential equations.

Sheffer sets [33] are a wide class of polynomials generalizing the Appell polynomial sequences [1]. A list of known Sheffer polynomials can be found in [6]. In preceding articles we have extended the Sheffer sets by introducing the notion of adjointness.

Adjointness is a general method of associating a family of Sheffer polynomials with another family of Sheffer's [24]. This definition is based on an equivalent characterization of Sheffer sets proven in [32, p. 18]. It simply follows by interchanging two basic elements of the generating function.

By using the general and efficient method already applied in preceding articles [17]-[19], [25]-[29], it is possible to construct the fundamental properties of polynomials adjoint to the classical ones, such as the recurrence relation, the shift operators and the differential equation.

We have already applied this technique in several cases [9, 14, 15, 31]. In this article we apply the same technique to other types of Sheffer polynomials, by considering the adjoint Peters and adjoint Pidduk polynomials.

2 Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [33] by means of the exponential generating function [35] of the type:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\,, \tag{1}$$

where

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} a_n\,\frac{t^n}{n!}\,, \qquad (a_0 \neq 0)\,, \\ H(t) &= \sum_{n=0}^{\infty} h_n\,\frac{t^n}{n!}\,, \qquad (h_0 = 0)\,. \end{aligned} \tag{2}$$

According to a different characterization (see [32, p. 18]), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} g_n\,\frac{t^n}{n!}\,, \qquad (g_0 \neq 0)\,, \\ f(t) &= \sum_{n=0}^{\infty} f_n\,\frac{t^n}{n!}\,, \qquad (f_0 = 0,\ f_1 \neq 0)\,. \end{aligned} \tag{3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e. such that $f\left(f^{-1}(t)\right) = f^{-1}\left(f(t)\right) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]}\,\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\,, \tag{4}$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]}\,, \qquad H(t) = f^{-1}(t)\,. \tag{5}$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} \sigma_n(x)\,\frac{t^n}{n!}\,. \tag{6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in [6].

3. Adjointness for Peters polynomials

According to the above considerations, Sheffer polynomials are characterized both by the ordered couples $(A(t), H(t))$, or by $(g(t), f(t))$.

Definition - Adjoint Sheffer polynomials are defined by interchanging the ordered couple $(A(t), H(t))$ with $(g(t), f(t))$, when writing the generating function.

Here and in the following the *tilde* "$\sim$", above the symbol of a polynomial set stands for the adjective "*adjoint*".

Here we consider the adjoint Peters polynomials, defined through their generating function, i.e. by putting

$$\begin{aligned} &A(t) = (1+e^{\lambda\, t})^{\mu}\,, \qquad H(t) = e^t - 1\,, \\ &G(t,x) = (1+e^{\lambda\, t})^{\mu}\ \exp\left[x(e^t-1)\right] = \sum_{k=0}^{\infty} \tilde{s}_k(x;\lambda,\mu)\,\frac{t^k}{k!}\,. \end{aligned} \tag{7}$$

3.1. A differential identity

Theorem 3.1. - *For any $k \geq 0$, the polynomials $\tilde{s}_k(x;\lambda,\mu)$ satisfy the differential identity:*

$$\tilde{s}'_k(x;\lambda,\mu) = \sum_{h=0}^{k-1}\binom{k}{h}\tilde{s}_h(x;\lambda,\mu)\,. \tag{8}$$

Proof. - By differentiating both sides of equation $(7)_2$ with respect to x, we find

$$\frac{\partial G}{\partial x} = (e^t-1)G(t,x) = e^t G(t,x) - G(t,x) = \sum_{k=1}^{\infty} \tilde{s}'_k(x;\lambda,\mu)\,\frac{t^k}{k!}\,. \tag{9}$$

i.e.

$$\sum_{k=1}^{\infty} \tilde{s}'_k(x;\lambda,\mu)\,\frac{t^k}{k!} = \sum_{k=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}\tilde{s}_h(x;\lambda,\mu)\,\frac{t^k}{k!} - \sum_{k=0}^{\infty}\tilde{s}_k(x;\lambda,\mu)\,\frac{t^k}{k!}\,.$$

