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Bipolar Fuzzy Ideals and Prime Ideals in Left Almost Rings

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ABSTRACT. A bipolar fuzzy subset is an extension of a fuzzy subset, that is an extension from mapping $f : R \to [0,1]$ to mapping $f : R \to [-1,1]$. Bipolar fuzzy has been studied in rings, but no research has been conducted on LA-ring. A non empty set $(R,+,)$ is called an LA-ring if $(R,+)$ is an LA-group and $(R,.)$ is an LA-semigroup. In this paper, we introduce the definitions and properties of bipolar fuzzy ideals, bipolar fuzzy completely prime ideals, and weakly bipolar fuzzy completely prime ideals in LA-rings. Ideal from bipolar fuzzy subset of the form $\gamma_I, t'A_M$, and bipolar fuzzy ideal from Cartesian product of bipolar fuzzy ideals are also given. Moreover, in this paper we study relationships between ideal and bipolar fuzzy ideal in LA-rings and the relationship between bipolar fuzzy ideal and bipolar fuzzy completely prime ideal in LA-rings.

1 Introduction

A fuzzy subset is a mapping from a non empty set X to [0,1] [4]. Many researchers have studied about fuzzy subset in rings such as Liu [14] who studied fuzzy subrings and fuzzy ideals in rings and Mukherjee and Sen [13] who studied fuzzy prime ideals in rings. Yusuf [11] in [9] introduced a new algebraic structure called left almost ring which is usually abbreviated as LA-ring. Furthermore, Yiarayong [9] developed the concept of LA-ring into

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fuzzy subset and introduced the definitions and properties of fuzzy LA-subring, fuzzy ideals, fuzzy completely prime and weakly fuzzy completely prime ideals in LA-rings.

Fuzzy subset then expanded on its codomain. A bipolar fuzzy subset is one of the expansion. There were many studies on bipolar fuzzy such as Yaqoob [8] that introduced bipolar fuzzy (ideal, bi-ideal, and interior ideal) in LA-semigroups. Then Muthuraj, et al. [10] introduced anti-bipolar fuzzy subrings and their properties. Also Anitha, et al. [6] discussed about bipolar fuzzy subsets in groups.

Although there have been studies that discussed about bipolar fuzzy subsets in rings, but there is no research that discuss about bipolar fuzzy subset in LA-ring. So, this is a new idea in the development of bipolar fuzzy subset.

2 Preliminaries

In this section we provide some definitions and examples that will be used in the next section. If a grupoid $(G,)$ hold the invertive law such that $(a.b).c = (c.b).a$, for all $a, b, c \in G$ then it is called an LA-semigrup $(G,.)$ [5]. If an LA-semigrup $(G,.)$: (i) has a left identity element e such that $e.a = a$ for all $a \in G$ and (ii) for all $a \in G$ there is a' such that $a'.a = e$, then $(G,.)$ is called an LA-group [7] in [12]. Yusuf [11] in [9] established the structure of an LA-ring, $(R,+,.)$ is called an LA-ring if: (i) $(R,+)$ is an LA-grup, (ii) $(R,.)$ is an LA-semigrup, and (iii) distributive laws of . over $+$ hold, such that for all $a, b, c \in R$, $(a+b).c = a.c + b.c$ and $a.(b+c) = a.b + a.c$.

Let a non empty set R be an LA-ring. If $e.a = a$ for all $a \in R$ then e is called left identity element in $R.R$ with a left identity element is called an LA-ring with left identity. $S \neq \emptyset \subseteq R$ is called a Left Almost-subring (LA-subring) of R if S is an LA-ring under binary operation induced by R [9]. An LA-subring S is called left ideal (right ideal) of R if $ra \in S(ar \in S)$ for all $r \in R$ and $a \in S$, S is called ideal of R if S is a left and right ideal of R [2] in [9].

Example 2.1. [1]: $R_1 = \{0,1,2,3,4\}$ with two binary operations defined in Table 2.1 and Table 2.2 is an LA-ring with 1 as the left identity element.

