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# Characteristic Cycles Integration on $D$-Modules to obtaining of Field Equations solutions on $\mathbb{L}$- Holomorphic Bundles

Prof. Dr. Francisco Bulnes*[0]

IINAMEI A. C., Research Department in Mathematics and Engineering, TESCHA
francisco.bulnes@tesch.edu.mx

**Abstract**

Considering certain derived categories on coherent $D$- modules is constructed a moduli space of equivalences between objects of a complex holomorphic bundle and a sheaf of coherent $D$- modules, which are determined for a generalized Penrose transform in the derived categories level, whose images are Hecke categories on $\mathbb{L}$- holomorphic bundles. These co-cycles represent solutions of the field equations where a particular case are the massless field equations with different helicities $h(k)$.

**Keywords:** Coherent $D$- Modules, Hecke Categories, Integration on Characteristics Cycles, $\mathbb{L}$- Holomorphic Bundles, Moduli Space.

**Mathematics Subject Classication:** 53D37; 11R39; 14D24; 83C60; 11S15

# 1 Introduction

Part of our research object is centered in the extension of the equivalences space of the type $H^1(U, O_{\mathbb{P}}(k)) \cong \ker(U, \Box_{h(k)})$, [1,2] under a more general context given through the language of the $D$- modules, searching extend our classification of differential operators of the field equations to context of the $G$- invariant holomorphic bundles and obtain a complete classification of all the different operators of the curved analogous on $\mathbb{M}$ [3-5]. Thus our moduli space will be the of equivalences of the conformal classes in the algebraic $D$-modules with coefficients in a coherent sheaf [5]. One way to the obtaining

[0]* ORCID: 0000-0002-2007-7801.

of this equivalences space, is analyse the origin of structure of the complexes that define the microscopic structure of the space-time $\mathbb{M}$. Through of this way we can establish in one natural form equivalences (isomorphisms) between derived categories [6-7] considering complexes analogous to $O^0_{\mathfrak{P}}(h) \rightarrow \ldots O^i_{\mathfrak{P}}(h) \rightarrow O^{i+1}_{\mathfrak{P}}(h) \rightarrow \cdots \rightarrow 0$, where $\mathfrak{P}$, is an integral transform associated to a double fibration, used to represented holomorphic solutions of the generalized wave equation [2,3], with helicities parameter $h$. To the micro-local context we need a regularity theorem subjacent to these integral transforms given on $D$- modules complexes $\cdots \rightarrow \mathfrak{C}_{j-1} \rightarrow \mathfrak{C}_j \rightarrow \mathfrak{C}_{j+1} \rightarrow \ldots$ where the solution classes of the corresponding integral transforms are in holomorphic vector $G$- bundles that correspond to $\mathbb{L}$- holomorphic bundles as consequence to the symplectic geometry born of the foliations in Lagrangian submanifolds of $\mathbb{M}$, inside the problem of the uniqueness of the integral transforms on the same coherent $D$- modules [3,4]. Likewise are proposed the derived categories in the geometrical Langlands ramifications problem as a special characterizing of the ramified field. Their ramifications can be identified as degenerated cycles corresponding to orbits of coherent $D$- modules of certain Moduli space that can be induced by an appropriate Zuckerman functor obtained by a generalized Penrose transform developed on derived categories of the moduli space $\mathcal{M}_{Flat}(G, C)$ [7, 8], to obtain classes of objects in a moduli space of fields where the Lagrangians are submanifolds of a Calabi-Yau manifold and have a field theory re-interpretation as D-branes as $\mathbb{P}$- modules [9-11].

# 2 Preliminary Definitions and Backgrounds

We consider the $D$- modules category $M^L_{qc}(D)$, the category of quasi-coherent left $D$- modules on $X$, being $X$, an analytic complex manifold. This category is isomorphic to the category $M^R_{qc}(D)$, of quasi-coherent right $D^{opp}$- modules on $X$, [4]. Likewise for a category $M_{qc}(D)$, we call by $M_{coh}(D)$, the corresponding subcategory of coherent $D$- modules.

We consider a correspondence

$$\begin{array}{ccc} & S & \\ \pi\swarrow & & \searrow\nu, \\ X & & Y \end{array} \tag{2.1}$$