Then, putting $\tilde{s}'_0(x;\lambda,\mu) = 0$, equation (9) becomes

$$\sum_{k=0}^{\infty} \tilde{s}'_k(x;\lambda,\mu)\,\frac{t^k}{k!} = \sum_{k=0}^{\infty}\left[\sum_{h=0}^{k}\binom{k}{h}\tilde{s}_h(x;\lambda,\mu) - \tilde{s}_k(x;\lambda,\mu)\right]\frac{t^k}{k!}\,,$$

so that the differential identity (8) follows.

3.2 Recurrence relation

First note that, putting

$$\frac{e^{\lambda t}}{1+e^{\lambda t}} = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}\,, \tag{10}$$

we find

$$\sum_{k=0}^{\infty} \lambda^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!} + \sum_{k=0}^{\infty} \lambda^k \frac{t^k}{k!} \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}\,,$$

i.e.

$$\sum_{k=0}^{\infty} \lambda^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!} + \sum_{k=0}^{\infty} \sum_{h=0}^{k} \binom{k}{h} \lambda^{k-h} \alpha_h \frac{t^k}{k!}\,,$$

so that the coefficients α_k are defined by the recurrence relation

$$\begin{cases} \alpha_0 = \dfrac{1}{2}\,, \\[2ex] \alpha_k = \dfrac{1}{2}\left[\lambda^k - \displaystyle\sum_{h=0}^{k-1} \binom{k}{h} \lambda^{k-h} \alpha_h\right]. \end{cases} \tag{11}$$

Theorem 3.2. - *For any $k \geq 0$, the polynomials $\tilde{s}_k(x;\lambda,\mu))$ satisfy the following recurrence relation:*

$$\tilde{s}_{k+1}(x;\lambda,\mu) = \mu\lambda \sum_{h=0}^{k} \binom{k}{h} \left[\alpha_{k-h} + x\right] \tilde{s}_h(x;\lambda,\mu)\,, \tag{12}$$

where the coefficients α_k are defined by equation (11)*.*

Proof. - Differentiating $G(t,x)$ with respect to t, we have

$$\frac{\partial G(t,x)}{\partial t} = \frac{\mu\lambda\, e^{\lambda t}}{1+e^{\lambda t}}\, G(t,x) + x\, e^t\, G(t,x) = \sum_{k=0}^{\infty} \tilde{s}_{k+1}(x;\lambda,\mu) \frac{t^k}{k!}\,, \tag{13}$$

i.e.

$$\mu\lambda \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!} \sum_{k=0}^{\infty} \tilde{s}_k(x;\lambda,\mu) \frac{t^k}{k!} + x \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{k=0}^{\infty} \tilde{s}_k(x;\lambda,\mu) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \tilde{s}_{k+1}(x;\lambda,\mu) \frac{t^k}{k!}\,,$$

$$\mu\lambda \sum_{k=0}^{\infty} \sum_{h=0}^{k} \binom{k}{h} \lambda^{k-h} \tilde{s}_h(x;\lambda,\mu) \frac{t^k}{k!} +$$

$$+\, x \sum_{k=0}^{\infty} \sum_{h=0}^{k} \binom{k}{h} \tilde{s}_h(x;\lambda,\mu) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \tilde{s}_{k+1}(x;\lambda,\mu) \frac{t^k}{k!}\,,$$

so that the recurrence relation (12) follows.

3.3 Shift operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\hat{P}\,(p_n(x)) = np_{n-1}(x)\,, \qquad \hat{M}\,(p_n(x)) = p_{n+1}(x)\,, \qquad (n = 1, 2, \dots). \tag{14}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by J.F. Steffensen [36], recently improved by G. Dattoli [11] and widely used in several applications (see e.g. [12, 13]).