Table 1: Table

(a) Table 2.1: Operation $+$ in R_1

+	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

(b) Table 2.2: Operation . in R_1

.	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Example 2.2. [9]: $R_2 = \{0,1,2,3,4,5\}$ with two binary operations defined in Table 2.3 and Table 2.4 is an LA-ring with 1 as the left identity element.

A bipolar fuzzy subset B of R is defined by $B = (f_B^+, f_B^-)$ where $f_B^+ : X \to [0,1]$ and $f_B^- : X \to [-1,0]$ [15]. An

Table 2: Table

(a) Table 2.3: Operation $+$ in R_2

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	5	0	1	2	3	4
2	4	5	0	1	2	3
3	3	4	5	0	1	2
4	2	3	4	5	0	1
5	1	2	3	4	5	0

(b) Table 2.4: Operation . in R_2

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

LA-ring R is a bipolar fuzzy subset of itself defined by $\Gamma = (R_\Gamma^+, R_\Gamma^-)$ where $R_\Gamma^+ : R \to 1$ and $R_\Gamma^- : R \to -1$. Let $B = (f_B^+, f_B^-)$ and $C = (g_C^+, g_C^-)$ be bipolar fuzzy subsets of an LA-ring, then the intersection and the union of B and C are defined as follows [3].

$$(B \cup C)(x) = ((f_B^+ \cup g_C^+)(x), (f_B^- \cap g_C^-)(x))$$

where

$$(f_B^+ \cup g_C^+)(x) = max\{f_B^+(x), g_C^+(x)\} \text{ and} (f_B^- \cap g_C^-)(x) = min\{f_B^-(x), g_C^-(x)\}.$$

and

$$(B \cap C)(x) = ((f_B^+ \cap g_C^+)(x), (f_B^- \cup g_C^-)(x))$$

where

$$(f^+ \cap g^+)(x) = min\{f_B^+(x), g_C^+(x)\} \text{ and} (f^- \cup g^-)(x) = max\{f_B^-(x), g_C^-(x)\}.$$

Definition 2.3. Let β be an index and $\{A_\alpha = (f_{A_\alpha}^+, f_{A_\alpha}^-) : \alpha \in \beta\}$ be a family of bipolar fuzzy subsets of R, then

$$\bigcap_{\alpha \in \beta} f_{A_\alpha}^-(x) = \inf\{f_{A_\alpha}^- : \alpha \in \beta\}$$

$$\bigcup_{\alpha \in \beta} f_{A_\alpha}^+(x) = \sup\{f_{A_\alpha}^+ : \alpha \in \beta\}$$

Definition 2.4. [6]: Let $B = (f_B^+, f_B^-)$ and $C = (g_C^+, g_C^-)$ be bipolar fuzzy subsets of set G and H respectively. $B \times C$ is the Cartesian product of B and C and defined as follows

$$B \times C = ((x,y), (f_B^+ \times g_C^+)(x,y), (f_B^- \times g_C^-)(x,y))$$

where $(f_B^+ \times g_C^+)(x,y) = \min\{f_B^+(x), g_C^+(y)\}$ and $(f_B^- \times g_C^-)(x,y) = \max\{f_B^-(x), g_C^-(y)\}$, for all $(x,y) \in G \times H$.

Definition 2.5. [8]: If $A = (f_A^+, f_A^-)$ and $B = (g_B^+, g_B^-)$ are bipolar fuzzy subsets of a non empty set X then $A \circ B$ is the product of A and B and defined as follows:

$$(A \circ B)(x) = ((f_A^+ \circ g_B^+)(x), (f_A^- \circ g_B^-)(x))$$

$$(f_A^+ \circ g_B^+)(x) = \begin{cases} \bigcup_{x=yz} \min\{(f_A^+ \circ g_B^+)(x)\}, & \text{if } \exists y,z \in R \ni x = yz \\ 0, & \text{if otherwise} \end{cases}$$

$$(f_A^- \circ g_B^-)(x) = \begin{cases} \bigcap_{x=yz} \max\{(f_A^- \circ g_B^-)(x)\}, & \text{if } \exists y,z \in R \ni x = yz \\ 0, & \text{if otherwise} \end{cases}$$