The manifolds $X$, and $Y$, when are analytic and complex manifolds, $\pi$, and $\nu$, are proper then $(\nu, \pi)$, induces a closed embedding

$$S \hookrightarrow X \times Y, \tag{2.2}$$

Also we difine $d_s = \dim_{\mathbb{C}} S$, with $d_{S\backslash Y} = d_S - d_Y$. We define the transform of a sheaf $\mathcal{F}$, on $X$, (more generality of an object of the derived category of sheaves) as:

$$\Phi_S \mathcal{F} = R\pi\nu^{-1}\mathcal{F}\left[d_{S\backslash Y}\right], \tag{2.3}$$

and we define the transform of $D_X$- module $\mathcal{M}$, like

$$\underline{\Phi}_{S*}\mathcal{M} = \underline{\pi}_* D\nu^*\mathcal{M}, \tag{2.4}$$

where $\underline{\pi}_*$, and $\nu^*$, denote the direct and inverse of $\pi$, and $\nu$, respectively, in the sense of the $D$- modules [3], and we consider also:

$$\Phi_S \mathcal{G} = R\nu!\pi^{-1}\mathcal{G}[d_{S\backslash X}], \tag{2.5}$$

to a sheaf $\mathcal{G}$, on $Y$. Then we have the formula:

$$\Phi_S RHom_{D_X}(\mathcal{M}, \mathcal{O}_X) = RHom_{D_Y}(\underline{\Phi}_S\mathcal{M}, \mathcal{O}_Y), \tag{2.6}$$

which is deduced the formula to the sheaf on $Y$ (coherent sheaf):

$$R\Gamma(X; RHom_{D_X}(\mathcal{M} \otimes \Phi_S\mathcal{G}, \mathcal{O}_X))[d_X] \cong R\Gamma(Y; RHom_{D_Y}(\underline{\Phi}_S\mathcal{M} \otimes \mathcal{G}, \mathcal{O}_Y))[d_Y], \tag{2.7}$$

This is the level of categorical equivalence of the transformation

$$\begin{array}{ccc} & \mathbb{F} & \\ {}^{\nu}\swarrow & & \searrow^{\pi,} \\ \mathbb{P} & & \mathbb{M} \end{array} \tag{2.8}$$

that defines the classical Penrose transform [1] [5]. We want said double fibration to the context of the right derived $D$- modules of the derived category $D^b(M^R_{qc}(D))$, because is necessary to give an equivalence with subcategory of the right $D$- modules that have support in $Y$, to guarantee the inverse image of Penrose transform that we denote as $\mathcal{P}^{-1}$ and with it obtain

[1]The Penrose transform is a geometrical integral transform that interprets elements of various analytic cohomology groups on open subsets of complex projective 3-dimensional space as solutions of linear differential equations on a Grassmannian of 2-dimensional planes in a 4-dimensional space. In mathematical physics we are interested to obtain solution clases of wave equation to different helicities.

an image of closed range to $\mathcal{P}$.

But we have the Kashiwara theorem which establishes the equivalence of the subcategory of the right $D$- modules that have support in $Y$ with the direct image of a right derived functor deduced of a closed immersion.

We enounce the Kashiwara thorem.

**Theorem 2.1 (Kashiwara)** *Let $i : Y \rightarrow X$, be the correspondence (2. 1), a closed immersion. Then the direct image functor $i_+$, is an equivalence of $M_{qc}^R(D^i)$ with the full subcategory of $M_{qc}^R(D)$, consisting of modules with support in $Y$.*

Proof. [5].

To define the images of direct functors to $D$- modules is necessary to use derived categories. For it, we consider the derived category $D^b(M_{qc}^R(D^\vee))$, the bounded derived category for right quasi-coherent $D^\nu$- modules of the form:

$$R_{\nu^+}(\mathcal{V}) = R\nu^*(\mathcal{V} \otimes_D^R \nu D_{Y \rightarrow X}), \tag{2.9}$$

where $\mathcal{V}$ is the characteristic manifold and $R$ is the right derived functor

$$R : M_{qc}^R(D^\vee) \rightarrow M_{qc}^R(D), \tag{2.10}$$

Also $r : Y \rightarrow X$, and $D_{Y \rightarrow X} = \nu^*(D) = O_Y \otimes_{\nu^{-1}} (O_X)\nu^{-1}D$. Then $D_{Y \rightarrow X}$, is a right $\nu^{-1}$, $D$- module to the right multiplication in the second functor.

The equivalences of the Kashiwara's theorem preserve coherence and holonomicity [4]. Then preserve conformability in $\mathbb{M}$, [2] [1-3, 5].

Likewise, the motivation to use the integral geometry methods in field theory to establish equivalences to determine solution classes to different field equations comes from the interpreting of the Grassmannian $G_{2,4}(\mathbb{C})$ as the complexification of the conformal compactification of Minkowski space. By the theorem 2. 1, and some results on involutive manifolds by Oshima [6], we can characterize certain spaces to the regularity of the Penrose transform images in $D$- modules.

[2] $\mathbb{M}$ is a complex Riemannian manifold of the 4-dimension. Of fact is the Kaehlerian manifold. To our study in integral geometry we consider $\mathbb{M} \cong G_{2,4}(\mathbb{C})$

A micro-local analysis carry us to generalize the Penrose transform in the derived categories context to the obtaining of a moduli space of equivalences between Lagrangian submanifolds and $m$- folds of a Calabi-Yau manifold. This precisely we want prove to field invariant on complex of certain sheaves whose germs are the differential operators of the field equations and where these germs correspond to elements of holomorphic $G$- bundles.

