Algebraic properties of the path complexes of cycles

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ABSTRACT.Let $G$ be a simple graph and $\Delta_t(G)$ be a simplicial complex whose facets correspond to the paths of length $t$ ($t \geq 2$) in $G$. It is shown that $\Delta_t(C_n)$ is matroid, vertex decomposable, shellable and Cohen-Macaulay if and only if $n = t$ or $n = t + 1$, where $C_n$ is an $n$-cycle. As a consequence we show that if $n = t$ or $t + 1$ then $\Delta_t(C_n)$ is partitionable and Stanley’s conjecture holds for $K[\Delta_t(C_n)]$.

Introduction

Let $R = K[x_1, \ldots, x_n]$, where $K$ is a field. Fix an integer $n \geq t \geq 2$ and let $G$ be a directed graph. A sequence $x_{i_1}, \ldots, x_{i_t}$ of distinct vertices is called a path of length $t$ if there are $t - 1$ distinct directed edges $e_1, \ldots, e_{t-1}$ where $e_j$ is a directed edge from $x_{i_j}$ to $x_{i_{j+1}}$. Then the path ideal of $G$ of length $t$ is the monomial ideal $I_t(G) = (x_{i_1} \ldots x_{i_t} : x_{i_1}, \ldots, x_{i_t} \text{ is a path of length } t \text{ in } G)$ in the polynomial ring $R = K[x_1, \ldots, x_n]$. The distance $d(x,y)$ of two vertices $x$ and $y$ of a graph $G$ is the length of the shortest path from $x$ to $y$. The path complex $\Delta_t(G)$ is defined by

$$\Delta_t(G) = (\{x_{i_1}, \ldots, x_{i_t} : x_{i_1}, \ldots, x_{i_t} \text{ is a path of length } t \text{ in } G\}).$$

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Path ideals of graphs were first introduced by Conca and De Negri [3] in the context of monomial ideals of linear type. Recently the path ideal of cycles has been extensively studied by several mathematicians. In [9], it is shown that $I_2(C_n)$ is sequentially Cohen-Macaulay, if and only if, $n = 3$ or $n = 5$. Generalizing this result, in [13], it is proved that $I_t(C_n)$, $(t > 2)$, is sequentially Cohen-Macaulay, if and only if $n = t$ or $n = t + 1$ or $n = 2t + 1$. Also, the Betti numbers of the ideal $I_t(C_n)$ and $I_t(L_n)$ is computed explicitly in [1]. In particular, it has been shown that:

**Theorem 0.1 ([1, Corollary 5.15]).** Let $n$, $t$, $p$ and $d$ be integers such that $n \geq t \geq 2$, $n = (t + 1)p + d$, where $p \geq 0$ and $0 \leq d < (t + 1)$.

(i) The projective dimension of the path ideal of a graph cycle $C_n$ or line $L_n$ is given by,

$$\text{pd } (I_t(C_n)) = \begin{cases} 2p, & d \neq 0 \\ 2p - 1, & d = 0 \end{cases} \quad \text{pd } (I_t(L_n)) = \begin{cases} 2p - 1, & d \neq t \\ 2p, & d = t \end{cases}$$

(ii) The regularity of the path ideal of a graph cycle $C_n$ or line $L_n$ is given by,

$$\text{reg } (I_t(C_n)) = (t - 1)p + d + 1 \quad \text{reg } (I_t(L_n)) = \begin{cases} p(t - 1) + 1, & d < t \\ p(t - 1) + t, & d = t \end{cases}$$

In [8] it has been shown that, $\Delta_t(G)$ is a simplicial tree if $G$ is a rooted tree and $t \geq 2$. One of interesting problems in combinatorial commutative algebra is the Stanley’s conjectures. The Stanley’s conjectures are studied by many researchers. Let $R$ be a $\mathbb{N}^n$-graded ring and $M$ a $\mathbb{Z}^n$-graded $R$-module. Then Stanley [10] conjectured that

$${\text{depth}}(M) \leq s\text{depth}(M)$$

He also conjectured in [11] that each Cohen-Macaulay simplicial complex is partitionable. Herzog, Soleyman Jahan and Yassemi in [7] showed that the conjecture about partitionability is a special case of the Stanley’s first conjecture. In this work, we study algebraic properties of $\Delta_t(C_n)$. In Section 1, we recall some definitions and results which will be needed later. In Section 2, for all $t > 2$ we show that the following conditions are equivalent:

(i) $\Delta_t(C_n)$ is matroid;
(ii) $\Delta_t(C_n)$ is vertex decomposable;
(iii) $\Delta_t(C_n)$ is shellable;
(iv) $\Delta_t(C_n)$ is Cohen-Macaulay;
(v) $n = t$ or $n = t + 1$.