3 Bipolar Fuzzy Ideals and Prime Ideals in Left Almost Rings

In this section the results and discussion of bipolar fuzzy ideals and prime ideals in left almost rings are given. We first introduce the concept of a bipolar fuzzy LA-subring of an LA-ring and then we give the definitions, properties and examples of bipolar fuzzy ideal, bipolar fuzzy completely prime ideal, and weakly bipolar fuzzy completely prime ideal in LA-rings. We prove that the Cartesian product of two or finitely bipolar fuzzy ideals is bipolar fuzzy ideal. We also prove that bipolar fuzzy subset of the forms γ_I and $t'A_M$ are also bipolar fuzzy ideals. We can see the relationships between ideal and bipolar fuzzy ideal in the theorem of γ_I. We also give the relationship between bipolar fuzzy ideal and bipolar fuzzy completely prime ideal in LA-rings in the second from the last theorem. In this section we use the notation of R which is mean LA-ring.

Definition 3.1. A bipolar fuzzy subset $B = (f_B^+, f_B^-)$ of R is called a bipolar fuzzy LA-subring of R if for all $x, y \in R$ if for all $x, y \in R$

1. $f_B^+(x-y) \geq \min\{f_B^+(x), f_B^+(y)\}$
2. $f_B^+(xy) \geq \min\{f_B^+(x), f_B^+(y)\}$
3. $f_B^-(x-y) \geq \max\{f_B^-(x), f_B^-(y)\}$
4. $f_B^-(xy) \geq \max\{f_B^-(x), f_B^-(y)\}$

Theorem 3.2. Let $(R_1, +, .)$ and $(R_2, \oplus, \odot)$ be LA-rings. If B and C are bipolar fuzzy LA-subrings of R_1 and R_2 respectively then $B \times C$ is an bipolar fuzzy LA-subring of $(R_1 \times R_2, \boxplus, \boxdot)$.

Proof. Let (x_1, y_1) and $(x_2, y_2) \in R_1 \times R_2$. Since B and C are bipolar fuzzy LA-subrings of R_1 and R_2 respectively then we have $f_B^+(x_1.x_2) \geq \min\{f_B^+(x_1), f_B^+(x_2)\}, f_B^-(x_1x_2) \leq \max\{f_B^+(x_1), f_B^+(x_2)\}, f_B^+(x_1 - x_2) \geq \min\{f_B^+(x_1), f_B^+(x_2)\}$, and $f_B^-(x_1 - x_2) \leq \max\{f_B^+(x_1), f_B^+(x_2)\}$,

and

$$g_C^+(y_1.y_2) \geq \min\{g_C^+(y_1), g_C^+(y_2)\}, g_C^-(y_1.y_2) \leq \max\{g_C^-(y_1), g_C^-(y_2)\},$$

$$g_C^+(y_1 - y_2) \geq \min\{g_C^+(y_1), g_C^+(y_2)\}, g_C^-(y_1 - y_2) \leq \max\{g_C^-(y_1), g_C^-(y_2)\}.$$

Hence

$$\begin{aligned}
(f_B^+ \times (g_C^+)(x_1, y_1)) \boxdot (x_2, y_2) &= (f_B^+ \times g_C^+)(x_1.x_2, y_1 \odot y_2) \\
&= \min\{f_B^+(x_1.x_2), g_C^+(y_1 \odot y_2)\} \\
&\geq \min\{\min\{f_B^+(x_1), f_B^+(x_2)\}, \min\{g_C^+(y_1), g_C^+(y_2)\}\} \\
&= \min\{\min\{f_B^+(x_1), g_C^+(y_1)\}, \min\{f_B^+(x_2), g_C^+(y_2)\}\} \\
&= \min\{(f_B^+ \times g_C^+)(x_1, y_1), (f_B^+ \times g_C^+)(x_2, y_2)\}
\end{aligned}$$