Y. Ben Cheikh [4] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article (Corollary 3.2), the following result is proved

Theorem 3.3. *Let $(p_n(x))$ denote a Boas-Buck polynomial set, i.e. a set defined by the generating function*

$$A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} p_n(x)\,\frac{t^n}{n!}\,, \tag{15}$$

where

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} \tilde{a}_n t^n\,, \qquad (\tilde{a}_0 \neq 0)\,, \\ \psi(t) &= \sum_{n=0}^{\infty} \tilde{\gamma}_n t^n\,, \qquad (\tilde{\gamma}_n \neq 0 \quad \forall n)\,, \end{aligned} \tag{16}$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} \tilde{h}_n\, t^{n+1}\,, \qquad (\tilde{h}_0 \neq 0)\,. \tag{17}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0\,, \qquad \sigma(x^n) = \frac{\tilde{\gamma}_{n-1}}{\tilde{\gamma}_n}\, x^{n-1}\,, \qquad (n = 1, 2, \dots). \tag{18}$$

Put

$$\sigma^{-1}(x^n) = \frac{\tilde{\gamma}_{n+1}}{\tilde{\gamma}_n}\, x^{n+1} \quad (n = 0, 1, 2, \dots). \tag{19}$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas-Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\hat{P} = f(\sigma)\,, \qquad \hat{M} = \frac{A'[f(\sigma)]}{A[f(\sigma)]} + xD_x H'[f(\sigma)]\sigma^{-1}\,, \tag{20}$$

where prime *denotes the ordinary derivatives with respect to t.*

Note that in our case we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by equation (18) simply reduces to the derivative operator D_x. Furthermore, we have:

$$A(t) = (1 + e^{\lambda\, t})^{\mu}\,, \qquad H(t) = e^t - 1\,, \qquad H'(t) = e^t$$

$$G(t,x) = (1 + e^{\lambda\, t})^{\mu}\ \exp\left[x(e^t - 1)\right] = \sum_{k=0}^{\infty} \tilde{s}_k(x;\lambda,\mu)\,\frac{t^k}{k!}\,,$$

$$\frac{A'(t)}{A(t)} = \mu\lambda\,\frac{e^{\lambda\, t}}{1 + e^{\lambda\, t}}\,,$$

$$f(t) = H^{-1}(t) = \log(t+1) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{k+1}}{k+1}\,,$$

so that we have the theorem

Theorem 3.4. *The adjoint Peters polynomial set* $\{\tilde{s}_k(x;\lambda,\mu))\}$ *is quasi-monomial under the action of the operators*

$$\begin{aligned} \hat{P} &= \log(D_x + 1) = \sum_{k=0}^{\infty} (-1)^k \frac{D_x^{k+1}}{k+1}\,, \\ \hat{M} &= \mu\lambda\,\frac{(D_x+1)^{\lambda}}{1 + (D_x+1)^{\lambda}} + x\,(D_x + 1)\,, \end{aligned} \tag{21}$$

where

$$(D_x + 1)^{\lambda} = \sum_{k=0}^{\infty} \binom{\lambda}{k} D_x^k\,.$$

3.4 Differential equation

According to the results of monomiality principle [11], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\,p_n(x) = n\,p_n(x)\,. \tag{22}$$

Recalling the series expansion

$$\frac{t}{1+t} = \sum_{k=1}^{\infty} (-1)^{k+1} t^k\,,$$

in the present case, we have the theorem

Theorem 3.5. *The Sheffer-type adjoint Peters polynomials* $\{\tilde{s}_k(x;\lambda,\mu))\}$ *satisfy the differential equation*

$$\left[\mu\lambda\,\frac{(D_x+1)^{\lambda}}{1 + (D_x+1)^{\lambda}} + x\,(D_x + 1)\right] \sum_{k=0}^{\infty} (-1)^k \frac{D_x^{k+1}}{k+1}\,\tilde{s}_n(x;\lambda,\mu) = n\,\tilde{s}_n(x;\lambda,\mu)\,,$$

i.e.