(see Theorem 2.6).

In Section 3 as an application of our results we show that if $n = t$ or $n = t + 1$ then $\Delta_t(C_n)$ is partitionable and Stanley’s conjecture holds for $K[\Delta_t(C_n)]$. 

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1 Preliminaries

In this section we recall some definitions and results which will be needed later.

Definition 1.1. A simplicial complex $\Delta$ over a set of vertices $V = \{x_1, \ldots, x_n\}$, is a collection of subsets of $V$, with the property that:

(a) $\{x_i\} \in \Delta$, for all $i$;

(b) if $F \in \Delta$, then all subsets of $F$ are also in $\Delta$ (including the empty set).

An element of $\Delta$ is called a face of $\Delta$ and complement of a face $F$ is $V \setminus F$ and it is denoted by $F^c$. Also, the complement of the simplicial complex $\Delta = \langle F_1, \ldots, F_q \rangle$ is $\Delta^c = \langle F_1^c, \ldots, F_q^c \rangle$. The dimension of a face $F$ of $\Delta$, $\dim F$, is $|F| - 1$ where, $|F|$ is the number of elements of $F$ and $\dim \emptyset = -1$. The faces of dimensions 0 and 1 are called vertices and edges, respectively. A non-face of $\Delta$ is a subset $F$ of $V$ with $F \not\in \Delta$. We denote by $\mathcal{N}(\Delta)$, the set of all minimal non-faces of $\Delta$. The maximal faces of $\Delta$ under inclusion are called facets of $\Delta$. The dimension of the simplicial complex $\Delta$, $\dim \Delta$, is the maximum of dimensions of its facets. If all facets of $\Delta$ have the same dimension, then $\Delta$ is called pure.

Let $\mathcal{F}(\Delta) = \{F_1, \ldots, F_q\}$ be the facet set of $\Delta$. It is clear that $\mathcal{F}(\Delta)$ determines $\Delta$ completely and we write $\Delta = \langle F_1, \ldots, F_q \rangle$. A simplicial complex with only one facet is called a simplex. A simplicial complex $\Gamma$ is called a subcomplex of $\Delta$, if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.

For $v \in V$, the subcomplex of $\Delta$ obtained by removing all faces $F \in \Delta$ with $v \in F$ is denoted by $\Delta \setminus v$. That is,

$$\Delta \setminus v = \{F \in \Delta: v \not\in F\}.$$ 

The link of a face $F \in \Delta$, denoted by $\text{link}_\Delta(F)$, is a simplicial complex on $V$ with the faces, $G \in \Delta$ such that, $G \cap F = \emptyset$ and $G \cup F \in \Delta$. The link of a vertex $v \in V$ is simply denoted by $\text{link}_\Delta(v)$.

$$\text{link}_\Delta(v) = \{F \in \Delta: v \not\in F, F \cup \{v\} \in \Delta\}.$$ 

Let $\Delta$ be a simplicial complex over $n$ vertices $\{x_1, \ldots, x_n\}$. For $F \subset \{x_1, \ldots, x_n\}$, we set:

$$x_F = \prod_{x_i \in F} x_i.$$ 

We define the facet ideal of $\Delta$, denoted by $I(\Delta)$, to be the ideal of $S$ generated by $\{x_F: F \in \mathcal{F}(\Delta)\}$. The non-face ideal or the Stanley-Reisner ideal of $\Delta$, denoted by $I_{\Delta}$, is the ideal of $S$ generated by square-free monomials $\{x_F: F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S/I_\Delta$ the Stanley-Reisner ring of $\Delta$.