$$\begin{aligned}
(f_B^- \times g_C^-)((x_1,y_1) \boxdot (x_2,y_2)) &= (f_B^- \times g_C^-)(x_1.x_2, y_1 \odot y_2) \\
&\leq \max\{\max\{f_B^-(x_1), f_B^-(x_2)\}, \min\{g_C^-(y_1), g_C^-(y_2)\}\} \\
&= \max\{(f_B^- \times g_C^-)(x_1,y_1), (f_B^- \times g_C^-)(x_2,y_2)\} \\
(f_B^+ \times g_C^+)((x_1,y_1) \boxminus (x_2,y_2)) &= (f_B^+ \times g_C^+)(x_1 - x_2, y_1 \ominus y_2) \\
&\geq \min\{\min\{f_B^+(x_1), f_B^+(x_2)\}, \min\{g_C^+(y_1), g_C^+(y_2)\}\} \\
&= \min\{(f_B^+ \times g_C^+)(x_1,y_1), (f_B^+ \times g_C^+)(x_2,y_2)\} \\
(f_B^- \times g_C^-)((x_1,y_1) \boxminus (x_2,y_2)) &= (f_B^- \times g_C^-)(x_1 - x_2, y_1 \ominus y_2) \\
&\leq \max\{\max\{f_B^-(x_1), f_B^-(x_2)\}, \min\{g_C^-(y_1), g_C^-(y_2)\}\} \\
&= \max\{(f_B^- \times g_C^-)(x_1,y_1), (f_B^- \times g_C^-)(x_2,y_2)\}
\end{aligned}$$

Thus, $B \times C$ is a bipolar fuzzy LA-subring of $(R_1 \times R_2, \boxplus, \boxdot)$.

Corollary 3.3. Let $A_1, A_2, ..., A_n$ be bipolar fuzzy LA-subrings of $R_1, R_2, ..., R_n$ respectively then $\prod_{i=1}^{n} A_i$ is a bipolar fuzzy LA-subring of $\prod_{i=1}^{n} R_i$.

Proof. It can be proved analog to Theorem 3.2.

Definition 3.4. A bipolar fuzzy subset $B = (f_B^+, f_B^-)$ of R is called a bipolar fuzzy left (right) ideal of R if for all $x, y \in R$

1. B is a bipolar fuzzy LA-subring of R
2. $f_B^+(xy) \geq f_B^+(y) (f_B^+(xy) \geq f_B^+(x))$
3. $f_B^-(xy) \geq f_B^-(y) (f_B^-(xy) \geq f_B^-(x))$

Definition 3.5. A bipolar fuzzy subset $B = (f_B^+, f_B^-)$ of R is called a bipolar fuzzy ideal of R if for all $x, y \in R$

1. B is a bipolar fuzzy LA-subring of R
2. $f_B^+(xy) \geq \max\{f_B^+(x), f_B^+(y)\}$
3. $f_B^-(xy) \geq \min\{f_B^-(x), f_B^-(y)\}$

Example 3.6. A bipolar fuzzy subset $B = (f_B^+, f_B^-)$ defined as follows is a bipolar fuzzy ideal of R_1

$$f_B^+(x) = \begin{cases} 0.8, \text{ if } x = 0 \\ 0.5, \text{ if } x \in \{1,2,3,4\} \end{cases}$$

and

$$f_B^-(x) = \begin{cases} -0.4, \text{ if } x = 0 \\ -0.1, \text{ if } x \in \{1,2,3,4\} \end{cases}$$

Example 3.7. A bipolar fuzzy subset $B = (f_A^+, f_A^-)$ defined as follows is a bipolar fuzzy ideal of R_2

$$f_A^+(x) = \begin{cases} 1, \text{ if } x = 0 \\ 0.8, \text{ if } x \in \{2,4\} \\ 0.5 \text{ if } x \in \{1,3,5\} \end{cases}$$

and

$$f_A^-(x) = \begin{cases} -0.8, \text{ if } x = 0 \\ -0.5, \text{ if } x \in \{2,4\} \\ -0.2 \text{ if } x \in \{1,3,5\} \end{cases}$$

Lemma 3.8. If $B = (f_B^+, f_B^-)$ and $C = (g_C^+, g_C^-)$ are bipolar fuzzy left ideals (right ideals, ideals) of $(R_1, +, .)$ and $(R_2, \oplus, \odot)$ respectively then $B \times C$ is a bipolar fuzzy left ideals (right ideals, ideals) of $(R_1 \times R_2, \boxplus, \boxdot)$.