$$\left[\mu\lambda\sum_{k=1}^{n}(-1)^{k+1}\sum_{h=0}^{n}\binom{\lambda k}{h}D_x^h + x\,(D_x+1)\right] \sum_{k=0}^{n-1}(-1)^k\frac{D_x^{k+1}}{k+1}\,\tilde{s}_n(x;\lambda,\mu) = n\,\tilde{s}_n(x;\lambda,\mu)\,, \tag{23}$$

because, for any fixed n, the series expansions in equation (23) *reduces to a finite sum when it are applied to a polynomial of degree n.*

3.5 First few values of $\tilde{s}_n(x;\lambda) := \tilde{s}_n(x;\lambda,1)$

$$\begin{aligned}
\tilde{s}_0(x;\lambda) &= 2,\\
\tilde{s}_1(x;\lambda) &= 2x+\lambda,\\
\tilde{s}_2(x;\lambda) &= 2x^2+2(\lambda+1)x+\lambda^2,\\
\tilde{s}_3(x;\lambda) &= 2x^3+3(\lambda+2)x+(3\lambda^2+3\lambda+2)x+\lambda^3,\\
\tilde{s}_4(x;\lambda) &= 2x^4+4(\lambda+3)x^3+2(3\lambda^2+6\lambda+7)x^2+2(2\lambda^3+3\lambda^2+2\lambda+1)x+\lambda^4,\\
\tilde{s}_5(x;\lambda) &= 2x^5+5(\lambda+4)x^4+10(\lambda^2+3\lambda+5)x^3+5(2\lambda^3+6\lambda^2+7\lambda+6)x^2+\\
&\quad+(5\lambda^4+10\lambda^3+10\lambda^2+5\lambda+2)x+\lambda^5,\\
\tilde{s}_6(x;\lambda) &= 2x^6+6(\lambda+5)x^5+5(3\lambda^2+12\lambda+26)x^4+10(2\lambda^3+9\lambda^2+15\lambda+18)x^3+\\
&\quad+(15\lambda^4+60\lambda^3+105\lambda^2+90\lambda+62)x^2+\\
&\quad+(6\lambda^5+15\lambda^4+20\lambda^3+15\lambda^2+6\lambda+2)x+\lambda^5
\end{aligned}$$

and, in particular,

3.6 First few values of $\tilde{s}_n(x) := \tilde{s}_n(x;1,1)$

$$\begin{aligned}
\tilde{s}_0(x) &= 2,\\
\tilde{s}_1(x) &= 2x+1,\\
\tilde{s}_2(x) &= 2x^2+4x+1,\\
\tilde{s}_3(x) &= 2x^3+9x^2+8x+1,\\
\tilde{s}_4(x) &= 2x^4+16x^3+32x^2+16x+1,\\
\tilde{s}_5(x) &= 2x^5+25x^4+90x^3+105x^2+32x+1,\\
\tilde{s}_6(x) &= 2x^6+36x^5+205x^4+440x^3+332x^2+64x+1,\\
\tilde{s}_7(x) &= 2x^7+49x^6+406x^5+1400x^4+2002x^3+1029x^2+128x+1,\\
\tilde{s}_8(x) &= 2x^8+64x^7+728x^6+3696x^5+8652x^4+8736x^3+3152x^2+256x+1
\end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha©.

4. Adjoint Pidduk polynomials

Here we consider the adjoint Pidduk polynomials, defined through their generating function, i.e. by putting

$$\begin{gathered}A(t)=\frac{2}{e^t+1}, \qquad H(t)=\frac{e^t-1}{e^t+1}, \\ G(t,x)=\frac{2}{e^t+1}\exp\left[x\left(\frac{e^t-1}{e^t+1}\right)\right]=\sum_{k=0}^{\infty}\tilde{p}_k(x)\,\frac{t^k}{k!}\,,\end{gathered} \tag{24}$$