# 3 Characteristic Cycles and Equivalences

The construction of characteristic cycles can be established when Kashiwara's theorem equivalences associate to every object of $\mathcal{G}$ (coherent sheaf on $Y$ ) of this category, a Lagrangian cycle $C_y(\mathcal{G}) \in \Lambda, \forall y \in Y$, into $\mathcal{M} = T^*Z$ which is a cone (invariant under rescalling of cotangent fibers) [7].

If $\mathcal{G}$ is the structure sheaf of a closed submanifold like the given by $\Lambda$ in the double fibration

$$\begin{array}{ccc} & \Lambda & \\ {}^{p_1|\Lambda}\swarrow & & \searrow^{p_2|\Lambda} \\ T \times X & & T \times Y \end{array} \qquad (3.1)$$

The consequences of this are:

i). $p_2 \mid \Lambda$, is a closed embedding identifying $\Lambda$, (deep space) to a closed regular involutive manifold $V \subset T^*Y$ (foliation).
ii). $p_1 \mid \Lambda$, is smooth and surjection. $T^*X$, could represent in mathematical physics an observable set of the space-time.

Then the cycle $C_y(\mathcal{G}) \in \Lambda, \forall y \in Y$ is justly the co-normal bundle and this tends to be singular. Then we can re-write the Kashiwaras theorem in derived categories as follows:

"The construction of a smoothed characteristic cycle gives origin to a full embedding of derived categories $D^b_{RS(V)}(\mathfrak{E}_X)$ (full subcategory of objects $\mathcal{M}$ that have cohomology groups with regular singularities along $V$) into $M_{RS(V)}(\mathfrak{E}_X)$ (sheaf of micro-differential $X$)"

The $\mathfrak{E}_X$- modules conform a support ring of the $D_X$- modules. A $\mathfrak{E}_X$-module is a ring of micro-differential formal operators of finite order on $T^*X$.

We consider $C_V$ =codim$V$, in $T^*Y$. then we have the following local model of correspondences:

**Lemma 3.1** *Assume that (2. 2) is a closed embedding and the assumptions given in i), and ii) above. Then for every* $(p, q) \in \Lambda$ *exists open subsets* $U_X, U_Y \subset T^*Y$, *with* $p \in U_X$, *and* $q \in U_Y$ *and* $Z$, *a complex manifold of dimension* $c_V$. *Also we define a contact transformation* $\psi : U_q \xrightarrow{\cong} U'_X \times T^*Z$, *such that* $Id_{U_X} \times \psi$, *induces an isomorphism of correspondences*

$$\begin{array}{ccccc} & \Lambda \cap (U_X \times U_Y) & & \Lambda_X \times T^*Z & \\ & p_1 \swarrow \quad \searrow p_2^{\alpha} & \xrightarrow{\cong} & p_1 \swarrow \quad \searrow p_2^{\alpha} & \\ U_X & & V \times U_q \qquad U_X & & U'_X \times T^*Z, \end{array} \tag{3.2}$$

where $\Lambda_X \in U_X \times U'_X$ is the graph of a contact transformation

$$\chi : U_X \xrightarrow{\cong} U'_X, \tag{3.3}$$

and $p_2^{\alpha}$, denotes the projection

$$U_X \times U'^{\alpha}_X \times T^*Z \xrightarrow{\cong} U'_X \times T^*Z \tag{3.4}$$

Then is clear that of the before lemma, we can characterize the cycles propitious to determine an full embedding of derived categories of the full subcategory of sheaves of $K$- vector spaces comprising complexes with a constructible cohomology as of the type $H^0(\underline{\Phi}_S(\mathcal{M})) = Mod_{RS(V)}(D_Y)$. The super-indice $^{\alpha}$, in the projection $p_2^{\alpha}$, defines a natural isomorphism which exist in the micro-local structure of the derived category of functors of the moduli space that will relate the derived categories of coherent $D$- modules with the categories of objects $\mathcal{M}$ that have cohomology groups with regular singularities along $V$. This will permit a classification of differential operators of the field equations in the holomorphic vector bundles, and likewise to obtain the integrals or solutions of the field equations through the solution classes defined by the corresponding cohomology groups. This groups are obtained in natural way through the different Penrose transform versions or related with other geometrical integral transforms as given for twistor transform, Radon transform, etcetera.

Let $f : S \to X$, be a morphism, and denote by $f!$, and $f^{-1}$ the proper direct image and inverse image for $D$- modules and we denote by $\boxtimes$, the exterior tensor product. To $\mathcal{M} \in D^b_{RS(V)}(D_X)$, and using (2. 7) we associate its dual

$$\mathcal{D}\mathcal{M} = RHom_{D_X}(\mathcal{M}, D_X \otimes_{O_X} \Omega^{\otimes} X), \tag{3.5}$$

Here $D_X \otimes_{O_X} \Omega^{\otimes} = O_X$, where $\Omega_X$, is the sheaf of holomorphic forms of maximal degree. We also set $\underline{D}\mathcal{M} = \underline{D}'\mathcal{M}[d_X]$. Thus $\underline{D}'\mathcal{M}$, and $\underline{D}\mathcal{M}$ belong to $D^b_{RS(V)}(D_X)$. We define $\mathcal{K} \in \mathcal{M}_\Lambda(D)$, to be a simple $D_{X\times Y}$- module along $\Lambda$. Remember that $\Lambda$ is the Lagrangian manifold of Lagrangian submanifolds that must be corresponded to the $m$- folds of the Calabi-Yau manifold $Y$.