Proof. By Theorem 3.2., we have $B \times C$ is a bipolar fuzzy LA-subring of $(R_1 \times R_2$. Since B and C are bipolar fuzzy ideals then

$$\begin{aligned}(f_B^+ \times g_C^+)((x_1,y_1) \boxdot (x_2,y_2)) &= (f_B^+ \times g_C^+)(x_1.x_2, y_1 \odot y_2) \\ &= \min\{f_B^+(x_1.x_2), g_C^+(y_1 \odot y_2)\} \\ &\geq \min\{f_B^+(x_2), g_C^+(y_2)\} \\ &= (f_B^+ \times g_C^+)(x_2,y_2)\end{aligned}$$

$$\begin{aligned}(f_B^+ \times g_C^+)((x_1,y_1) \boxdot (x_2,y_2)) &= (f_B^+ \times g_C^+)(x_1.x_2, y_1 \odot y_2) \\ &= \min\{f_B^+(x_1.x_2), g_C^+(y_1 \odot y_2)\} \\ &\geq \min\{f_B^+(x_1), g_C^+(y_1)\} \\ &= (f_B^+ \times g_C^+)(x_1,y_1)\end{aligned}$$

Since $(f_B^+ \times g_C^+)((x_1,y_1) \boxdot (x_2,y_2)) \geq (f_B^+ \times g_C^+)(x_2,y_2)$ and $(f_B^+ \times g_C^+)((x_1,y_1) \boxdot (x_2,y_2)) \geq (f_B^+ \times g_C^+)(x_1,y_1)$ then $(f_B^+ \times g_C^+)((x_1,y_1) \boxdot (x_2,y_2)) \geq \max\{(f_B^+ \times g_C^+)(x_1,y_1), (f_B^+ \times g_C^+)(x_2,y_2)\}$. And we have

$$\begin{aligned}(f_B^+ \times g_C^+)((x_1,y_1) \boxdot (x_2,y_2)) &= (f_B^+ \times g_C^+)(x_1.x_2, y_1 \odot y_2) \\ &= \max\{f_B^-(x_1.x_2), g_C^-(y_1 \odot y_2)\} \\ &\geq \min\{f_B^+(x_2), g_C^+(y_2)\} \\ &= (f_B^- \times g_C^-)(x_2,y_2)\end{aligned}$$

$$\begin{aligned}(f_B^+ \times g_C^+)((x_1,y_1) \boxdot (x_2,y_2)) &= (f_B^+ \times g_C^+)(x_1.x_2, y_1 \odot y_2) \\ &= \max\{f_B^-(x_1.x_2), g_C^-(y_1 \odot y_2)\} \\ &\geq \min\{f_B^-(x_1), g_C^+(y_1)\} \\ &= (f_B^- \times g_C^-)(x_1,y_1)\end{aligned}$$

Since $(f_B^- \times g_C^-)((x_1,y_1) \boxdot (x_2,y_2)) \leq (f_B^- \times g_C^-)(x_2,y_2)$ and $(f_B^- \times g_C^-)((x_1,y_1) \boxdot (x_2,y_2)) \leq (f_B^- \times g_C^-)(x_1,y_1)$ then $(f_B^- \times g_C^-)((x_1,y_1) \boxdot (x_2,y_2)) \leq \min\{(f_B^- \times g_C^-)(x_1,y_1), (f_B^- \times g_C^-)(x_2,y_2)\}$. Thus, $B \times C$ is a bipolar fuzzy ideal of $R_1 \times R_2$. Based on the steps above, it follows that if B and C are bipolar fuzzy left (right) ideals then $B \times C$ is a bipolar fuzzy left (right) ideal of $R_1 \times R_2$.