Definition 1.2. A simplicial complex $\Delta$ on $\{x_1, \ldots, x_n\}$ is said to be a matroid if, for any two facets $F$ and $G$ of $\Delta$ and any $x_j \in F$, there exists a $x_j \in G$ such that $(F \setminus \{x_j\}) \cup \{x_j\}$ is a facet of $\Delta$.

Definition 1.3. A simplicial complex $\Delta$ is recursively defined to be vertex decomposable, if it is either a simplex, or else has some vertex $v$ so that,

(a) Both $\Delta \setminus v$ and $\text{link}_\Delta(v)$ are vertex decomposable, and

(b) No face of $\text{link}_\Delta(v)$ is a facet of $\Delta \setminus v$. 

A vertex \( v \) which satisfies in condition (b) is called a *shelling vertex*.

**Definition 1.4.** A simplicial complex \( \Delta \) is *shellable*, if the facets of \( \Delta \) can be ordered \( F_1, \ldots, F_s \) such that, for all \( 1 \leq i < j \leq s \), there exists some \( v \in F_j \setminus F_i \) and some \( l \in \{1, \ldots, j-1\} \) with \( F_j \setminus F_i = \{v\} \).

A simplicial complex \( \Delta \) is called disconnected, if the vertex set \( V \) of \( \Delta \) is a disjoint union \( V = V_1 \cup V_2 \) such that no face of \( \Delta \) has vertices in both \( V_1 \) and \( V_2 \). Otherwise \( \Delta \) is connected. It is well-known that

\[ \text{matroid} \implies \text{vertex decomposable} \implies \text{shellable} \implies \text{Cohen-Macaulay} \]

**Definition 1.5.** Given a simplicial complex \( \Delta \) on \( V \), we define \( \Delta^\vee \), the *Alexander dual* of \( \Delta \), by

\[ \Delta^\vee = \{ V \setminus F : F \notin \Delta \} \]

It is known that for the complex \( \Delta \) one has \( I_{\Delta^\vee} = I(\Delta^c) \). Let \( I \neq 0 \) be a homogeneous ideal of \( S \) and \( N \) be the set of non-negative integers. For every \( i \in N \cup \{0\} \), one defines:

\[ t^S_i(I) = \max\{j : \beta^S_{i,j}(I) \neq 0\} \]

where \( \beta^S_{i,j}(I) \) is the \( i, j \)-th graded Betti number of \( I \) as an \( S \)-module. The *Castelnuovo-Mumford regularity* of \( I \) is given by:

\[ \text{reg}(I) = \sup\{t^S_i(I) - i : i \in \mathbb{Z}\} \]

We say that the ideal \( I \) has a \( d \)-linear resolution, if \( I \) is generated by homogeneous polynomials of degree \( d \) and \( \beta^S_{i,j}(I) = 0 \), for all \( j \neq i + d \) and \( i \geq 0 \). For an ideal which has a \( d \)-linear resolution, the Castelnuovo-Mumford regularity would be \( d \). If \( I \) is a graded ideal of \( S \), then we write \( (l_d) \) for the ideal generated by all homogeneous polynomials of degree \( d \) belonging to \( I \).

**Definition 1.6.** A graded ideal \( I \) is componentwise linear if \( (l_d) \) has a linear resolution for all \( d \).

Also, we write \( l_{(d)} \) for the ideal generated by the squarefree monomials of degree \( d \) belonging to \( I \).

**Definition 1.7.** A graded \( S \)-module \( M \) is called *sequentially Cohen-Macaulay* (over \( K \)), if there exists a finite filtration of graded \( S \)-modules,

\[ 0 = M_0 \subset M_1 \subset \cdots \subset M_r = M \]

such that each \( M_i/M_{i-1} \) is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

\[ \dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}) \]

The Alexander dual, allows us to make a bridge between (sequentially) Cohen-Macaulay ideals and (componentwise) linear ideals.

**Definition 1.8 (Alexander duality).** For a square-free monomial ideal \( I = (M_1, \ldots, M_q) \subseteq S = K[x_1, \ldots, x_n] \), the *Alexander dual* of \( I \), denoted by \( I^\vee \), is defined to be:

\[ I^\vee = P_{M_1} \cap \cdots \cap P_{M_q} \]

where, \( P_{M_i} \) is prime ideal generated by \( \{x_j : x_j \in M_i\} \).
Theorem 1.9 ([6, Proposition 8.2.20], [4, Theorem 3]). Let \( I \) be a square-free monomial ideal in \( S = \mathbb{K}[x_1, \ldots, x_n] \).