In particular $\mathcal{K}$, is regular holonomic and hence $\underline{D}\mathcal{K}$, is concentrated in degree zero. For $\mathcal{M} \in D^b_{RS(V)}(D_X)$, and $\mathcal{N} \in D^b_{RS(V)}(D_Y)$, we have

$$\Phi_{\mathcal{K}}\mathcal{M} = \underline{q_2}!(\mathcal{K}^L \otimes O_{X\times Y}\underline{q}_1^{-1}\mathcal{M}), \tag{3.6}$$

where $q_1$, and $q_2$, are defined by $X \xrightarrow{q_1} X \times Y \xrightarrow{q_2} Y$, and also the inverse functor

$$\Psi_{\mathcal{K}}\mathcal{N} = \underline{q_1}!(\underline{D}\mathcal{K} \otimes^L O_{X\times Y}\underline{q}_2^{-1}\mathcal{N})[d_X - d_Y], \tag{3.7}$$

where

$$\Phi_{\mathcal{K}}\mathcal{M} = H^j\underline{\Phi}_{\mathcal{K}}\mathcal{M}, \tag{3.8}$$

and

$$\Psi^j_{\mathcal{K}}\mathcal{N} = H^j\Psi_{\mathcal{K}}\mathcal{N}, \tag{3.9}$$

Therefore by this way we are establishing and constructing the more general version of $D$- modules transform (possibly a generalized Penrose transform with support supp$\mathcal{K}$ ) that we need to the field theory phenomena in the Universe $\mathbb{M}$. We are interested in calculate explicitely the image of a $D_X$- module associated to a line bundle to be consistent in “quantization” in equivalences to obtain solution classes defined by cohomology groups to the field equations.

The before functor (3. 9) through of supp$\mathcal{K}$ ,must help to obtain of equivalences through the isomorphisms that we need. A result to the respect is:

**Theorem 3.1.** *Assume that $q_1$, and $q_2$, are proper on supp$\mathcal{K}$, and assume i) and ii) given in the beginning of this section. Let $\mathcal{M}$ be a simple $D_X$- module along $T^*Y$ and let $\mathcal{N}$, be a simple $D_Y$- module along $V$. Then*

a) $\underline{\Phi}^0_{\mathcal{K}}$, and $\underline{\Psi}^0_{\mathcal{K}}$, send isomorphisms (isomorphism modulo flat connections $\varphi : \mathcal{M} \to \mathcal{N}$) to isomorphisms (isomorphism modulo flat connections $\psi : \mathcal{N} \to \mathcal{M}$).

b) $\underline{\Phi}^0_{\mathcal{K}}\mathcal{M}$, is simple along $V$, and $\underline{\Psi}^0_{\mathcal{K}}\mathcal{N}$, is simple along $T^*X$. Moreover $\underline{\Phi}^j_{\mathcal{K}}\mathcal{M}$, and $\underline{\Psi}^j_{\mathcal{K}}\mathcal{N}$, are flat connections for $j \neq 0$.

c) The natural adjunction morphisms $\mathcal{M} \to \underline{\Psi}^0_{\mathcal{K}}\underline{\Phi}^0_{\mathcal{K}}\mathcal{M}$, and $\underline{\Phi}^0_{\mathcal{K}}\underline{\Psi}^0_{\mathcal{K}}\mathcal{N} \to \mathcal{N}$, are isomorphisms modulo flat connections. In particular, the functors

$$Mod_{Coh}(T^*X, O_X) \underset{\Phi^0_{\mathcal{K}}}{\overset{\Psi^0_{\mathcal{K}}}{\longleftarrow\!\!\longrightarrow}} Mod_{Coh}(V, O_Y), \tag{3.10}$$

Are quasi-inverse to each other, and thus establish equivalence of categories. In $D$- modules theory the category given by $Mod_{Coh}(V, O_Y)$ is of the simple $D_Y$- modules along $V$.

The following step is give a result of equivalences between categories that suggest the extension to the before functors to the category of the vector bundle of lines where will can to be given a classification of the differential operators belonging to sheaves defined in the section 2, of this work, and have been published and reported in [8, 9].