Example 3.9. If B is a bipolar fuzzy ideal as defined in the Example 3.6 and A is a bipolar fuzzy ideal as defined in the Example 3.7 then $B \times A$ is a bipolar fuzzy ideal of $R_1 \times R_2$.

Corollary 3.10. If $A_1, A_2, ..., A_n$ are bipolar fuzzy left ideals (right ideals, ideals) of $R_1, R_2, ..., R_n$ respectively then

$\prod_{i=1}^{n} A_i$ is a bipolar fuzzy left ideal (right ideal, ideal) of $\prod_{i=1}^{n} R_i$.

Proof. It can be proved analog to Lemma 3.8.

Theorem 3.11. If I is a non empty subset of R and0 $\gamma_I = (f^+_{\gamma_I}, f^-_{\gamma_I})$ defined as follows is a bipolar fuzzy subset of R, where

$$f^+_{\gamma_I}(x) = \begin{cases} m^+ \in (0,1], \text{ if } x \in I \\ 0, \quad \text{if } x \notin I \end{cases}$$

and

$$f^-_{\gamma_I}(x) = \begin{cases} m^- \in [-1,0), \text{ if } x \in I \\ 0, \quad \text{if } x \notin I \end{cases}$$

Then the following properties hold.

1. I is an LA-subring of R if and only if γ_I is a bipolar fuzzy LA-subring of R
2. I is a left ideal of R if and only if γ_I is a bipolar fuzzy left ideal of R
3. I is a right ideal of R if and only if γ_I is a bipolar fuzzy right ideal of R
4. I is an ideal of R if and only if γ_I is a bipolar fuzzy ideal of R

Proof.

1. $(\Rightarrow)$ Let $x, y \in R$.
   - If $x \in I$ or $y \in I$ then we have $f^+_{\gamma_I}(x) = 0$ and $f^-_{\gamma_I}(x) = 0$ or $f^+_{\gamma_I}(y) = 0$ and $f^-_{\gamma_I}(y) = 0$, this implies $f^+_{\gamma_I}(xy) \geq 0 = \min\{f^+_{\gamma_I}(x), f^+_{\gamma_I}(y)\}, f^-_{\gamma_I}(xy) \leq 0 = \max\{f^-_{\gamma_I}(x), f^-_{\gamma_I}(y)\}, f^+_{\gamma_I}(x-y) \geq 0 = \min\{f^+_{\gamma_I}(x), f^+_{\gamma_I}(y)\}$, and $f^-_{\gamma_I}(x-y) \leq 0 = \max\{f^-_{\gamma_I}(x), f^-_{\gamma_I}(y)\}$.
   - If $x, y \in I$ then we have $xy \in I$ and $x - y \in I$. Thus $f^+_{\gamma_I}(xy) = m^+ = \min\{f^+_{\gamma_I}(x), f^+_{\gamma_I}(y)\}, f^-_{\gamma_I}(xy) = m^- = \max\{f^-_{\gamma_I}(x), f^-_{\gamma_I}(y)\}, f^+_{\gamma_I}(x-y) = m^+ = \min\{f^+_{\gamma_I}(x), f^+_{\gamma_I}(y)\}$, and $f^-_{\gamma_I}(x-y) = m^- = \max\{f^-_{\gamma_I}(x), f^-_{\gamma_I}(y)\}$.
   - If $x, y \notin I$ then we have the same results as if $x \in I$ or $y \in I$.