(i) The ideal \( I \) is componentwise linear ideal if and only if \( S/I' \) is sequentially Cohen-Macaulay.

(ii) The ideal \( I \) has a \( q \)-linear resolution if and only if \( S/I' \) is Cohen-Macaulay of dimension \( n-q \).

Remark 1.10. Two special cases, we will be considering in this paper, are when \( G \) is a cycle \( C_n \), or a line graph \( L_n \) on vertices \( \{x_1, \ldots, x_n\} \) with edges

\[
E(C_n) = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_n, x_1)\};
\]

\[
E(L_n) = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)\}.
\]

2 Vertex decomposability path complexes of cycles

As the main result of this section, it is shown that \( \Delta_t(C_n) \) is matroid, vertex decomposable, shellable and Cohen-Macaulay if and only if \( n = t \) or \( n = t + 1 \). For the proof we shall need the following lemmas and propositions.

Lemma 2.1. Let \( \Delta_t(L_n) \) be a simplicial complex on \( \{x_1, \ldots, x_n\} \) and \( 2 \leq t \leq n \). Then \( \Delta_t(L_n) \) is vertex decomposable.

Proof. If \( t = n \), then \( \Delta_n(L_n) \) is a simplex which is vertex decomposable. Let \( 2 \leq t < n \) then one has

\[
\Delta_t(L_n) = \langle\{x_1, \ldots, x_t\}, \{x_2, \ldots, x_{t+1}\}, \ldots, \{x_{n-t+1}, \ldots, x_n\}\rangle.
\]

So \( \Delta_t(L_n) \setminus x_n = \langle\{x_1, \ldots, x_t\}, \{x_2, \ldots, x_{t+1}\}, \ldots, \{x_{n-t}, x_{n-1}\}\rangle \). Now we use induction on the number of vertices of \( L_n \) and by induction hypothesis \( \Delta_t(L_n) \setminus x_n \) is vertex decomposable. On the other hand, it is clear that \( \text{link}_{\Delta_t(L_n)} \{x_n\} = \langle\{x_{n-t+1}, \ldots, x_{n-1}\}\rangle \). Thus \( \text{link}_{\Delta_t(L_n)} \{x_n\} \) is a simplex which is not a facet of \( \Delta_t(L_n) \setminus x_n \). Therefore \( \Delta_t(L_n) \) is vertex decomposable.

Lemma 2.2. Let \( \Delta_2(C_n) \) be a simplicial complex on \( \{x_1, \ldots, x_n\} \). Then \( \Delta_2(C_n) \) is vertex decomposable.

Proof. Since \( \Delta_2(C_n) = \langle\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\rangle \) then we have

\[
\Delta_2(C_n) \setminus x_n = \langle\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-2}, x_{n-1}\}\rangle.
\]

By lemma 2.1 \( \Delta_2(C_n) \setminus x_n \) is vertex decomposable. Also it is trivial that \( \text{link}_{\Delta_2(C_n)} \{x_1\} = \langle\{x_{n-1}\}, \{x_1\}\rangle \) is vertex decomposable and no face of \( \text{link}_{\Delta_2(C_n)} \{x_1\} \) is a facet of \( \Delta_2(C_n) \setminus x_n \). Therefore \( \Delta_2(C_n) \) is vertex decomposable.

Lemma 2.3. Let \( \Delta_t(C_n) \) be a simplicial complex on \( \{x_1, \ldots, x_n\} \) and \( 3 \leq t \leq n - 2 \). Then \( \Delta_t(C_n) \) is not Cohen-Macaulay.