4.1. A differential identity

Theorem 4.1. *- For any $k \geq 0$, the polynomials $\tilde{p}_k(x)$ satisfy the differential identity:*

$$\tilde{p}'_k(x)=\sum_{h=0}^{k}\binom{k}{h}b_{k-h}\tilde{p}_k(x)\,, \tag{25}$$

where the coefficients b_k are the solution of the triangular system

$$\begin{cases} b_0=0\,, \\ b_k=\dfrac{1}{2}\left[1-\displaystyle\sum_{h=0}^{k-1}\binom{k}{h}b_h\right], \end{cases} \tag{26}$$

more precisely, because of the symmetry of the function involved, the preceding equation (9) *becomes, $\forall k \geq 0$:*

$$\begin{cases} b_{2k}=0\,, \\ b_{2k+1}=\dfrac{1}{2}\left[1-\displaystyle\sum_{\ell=0}^{\left[\frac{2k-1}{2}\right]}\binom{2k+1}{2\ell+1}b_{2\ell+1}\right], \end{cases} \tag{27}$$

Proof. - By differentiating both sides of equation $(7)_2$ with respect to x, we find

$$\frac{\partial G}{\partial x}=\frac{e^t-1}{e^t+1}\,G(t,x)\,. \tag{28}$$

Putting

$$\frac{e^t-1}{e^t+1}=\sum_{k=0}^{\infty}b_k\,\frac{t^k}{k!}\,, \tag{29}$$

we find

$$e^t-1=\sum_{k=0}^{\infty}\frac{t^k}{k!}-1=(e^t+1)\sum_{k=0}^{\infty}b_k\,\frac{t^k}{k!}=\sum_{k=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}b_h\,\frac{t^k}{k!}+\sum_{k=0}^{\infty}b_k\,\frac{t^k}{k!}\,,$$

$$\sum_{k=0}^{\infty}(1-b_k)\frac{t^k}{k!}-\sum_{k=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}b_h\,\frac{t^k}{k!}=1\,,$$

$$\sum_{k=0}^{\infty}\left[1-b_k-\sum_{h=0}^{k}\binom{k}{h}b_h\right]\frac{t^k}{k!}=1\,,$$

and therefore the coefficients b_k satisfy equations (26)-(27).

Then, putting $\tilde{p}_0'(x) = 0$, equation (11) becomes

$$\frac{\partial G}{\partial x} = \sum_{k=0}^{\infty} \tilde{p}_k'(x) \frac{t^k}{k!} = \sum_{k=0}^{\infty} b_k \frac{t^k}{k!} \sum_{k=0}^{\infty} \tilde{p}_k'(x) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \sum_{h=0}^{k} \binom{k}{h} b_{k-h}\, \tilde{p}_h(x) \frac{t^k}{k!}\,,$$

so that the differential identity (25) follows.

4.2 Recurrence relation

First note that, as a consequence of equation (12), we have

$$\frac{e^t}{(e^t+1)^2} = \frac{1}{2} \frac{d}{dt} \left(\frac{e^t - 1}{e^t + 1} \right) = \frac{1}{2} \sum_{k=0}^{\infty} b_{k+1} \frac{t^k}{k!}\,, \tag{30}$$

Theorem 4.2. *- For any $k \geq 0$, the polynomials $\tilde{p}_k(x)$ satisfy the following recurrence relation:*

$$\tilde{p}_{k+1}(x) = \sum_{h=0}^{k} \binom{k}{h} \left[\frac{x}{2}\, b_{k-h+1} - a_{k-h} \right] \tilde{p}_h(x)\,, \tag{31}$$

where the coefficients a_k are the solution of the triangular system

$$\begin{cases} a_0 = 1/2\,, \\ \\ \displaystyle\sum_{h=0}^{k-1} \binom{k}{h} a_h + 2a_k = 1\,, \qquad \forall k \geq 1\,. \end{cases} \tag{32}$$

Proof. - Differentiating $G(t,x)$ with respect to t, we have

$$\frac{\partial G(t,x)}{\partial t} = \frac{e^t}{(e^t+1)^2}\, x\, G(t,x) - \frac{e^t}{e^t+1}\, G(t,x)\,, \tag{33}$$

Putting

$$\frac{e^t}{e^t+1} = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}\,, \tag{34}$$

it is easily seen that the coefficients a_k are the solution of the triangular system (32).