**Theorem 3. 2** [3].*With the same hypothesis as in the before theorem 3. 1, assume also $d \geqslant 3$. With the notation of the theorem 2. 1, given in the section 2, then the following correspondence is an equivalence of categories*

$$Mod_{Coh}(D\mathcal{L}) \underset{G\Phi^0_{\mathcal{K}}}{\overset{\Psi^0_{\mathcal{K}}F}{\longleftarrow\!\!\longrightarrow}} Mod_{Coh}(V, O_Y), \tag{3.11}$$

A classification through generalized Verma modules to the solution class of field equations has been the work presented and published in the references mentioned.

As a corollary, using the Penrose transform, which can be of Radon type in a Lagrangian manifold $\Lambda$, we can obtain on the complex Minkowski space $\mathbb{M}$, the simple $D_{\mathbb{M}}$- modules along the characteristic manifold $V$, of the wave equation, which are classified by (half) integers called helicities $h(k)$.

Then the moduli space that we want must be the space of equivalences using geometrical additional hypothesis. $Mod_{Coh}(D\mathcal{L})$, is the full subcategory of $Mod_{Coh}(D_X)$, whose open sets are of the $D\mathcal{L}$ type to some bundle of lines $\mathcal{L}$.

In particular if $\mathcal{N}$, is a simple $D_Y$- module along $V$, exist an unique (up to $O_X$- linear isomorphism) lines bundle $\mathcal{L}$, on $X$, such that $\mathcal{N} \cong \underline{\Phi}^0_{\mathcal{K}} D\mathcal{L}$, in $Mod_{Coh}(D_Y, O_Y)$.

Then applying the theorem 2. 1, which says that simple $D_Y$- modules along $V$, are classified to flat connections for category of holomorphic vector bundles with flat connections, then we have the coherent $D$- modules space:

$$\begin{aligned} Mod_{Coh}(V, O_Y) = \{ M(D_Y - modules/singularities \\ to\ along\ the\ involutive\ manifold\ V/\ flat\ connections)\}, \end{aligned} \tag{3.12}$$

Then the theorem 2. 1, concludes that

$$\Phi^0_{\mathcal{K}} D\mathcal{L} \leftarrow \underline{\Phi}^0_{\mathcal{K}}(\Psi^0_{\mathcal{K}}\mathcal{N}) \rightarrow \mathcal{N}, \tag{3.13}$$

where we are interested in the image of a $D_Y$- module associated to a line bundle to the "quantization" in the field theory required.

To this "quantization" of these equivalences we consider a coherent $D$- module $\mathcal{M}$, to be a simple $D_X$- module along $T^*X$, and let $\mathcal{N}$, a simple $D_Y$- module along $V$. Then $D'\mathcal{M} \boxtimes \mathcal{N}$, is a simple $D_{X\times Y}$- module along $T^*X \times V$. Here the important notion is that $\forall p \in \Lambda$, a section $s \in Hom_{D_{X\times Y}}(D'\mathcal{M} \boxtimes \mathcal{N}, \mathcal{K})$, which is not degenerated at $p$, if for a simple generator $u$, of $D'\mathcal{M} \boxtimes \mathcal{N}$, at $p$, the composition $s(u)$, is a non-degenerated section of $\mathcal{K}$, at $p$. In little words: locally simple modules admit simple generators and can be checked immediately that this definition does not depend on the choice of such generators.

The before stays with more precision considering that $D'\mathcal{M} \boxtimes \mathcal{N}$ , is non-degenerated on $\Lambda$, if $_S$, is non-degenerated of any $p \in \Lambda$, then exists a natural isomorphism

$$\alpha : Hom_{D_{X\times Y}}(D'\mathcal{M} \boxtimes \mathcal{N}, \mathcal{K}) \to Hom_{D_Y}(\mathcal{N}, \underline{\Phi}_{\mathcal{K}}(\mathcal{M})), \tag{3.14}$$

In this level, the equivalences stay determined to the $D$- modules context and a section $_S$, defines a $D_Y$- linear morphism

$$\alpha(s) : \mathcal{N} \to \underline{\Phi}_{\mathcal{K}}(\mathcal{M}), \tag{3.15}$$

Then we to obtain the differential operators classification in the flat connections with the determinant condition of non-degeneration of $_S$, in $\Lambda$. If furthermore we consider additionally the geometrical hypothesis established by the geometrical Langlands duality, then in the holomorphic vector bundles context we can establish equivalences of the coherent $D$- modules defined before (on $\Lambda$) and holomorphic $G$- bundles.

As last result to preparing of these equivalences, we have the following result.

**Theorem 3. 3.** [3, 10].*With the above notations, if s, is non-degenerated on* $\Lambda$*, then* $\alpha(s)$*, is an isomorphism modulo flat connections.*

We can observe that when $\Lambda$, is the graph of a contact transformation (where $V$, is open in $T^*Y$) the theorem 3. 3, is reduced to the so-called “quantizing contact transformation”, in pseudo-differential equations.