Proof. It suffices to show that \( I_{\Delta_t(C_n)^{\vee}} \) has not a linear resolution. Since \( I_{\Delta_t(C_n)^{\vee}} = I(\Delta_t(C_n)^\Delta) \) then one can easily check that \( I_{\Delta_t(C_n)^{\vee}} = I_{n-1}(C_n) \). By Theorem 0.1 we have

\[
\text{reg}(I_{\Delta_t(C_n)^{\vee}}) = (n - t - 1)p + d + 1.
\]

Since \( 3 \leq t \leq n - 2 \) then one has \( \text{reg}(I_{\Delta_t(C_n)^{\vee}}) \neq n - t \) and by Theorem 1.9 \( \Delta_t(C_n) \) is not Cohen-Macaulay.
Proof. By lemma 2.3 it suffices to show that if \( n = t \) or \( t + 1 \), then \( \Delta_t(C_n) \) is vertex decomposable. If \( n = t \), then \( \Delta_n(C_n) \) is a simplex which is vertex decomposable.

If \( t = n - 1 \), then we have

\[
\Delta_{n-1}(C_n) = \langle \{x_1, \ldots, x_{n-1}\}, \{x_2, \ldots, x_n\}, \{x_3, \ldots, x_n, x_1\}, \ldots, \{x_{n-1}, x_1, \ldots, x_{n-2}\} \rangle.
\]

Now we use induction on the number of vertices of \( C_n \) and show that \( \Delta_{n-1}(C_n) \) is vertex decomposable. It is clear that \( \Delta_{n-1}(C_n) \setminus \{x_n\} = \langle \{x_1, \ldots, x_{n-1}\} \rangle \) is a simplex which is vertex decomposable.

On the other hand,

\[
\text{link}_{\Delta_{n-1}(C_n)} \{x_n\} = \langle \{x_1, \ldots, x_{n-2}\}, \ldots, \{x_{n-1}, x_1, \ldots, x_{n-3}\} \rangle = \Delta_{n-2}(C_{n-1}).
\]

By induction hypothesis \( \text{link}_{\Delta_{n-1}(C_n)} \{x_n\} \) is vertex decomposable. It is easy to see that no face of \( \text{link}_{\Delta_{n-1}(C_n)} \{x_n\} \) is a facet of \( \Delta_{n-1}(C_n) \setminus \{x_n\} \). Therefore \( \Delta_{n-1}(C_n) \) is vertex decomposable. \( \square \)

Proposition 2.5. \( \Delta_2(C_n) \) is a matroid if and only if \( n = 3 \) or \( 4 \).

Proof. If \( n = 3 \) or \( 4 \), then it is easy to see that \( \Delta_2(C_n) \) is a matroid. Now we prove the converse. It suffices to show that \( \Delta_2(C_n) \) is not a matroid for all \( n \geq 5 \). We consider two facets \( \{x_1, x_2\} \) and \( \{x_{n-1}, x_n\} \). Then we have

\[
(\{x_1, x_2\} \setminus \{x_1\}) \cup \{x_{n-1}\} = \{x_2, x_{n-1}\} \quad \text{and} \quad (\{x_1, x_2\} \setminus \{x_1\}) \cup \{x_n\} = \{x_2, x_n\}.
\]

Since \( \{x_2, x_{n-1}\} \) and \( \{x_2, x_n\} \) are not the facets of \( \Delta_2(C_n) \). So \( \Delta_2(C_n) \) is not matroid for all \( n \geq 5 \). \( \square \)

For the simplicial complexes one has the following implication:

\[
\text{Matroid} \Rightarrow \text{vertex decomposable} \Rightarrow \text{shellable} \Rightarrow \text{Cohen-Macaulay}
\]

Note that these implications are strict, but by the following theorem, for path complexes, the reverse implications are also valid.

Theorem 2.6. Let \( t \geq 3 \). Then the following conditions are equivalent:

(i) \( \Delta_t(C_n) \) is matroid;

(ii) \( \Delta_t(C_n) \) is vertex decomposable;

(iii) \( \Delta_t(C_n) \) is shellable;

(iv) \( \Delta_t(C_n) \) is Cohen-Macaulay;

(v) \( n = t \) or \( t + 1 \).

Proof. (i) \( \Rightarrow \) (ii), (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (iv) is well-known.

(iv) \( \Rightarrow \) (v): Follows from Lemma 2.3 and Proposition 2.4.

(v) \( \Rightarrow \) (i): If \( n = t \), then \( \Delta_t(C_n) \) is a simplex which is a matroid.