Therefore we find

$$\frac{\partial G(t,x)}{\partial t} = \sum_{k=0}^{\infty} \tilde{p}_{k+1}(x) \frac{t^k}{k!} = \frac{x}{2} \sum_{k=0}^{\infty} b_{k+1} \frac{t^k}{k!} \sum_{k=0}^{\infty} \tilde{p}_k(x) \frac{t^k}{k!} - \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \sum_{k=0}^{\infty} \tilde{p}_k(x) \frac{t^k}{k!}\,,$$

i.e.

$$\sum_{k=0}^{\infty} \tilde{p}_{k+1}(x) \frac{t^k}{k!} = \frac{x}{2} \sum_{k=0}^{\infty} \sum_{h=0}^{k} \binom{k}{h} b_{k-h+1}\, \tilde{p}_h(x) \frac{t^k}{k!} - \sum_{k=0}^{\infty} \sum_{h=0}^{k} \binom{k}{h} a_{k-h}\, \tilde{p}_h(x) \frac{t^k}{k!}\,,$$

so that the recurrence relation (31) follows.

Note that, even in this case, we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by equation (18) simply reduces to the derivative operator D_x. Furthermore, we have:

$$A(t) = \frac{2}{e^t+1}, \qquad H(t) = \frac{e^t-1}{e^t+1}, \qquad \frac{A'(t)}{A(t)} = -\frac{e^t}{e^t+1},$$

$$f(t) = H^{-1}(t) = \log\left(\frac{1+t}{1-t}\right) = 2\,\text{settanh}\, t = 2\sum_{k=0}^{\infty}\frac{t^{2k+1}}{2k+1},$$

$$H'(t) = \frac{2e^t}{(e^t+1)^2},$$

so that we have the theorem

Theorem 4.3. *The adjoint Pidduk polynomial set* $\{\tilde{p}_n(x)\}$ *is quasi-monomial under the action of the operators*

$$\begin{aligned} \hat{P} &= \log\left(\frac{1+D_x}{1-D_x}\right) = 2\sum_{k=0}^{\infty}\frac{D_x^{2k+1}}{2k+1}, \\ \hat{M} &= \tfrac{1}{2}\left[-(1+D_x) + x(1-D_x^2)\right]. \end{aligned} \tag{35}$$

4.3 Differential equation

According to the results of monomiality principle [11], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\, p_n(x) = n\, p_n(x). \tag{36}$$

In the present case, we have

Theorem 4.4. *The Sheffer-type adjoint Pidduk polynomials* $\{\tilde{p}_n(x)\}$ *satisfy the differential equation*

$$\left[x\,(1-D_x^2) - (1+D_x)\right]\sum_{k=0}^{\infty}\frac{D_x^{2k+1}}{2k+1}\,\tilde{p}_n(x) = n\,\tilde{p}_n(x)$$

i.e.

$$\left[x\,(1-D_x^2) - (1+D_x)\right]\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{D_x^{2k+1}}{2k+1}\,\tilde{p}_n(x) = n\,\tilde{p}_n(x), \tag{37}$$

because, for any fixed n, the last series expansion reduces to a finite sum, with upper limit $\left[\frac{n-1}{2}\right]$, when it is applied to a polynomial of degree n.

4.4 First few values of the adjoint Pidduk polynomials

$$\tilde{p}_0(x) = 1$$
$$\tilde{p}_1(x) = \frac{1}{2}(x-1)$$
$$\tilde{p}_2(x) = \frac{1}{4}(x^2-2x)$$
$$\tilde{p}_3(x) = \frac{1}{8}(x^3-3x^2-2x+2)$$
$$\tilde{p}_4(x) = \frac{1}{16}(x^4-4x^3-8x^2+16x)$$
$$\tilde{p}_5(x) = \frac{1}{32}(x^5-5x^4-20x^3+60x^2+16x-16)$$
$$\tilde{p}_6(x) = \frac{1}{64}(x^6-6x^5-40x^4+160x^3+136x^2-272x)$$
$$\tilde{p}_7(x) = \frac{1}{128}(x^7-7x^6-70x^5+350x^4+616x^3-1848x^2-272x+272)$$
$$\tilde{p}_8(x) = \frac{1}{256}(x^8-8x^7-112x^6+672x^5+2016x^4-8064x^3-3968x^2+7936x)$$

Further values can be easily achieved by using Wolfram Alpha©.