# 4 Main Result and a Lemma on Equivalence of Characteristic Cycle Spaces

We define the category $Mod^b_{good}(D_X; T^*X)$, as the localization of $Mod^b_{good}(D_X)$, for the thick sub-category of holomorphic bundles endowed with the flat connection belonging to the $D$- module:

$$M_X = \{\mathcal{M} \in Mod_{good}(D_X) \mid char(\mathcal{M}) \subset T^*X \times X\}, \tag{4.1}$$

In particular the objects of $Mod^b_{good}(D_X; T^*X)$, are the same objects of $Mod_{good}(D_X)$, and a morphism $w : \mathcal{M} \to \mathcal{M}'$ in $Mod_{good}(D_X)$, is transformed in an isomorphism in $Mod^b_{good}(D_X; T^*X)$, if $ker\ \ w$, and $coker\ \ w$, correspond to (4.1). [3]

[3]Equivalent to say that $\mathcal{E}w : \mathfrak{C}\mathcal{M} \to \mathfrak{C}\mathcal{M}'$, is an isomorphism on $T^*X$. Here each element $\mathfrak{C}$ belongs to a chain complex category $Kom(A)$, where $A$, is an Abelian category.

In a similar way we define $M_Y$, and the category $Mod^b_{good}(D_Y; T^*Y)$, which are obtained by localization of objects in the category $Mod_{RS(V)}(D_Y)$, respect to $M_Y$. Remember that we want all objects of full embedding of derived categories of the full subcategory of sheaves of $K$- vector spaces comprising complexes with a constructible cohomology as of the type $H^0(\underline{\Phi}_S(\mathcal{M})) = Mod_{RS(V)}(D_Y)$.

In the context of the generalized $D$- modules, the generalized Penrose transform version carries to a functor

$$\underline{\Phi}_S + additional \quad geometrical \quad hypothesis, \tag{4.2}$$

which establishes the equivalenes of the moduli space

$$\mathscr{M}_{d[S/Y]} = \{M(D_Y - modules/singularities \ along \ the \ involutive \ manifold \ V/ \ flat \ \ connections \cong H^\bullet(\underline{U}, \mathcal{O})\}, \tag{4.3}$$

considering the moduli space like base

$$\mathscr{M}_{d[S/Y]} = \{M(D_X - modules/flat \ \ conections \cong Ker(U, \Box_{h(k)}))\} \tag{4.4}$$

Then the cohomology on moduli spaces is the cohomology on the space-time $\mathbb{M}$, with an equivalence like given in the classic Penrose transform to a more general cohomology group that $H^\bullet(\underline{U}, O_{\mathbb{P}}(k))$ , and whose dimension can be calculated by intersection methods.

We consider the cohomology groups of zero dimension of the functors $\underline{\Phi}_S$, and $\underline{\Psi}_{S^{\frown}}$, we get functors that we will can denote by $\underline{\Phi}^0_S$, and $\underline{\Psi}^0_{S^{\frown}}$, respectively. In other words we obtain the images:

$$\underline{\Phi}^0_S = H^0(\underline{\Phi}_S(\mathcal{M})), \qquad \underline{\Psi}^0_{S^{\frown}} = H^0(\underline{\Psi}_{S^{\frown}}(\mathcal{M})), \tag{4.5}$$

where using the theorem 2. 1, (using the local and microlocal analysis of the before sections) are established the correspondences between categories:

$$Mod_{good}(D_X) \underset{\Phi^0_{\mathcal{K}}}{\overset{\Psi^0_{\mathcal{K}}}{\longleftarrow \atop \longrightarrow}} Mod_{RS(V)}(D_Y), \tag{4.6}$$

By (2. 1), (4. 3) and (4. 4) these equivalences conform the moduli space [11]:

$$\mathscr{M} = \{\mathcal{M} \mid \Gamma H^{\bullet}_{D_X}(Hom(Mod_{good})(D_X)), Mod_{RS(V)}(D_Y)) = \tag{4.7}$$
$$H^0(\underline{\Phi}_S(\mathcal{M})), \forall D_X, D_Y \in M(derived \;\; D-modules)\},$$

The additional geometrical hypothesis in the functor (4. 2) comes established by the Langlands geometrical duality [29] which says that the derived category of coherent sheaves on a moduli space $\mathscr{M}_{Flat}[^{L}G, C]$, where $C$, is the complex given by

$$C : \cdots \xrightarrow{d^{j-1}} \mathfrak{C}^{j-1} \xrightarrow{d^j} \mathfrak{C}^{j} \xrightarrow{d^{j+1}} \mathfrak{C}^{j+1} \xrightarrow{d^{j+2}} \mathfrak{C}^{j+2} \xrightarrow{d^{j+3}} \ldots, \tag{4.8}$$

is equivalent to the derived category of $D$- modules on the moduli space of holomorphic vector $G$- bundles given by $B_G(C)$, [23]. These equivalences permit to map points of $\mathscr{M}_{Flat}[^{L}G, C]$, to a Hecke eigen-sheaf given by $B_G(C)$.