If \( n = t + 1 \), then

\[
\Delta_t(C_n) = \langle \{x_1, \ldots, x_t\}, \{x_2, \ldots, x_{t+1}\}, \{x_3, \ldots, x_{t+1}, x_1\}, \ldots, \{x_{t+1}, x_1, \ldots, x_{t+1}\} \rangle.
\]
For any two facets $F$ and $G$ of $\Delta_t(C_n)$ one has $|F \cap G| = t - 1$. We claim that for any two facets $F$ and $G$ of $\Delta_t(C_n)$ and any $x_i \in F$, there exists a $x_j \in G$ such that $(F \setminus \{x_j\}) \cup \{x_i\}$ is a facet of $\Delta_t(C_n)$. We have to consider two cases. If $x_i \in F$ and $x_j \notin G$, then we choose $x_j \in G$ such that $x_j \notin F$. Thus $(F \setminus \{x_j\}) \cup \{x_j\} = G$ which is a facet of $\Delta_t(C_n)$.

For other case, if $x_i \in F$ and $x_j \in G$, then we choose $x_j \in G$ such that $x_j$ is the same $x_i$. Therefore $(F \setminus \{x_i\}) \cup \{x_j\} = F$ is a facet of $\Delta_t(C_n)$ which completes the proof. 

\[ \square \]

### 3 Stanley Decompositions

Let $R$ be any standard graded $K$-algebra over an infinite field $K$, i.e., $R$ is a finitely generated graded algebra $R = \bigoplus_{i \geq 0} R_i$ such that $R_0 = K$ and $R$ is generated by $R_1$. There are several characterizations of the depth of such an algebra. We use the one that depth$(R)$ is the maximal length of a regular $R$-sequence consisting of linear forms. Let $x_F = \cap_{i \in F} x_i$ be a squarefree monomial for some $F \subseteq [n]$ and $Z \subseteq \{x_1, \ldots, x_n\}$. The $K$-subspace $x_FK[Z]$ of $S = K[x_1, \ldots, x_n]$ is the subspace generated by monomials $x_Fu$, where $u$ is a monomial in the polynomial ring $K[Z]$. It is called a squarefree Stanley space if $\{x_i : i \in F\} \subseteq Z$. The dimension of this Stanley space is $|Z|$. Let $\Delta$ be a simplicial complex on $\{x_1, \ldots, x_n\}$. A squarefree Stanley decomposition $D$ of $K[\Delta]$ is a finite direct sum $\bigoplus_i u_iK[Z]$ of squarefree Stanley spaces which is isomorphic as a $\mathbb{Z}^n$-graded $K$-vector space to $K[\Delta]$, i.e.

\[ K[\Delta] \cong \bigoplus_i u_iK[Z] \]

We denote by sdepth$(D)$ the minimal dimension of a Stanley space in $D$ and we define sdepth$(K[\Delta]) = \max\{\text{sdepth}(D)\}$, where $D$ is a Stanley decomposition of $K[\Delta]$. Stanley conjectured in [10] the upper bound for the depth of $K[\Delta]$ as the following:

\[ \text{depth}(K[\Delta]) \leq \text{sdepth}(K[\Delta]) \]

Also we recall another conjecture of Stanley. Let $\Delta$ be again a simplicial complex on $\{x_1, \ldots, x_n\}$ with facets $G_1, \ldots, G_t$. The complex $\Delta$ is called partitionable if there exists a partition $\Delta = \bigcup_{i=1}^t [F_i, G_i]$ where $F_i \subseteq G_i$ are suitable faces of $\Delta$. Here the interval $[F_i, G_i]$ is the set of faces $\{H \in \Delta : F_i \subseteq H \subseteq G_i\}$. In [11] and [12] respectively Stanley conjectured each Cohen-Macaulay simplicial complex is partitionable. This conjecture is a special case of the previous conjecture. Indeed, Herzog, Soleyman Jahan and Yassemi [7] proved that for Cohen-Macaulay simplicial complex $\Delta$ on $\{x_1, \ldots, x_n\}$ we have that depth$(K[\Delta]) \leq \text{sdepth}(K[\Delta])$ if and only if $\Delta$ is partitionable.

Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable. Then as a consequence of our results we obtain:

**Corollary 3.1.** if $n = t + 1$ then $\Delta_t(C_n)$ is partitionable and Stanley’s conjecture holds for $K[\Delta_t(C_n)]$.

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