5 Conclusion

We have introduced two set of adjoint Sheffer polynomials, called adjoint Peters and adjoint Pidduk polynomials, in the framework of a general technique associating a new Sheffer polynomial set starting from a given one [24]. We have derived their main characteristics by using the monomiality principle, introduced by G. Dattoli [11], and the possibility to construct the shift operators by means of preceding results by Y. Ben Cheihk [4]. We have found, as a consequence, the differential equations satisfied by the considered polynomials.
In fact, it has been shown that, for the polynomials of the Sheffer class, the differential equation follows from the basic elements of their generating function, in a constructive way, by using a simple and efficient method based on the monomiality principle.
This technique has been already used in several cases [9, 14, 15, 31] and can be applied for every Sheffer set. Furthermore, more generally, it could be used even for the Boas-Buck polynomials, provided that the general shift operators defined in the Ben Cheikh article [4] can be explicitly derived.
It is worth noting that, using the technique applied in this article, it is possible to avoid the use of the linear algebraic approach described in [10, 37, 37, 38], which makes use of determinantal forms of the considered polynomials, a method that seems to be less natural than the one described above, as it requires elements foreign to the theory of polynomials.

References

[1] Appell, P.: Sur une classe de polynômes, Ann. Sci. Ecole Norm. Sup. (2) 9, 119–144 (1880).

[2] Bell, E.T.: Exponential polynomials, Annals of Mathematics 35, 258–277 (1934).

[3] Bell, E.T.: The Iterated Exponential Integers, Annals of Mathematics 39, 539–557 (1938).

[4] Ben Cheikh, Y.: Some results on quasi-monomiality, Appl. Math. Comput. 141, 63–76 (2003).

[5] Boas, R.P., Buck, R.C.: Polynomials defined by generating relations, Amer. Math. Monthly 63, 626–632 (1958).

[6] Boas, R.P., Buck, R.C.: Polynomial Expansions of Analytic Functions, Springer-Verlag, Berlin, Gottingen, Heidelberg, New York, 1958.

[7] Brenke, W.C.: On generating functions of polynomial systems, Amer. Math. Monthly 52, 297–301 (1945).

[8] Bretti, G., Natalini, P., Ricci, P.E.: A new set of Sheffer-Bell polynomials and logarithmic numbers, Georgian Math. J., https://doi.org/10.1515/gmj-2019-2007

[9] Bretti, G., Natalini, P., Ricci, P.E.: New sets of Euler-type polynomials, J. Ana. Num. Theor. 6 (No. 2), 51–54 (2018).

[10] Costabile, F.A.; Longo, E. A new recurrence relation and related determinantal form for Binomial type polynomial sequences, Mediterr. J. Math. 13, 4001-4017 (2016).

[11] Dattoli, G.: Hermite-Bessel and Laguerre-Bessel functions: A by-product of the monomiality principle, in Advanced Special Functions and Applications (Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics; Melfi, 9-12 May, 1999), in: Cocolicchio, D., Dattoli, G., Srivastava, H.M., (Editors), Aracne Editrice, Roma, pp. 147–164, 2000.

[12] Dattoli, G., Ricci, P.E., Srivastava, H.M., (Editors): Advanced Special Functions and Related Topics in Probability and in Differential Equations, (Proceedings of the Melfi School on Advanced Topics in Mathematics and Physics; Melfi, June 24-29, 2001), in: Applied Mathematics and Computation 141 (No. 1), 1–230 (2003).

[13] Dattoli, G., Germano, B., Martinelli, M.R., Ricci, P.E.: Monomiality and partial differential equations, Math. Comput. Modelling 50, 1332–1337 (2009).