**Theorem (F. Bulnes) 4. 1.***The moduli space obtained by this way, is the moduli space $\mathscr{M}_{Flat}[^{L}G, C]$, where characteristic cycles are the Lagrangian submanifolds $\Lambda$, defined with a complex $C$, of certain special sheaf of holomorphic $G$- bundles (eigensheaf of Hecke).*

*Proof.* [7]. ◆

The Hecke eigensheaf $B_G(C)$, will be conformed by the Lagrangians $C(\mathcal{G})$, whose sheaf $\mathcal{G}$ must be to the Lagrangians of $Y$. Remember that $\Lambda$ is the Lagrangian manifold of Lagrangian submanifolds that must be corresponded to the $m$- folds of the Calabi-Yau manifold $Y$.

Then an application in field theory on the space-time $\mathbb{M}$, which can be identified by the Hitchin moduli space:

$$\mathbb{M} =: \mathscr{M}_H[G, C], \tag{4.9}$$

can be of fact the existence of equivalences between $D$- modules of $B_G(C)$, and some $D$- branes on $\mathscr{M}_H[G, C]$. This carry us to an example in which characteristic cycles as Lagrangians can have their equivalent in a flat space $\mathbb{P}^{n+4d}$, of a corresponding supertwistor space $\mathbb{PT}$, as lines bundles, which is the demonstration of the equivalence between objects of certain class as singularities of the space-time and objects of the moduli space (4. 3). Remember that in the more simple case we can have polynomials of a homogeneous lines bundle corresponding to the complex projective space $\mathbb{P}^3$, where zeros of the polynomial are identified for the singularities of the space-time.

Then a corollary, is the following integration on complex super-projective space to tack the $SO(4, \mathbb{R})$, space-time through $D$- branes, and the problem of singularities stays saved using a criteria of dimension of the cotangent space corresponding to the singularity [7]. This is equivalent to the Langlands ramification problem with a singular connection [13]. But this problem has been solvented through to define connections with regular singularities [13].

We can give the following technical lemma on equivalence between objects of the moduli space (4. 3) and the singularities (holes of the space-time $\mathbb{M}$, with the corresponding characterization given by (4. 9))

**Lemma (F. Bulnes) 4. 2.** *Characteristic cycles in $C(\mathcal{G})$, as Lagrangians have their equivalent in a flat space $\mathbb{P}^{n+4d}$, (corresponding to the supertwistor space $\mathbb{PT}$,) as lines bundles in $\mathbb{P}^{\smile}$. The cycles in $C(\mathcal{G})$, are solutions to the field equation on $\mathbb{L}$-. Holomorphic Bundles, to the space-time $\mathbb{M}$, which includes singularities.*

*Proof.* We consider the equivalences between the moduli spaces [7, 14]:

$$\mathscr{M}_{G-equi\ variant}[G/H, C^{\wedge}] \cong \mathscr{M}_{G-equi\ variant}[CY, C^{\smile}], \tag{4.10}$$

where by the construction given in the section 3, and fixing a holomorphic vector bundle $E$, we have:

$$C^{\wedge} = CC(\mathcal{G}) = \left\{ C_y(\mathcal{G}) \in \Lambda \mid I(\omega) = \int\limits_{C_y(\mathcal{G}), y \in Y} Tr(\omega) \wedge vol, \forall \omega \in \Omega^{0,\bullet}(Y, End(E)) \right\} \tag{4.11}$$

which is the category of Lagrangian submanifolds of $\Lambda$. The complex $C^{\smile}$, is the complex where $Y$- manifolds (that are orbifolds of a $CY$- manifold) are as $\mathbb{P}$- modules. Then their elements are $D$- branes in the orbifolds like $D$-modules on lines bundles in $\mathbb{P}^{\smile}$. In (4.11), the complex $C$, follows being the complex defined (4. 8) and $Tr(\omega) \wedge vol$, is the Lagrangian that defines the action of each section of the holomorphic vector bundle on $\Lambda$. But this last is a $\mathbb{L}$-. Holomorphic Bundle,

To the inverse process we have that the $dg$- algebra is the $dg$- algebra de Dolbeault of $(0, \bullet)$- forms $\Omega^{0,\bullet}(Y, End(E))$, which clarly is una $dg$- algebra of endomorphisms of a generator of the derived category $D^b(Coh(Y))$, where $Y$, is Calabi-Yau. But $\Omega^{0,\bullet}(Y, End(E))$ is holomorphically smooth compact finite dimensional. Then the existence of a non-degenerated cohomology class $[\varphi]$, defined by the functor image $\underline{\Phi}_S(Mod_{good}(D_X))$, of cycles is guarantied and (4. 7) is satisfied. Then

$$\Gamma H^{\bullet}_{D_X} Hom(Mod_{good}(D_X), Mod_{RS(V)}(D_Y)) = H^0(\underline{\Phi}_S(\mathcal{M})), \tag{4.12}$$

records in a moduli space of holomorphic bundles which is an extension of the transformed cycles by the classic Penrose transform. By the theorem 3. 3, the equivalence established in (4. 12) is an isomorphism modulo flat connections. ◆

# 5 Conclusions

The equivalence between two derived categories of two objects class has been demonstrated on the base to construct and determine a moduli space as equivalences space of algebraic objects as coherent $D$- modules and geometrical objects as G-invariant holomorphic vector bundles, where are established the field equations solutions to different parameters of helicities considering isomorphisms modulo flat connections.
This can bring in consequence an adequate classification of differential operators to different field equations considering their corresponding classifying spaces through Verma modules in the context of cohomology groups, that define the integrals or solutions to these equations [9]. Likewise, this method, which use inverse methods in geometry, could extend the integral geometry methods used to explore the Universe through a integrals theory which can join the macroscopic and microscopic aspects of the space-time.
The limitations can be, how much can we extend the holomorficity of the structure of the time-space model to obtain integrals of field equations in an even meromorphic context, to include many of the aspects of field theory that occur in the Universe. Possibly using ramifications of the Langlands correspondences in the meromorphic context, where can be constructed a generalized Penrose transform as the extended Ward-Penrose transform could be useful in this case. The power of the methods is the obtaining of solution classes through the functors, which even can be in the derived geometry context to the stacks and infinite algebras establishing correspondence between operators of these algebras and complex vector bundles of the stacks. Then can be constructed a Universe theory through integration invariants as is explained in integral geometry. The lemma proposed is a technical lemma which goes focused to prove that each Hecke eigensheaf will be conformed by the Lagrangians whose sheaf must be to the Lagrangians of a Calabi-Yau manifold. The Hecke eigensheaf will be obtained through

functors constructed via integral transforms on algebraic $D$-modules. The lemma establishes the correspondence between distinct objects context (algebraic and geometrical objects) in a clear example of the power of this method.

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**Technical Notation**

$D_X$- It's the sheaf of rings of holomorphic linear operators (a $D$- module).

$\mathcal{G}$- Coherent sheaf of the $D_Y$- modules $\underline{\Phi}_{S*}\mathcal{M}$.

*dg- Algebra* - Differential graded algebra.

$\Lambda$- Manifold of Lagrangian submanifolds.

$\square_{h(k)}$- Wave operator. This is a differential operator that composes the wave equation on the space $\mathbb{R}^4$, on different helicities $h(k), \forall k \in \mathbb{Z}^+$.

$O_X$- Sheaf of holomorphic functions on a complex manifold $X$.

$\mathbb{M}$- Complex Riemannian manifold that models the space-time which is defined formally as $\mathbb{M}- \cong \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C}$.

$C(\mathcal{G})$- Cycles space on the sheaf $\mathcal{G}$.

$End(E)$- Edomorphisms spaces of a holomorphic vector bundle $E$.

$\boxtimes$- Tensor exterior product of modules.

$H^0(\underline{\Phi}_S(\mathcal{M}))$- Cohomology group or solution space of integral objects in the category $Mod_{RS(V)}(D_Y)$. These are extended solutions that we want to the field equation (wave equation) in the space-time including singularities (sources or holes), using the integral transforms scheme.

$CY$- Calabi Yau manifold. It's a microscopic model of space-time developed through strings theory.

$B_G(C)$- Hecke eigensheaf which has nature $D$- modules space on the moduli space of holomorphic vector $G$- bundles on the complex defined in (4. 8).

$D$- branes- It's a $D$- module defined on $\mathbb{L}$- Holomorphic Bundle or also can be defined as $D$- modules on lines bundles in a dual of a projective space. The $D$- branes, from a point of view of the pure field theory are fields that can stack geometrical objects in the space-time, for example hypersurfaces.

$\mathscr{M}_H[G, C]$- Hitchin moduli space, which establishes equivalences between algebraic and geometrical objects in the space-time, for example dimensions of algebras, cohomology classes, holomorphic vector bundles or $G$- invariant local systems. The complex Riemannian manifold model of the space-time can be identified by the corresponding Hitchin moduli space. It's very useful to define identities and relations with other moduli space.

$\mathbb{P}^\bullet$- Complex projective space of certain dimension $\bullet$.

$D^b(Coh(Y))$- Derived category of the coherent $D$- modules that are solutions (as orbifolds) in a Calabi-Yau manifold.

$\mathscr{M}_{d[S/Y]}$- Moduli space of equivalences established or determined by the functors (4. 2). This moduli space establishes the solutions of the field equations through kernels of the field equations whose cohomology group is $H^\bullet(\underline{U}, O)$.

$Mod_{Coh}(D\mathcal{L})$- Category of the coherent $D$- modules on the lines bundle $\mathcal{L}$. This category is the full subcategory of $Mod_{Coh}(D_X)$. The corresponding differential operators are constructed on a graded algebra $D\mathcal{L}$.

$Mod_{Coh}(V, O_Y)$- Category of the simple $D_Y$- modules along $V$.

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