[14] Natalini, P., Bretti, G., Ricci, P.E.: Adjoint Hermite and Bernoulli polynomials, Bull. Allahabad Math. Soc., (Dharma Prakash Gupta Memorial) 33 (No. 2), 251–264 (2018).

[15] Natalini, P., Bretti, G., Ricci, P.E.: Adjoint Poisson-Charlier and Related polynomials, Arabian J. of Math. (2019), (submitted).

[16] Natalini, P., Ricci, P.E.: Remarks on Bell and higher order Bell polynomials and numbers, Cogent Mathematics 3, 1–15 (2016).

[17] Natalini, P., Ricci, P.E.: Bell-Sheffer exponential polynomials of the second kind, Georgian Math. J., (to appear) (2019).

[18] Natalini, P., Ricci, P.E.: New Bell-Sheffer polynomial sets Axioms, 7, 71 (2018); doi:10.3390/axioms7040071

[19] Natalini, P., Ricci, P.E.: Adjoint Appell-Euler and first kind Appell-Bernoulli polynomials, Appl. Appl. Math. (AAM), (to appear) (2019).

[20] Pinelas, S., Ricci, P.E.: On Sheffer polynomial families 4Open, 2, 4 (2019); available on line at https://doi.org/10.1051/fopen/2019004

[21] Qi, F., Niu, D.-W., Lim D., Guo, B.-N.: Some properties and an application of multivariate exponential polynomials, HAL archives (2018); available online at https://hal.archives-ouvertes.fr/hal-01745173.

[22] Qi, F.: Integral representations for multivariate logarithmic potentials, J. Comput. Appl. Math. 336, 54–62 (2018); available on line at https://doi.org/10.1016/j.cam.2017.11.047.

[23] Qi, F.: On multivariate logarithmic polynomials and their properties, Indag. Math. 336, (2018), (in press); available on line at https://hal.archives-ouvertes.fr/hal-01745173.

[24] Ricci, P.E.: Adjointness for Sheffer polynomials, Appl. Anal. Discr. Math., (2018), (submitted).

[25] Ricci, P.E.: Logarithmic-Sheffer polynomial sets Jñānābha, 48 (No. 1), 119–128 (2018).

[26] Ricci, P.E.: Logarithmic-Sheffer polynomials of the second kind Tbilisi Math J., 11(3) 95-106 (2018).

[27] Ricci, P.E.: Generalized Hermite polynomial families Jñānābha, 49 (1), 80–88 (2019).

[28] Ricci, P.E.: Mittag-Leffler polynomial differential equation Jñānābha, 48 (2), 1–6 (2018).

[29] Ricci, P.E.: Differential equations for classical and non-classical polynomial sets. A survey, Axioms, 8, 50 (2019), doi:10.3390/axioms8020050

[30] Ricci, P.E., Natalini, P., Bretti, G.: Sheffer and Brenke polynomials associated with generalized Bell numbers, Jñānābha, 47 (No. 2), 337–352 (2017).

[31] Ricci, P.E., Bretti, G., Natalini, P.: New sets of Hahn-type polynomials, Bull. Allahabad Math. Soc., 34, Part 1, 135–146 (2019).

[32] Roman, S.M.: The Umbral Calculus, Academic Press, New York, 1984.

[33] Sheffer, I.M.: Some properties of polynomials sets of zero type, Duke Math. J. 5, 590–622 (1939).

[34] Sloane, N.J.A.: The On-Line Encyclopedia of Integer Sequences. Published electronically at http://oeis.org, 2016.

[35] Srivastava, H.M., Manocha, H.L.: A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[36] Steffensen, J.F.: The poweroid, an extension of the mathematical notion of power, Acta Math. 73, 333–366 (1941).

[37] Yang, Y.; Youn, H.: Appell polynomial sequences: a linear algebraic approach, JP J. Algebra Number Theory Appl, 13, 6598 (2009).

[38] Yang, S.L.: Recurrence relations for Sheffer sequences, Linear Algebra Appl. 437, 2986–2996 (2012).