   Thus for all $x, y \in R$. $f^+_{\gamma_I}(xy) \geq \min\{f^+_{\gamma_I}(x), f^+_{\gamma_I}(y)\}, f^-_{\gamma_I}(xy) \leq \max\{f^-_{\gamma_I}(x), f^-_{\gamma_I}(y)\}, f^+_{\gamma_I}(x-y) \geq \min\{f^+_{\gamma_I}(x), f^+_{\gamma_I}(y)\}$, and $f^-_{\gamma_I}(x-y) \leq \max\{f^-_{\gamma_I}(x), f^-_{\gamma_I}(y)\}$. Hence γ_I is a bipolar fuzzy LA-subring of R.
   $(\Leftarrow)$ Let $x, y \in I$, then we have $f^+_{\gamma_I}(xy) \geq \min\{f^+_{\gamma_I}(x), f^+_{\gamma_I}(y)\} = m^+, f^-_{\gamma_I}(xy) \leq \max\{f^-_{\gamma_I}(x), f^-_{\gamma_I}(y)\} = m^-, f^+_{\gamma_I}(x-y) \geq \min\{f^+_{\gamma_I}(x), f^+_{\gamma_I}(y)\} = m^+$, and $f^-_{\gamma_I}(x-y) \leq \max\{f^-_{\gamma_I}(x), f^-_{\gamma_I}(y)\} = m^-$. Thus $f^+_{\gamma_I}(xy) = m^+, f^-_{\gamma_I}(xy) = m^-, f^+_{\gamma_I}(x-y) = m^+, f^-_{\gamma_I}(x-y) = m^-$, which implies $xy \in I$ and $x - y \in I$. Hence I is an LA-subring of R.
2. $(\Rightarrow)$ Since I is a left ideal of R then I is an LA-subring of R. Thus by point 1 implies that γ_I is a bipolar fuzzy LA-subring of R. Let $x, y \in R$.
   - if $y \notin I$ then $f^+_{\gamma_I}(y) = 0, f^-_{\gamma_I}(y) = 0, f^+_{\gamma_I}(xy)0 = f^+_{\gamma_I}(y)$, and $f^-_{\gamma_I}(xy) \leq 0 = f^-_{\gamma_I}(y)$.
   - If $y \in I$ then $xy \in I$, which implies that $f^+_{\gamma_I}(xy) = m^+ = f^+_{\gamma_I}(y), f^-_{\gamma_I}(xy) = m^- = f^-_{\gamma_I}(y)$. Since for all $x, y \in R$, $f^+_{\gamma_I}(xy) \geq f^+_{\gamma_I}(y)$ and $f^-_{\gamma_I}(xy) \leq f^-_{\gamma_I}(y)$ then γ_I is a bipolar fuzzy left ideal of R.

($\Leftarrow$) Since γ_I is a bipolar fuzzy left ideal of R then γ_I is a bipolar fuzzy LA-subring of R and by point 1 implies that I is an LA-subring of R. Let $x \in R, y \in I$. Since $y \in I$ then $f^+_{\gamma_I}(y) = m^+$ and $f^-_{\gamma_I}(y) = m^-$, which imply that $f^+_{\gamma_I}(xy) \geq f^+_{\gamma_I}(y) = m^+$ and $f^-_{\gamma_I}(xy) \leq f^-_{\gamma_I}(y) = m^-$. Then we have $f^+_{\gamma_I}(xy) = m^+$ and $f^-_{\gamma_I}(xy) = m^-$. So $xy \in I$. Since I is an LA-subring of R and for all $x \in R, y \in I$ implies $xy \in I$, then I is a left ideal of R.

3. ($\Rightarrow$) Let $x, y \in R$.
   - If $x \notin I$ then we have $f^+_{\gamma_I}(x) = 0, f^-_{\gamma_I}(x) = 0$, and $f^+_{\gamma_I}(xy) \geq 0 = f^+_{\gamma_I}(x), f^-_{\gamma_I}(xy) \leq 0 = f^-_{\gamma_I}(x)$.
   - If $x \in I$ then $xy \in I$, so we have $f^+_{\gamma_I}(xy) = m^+ = f^+_{\gamma_I}(x), f^-_{\gamma_I}(xy) = m^- = f^-_{\gamma_I}(x)$. Thus for all $x, y \in R$, $f^+_{\gamma_I}(xy) \geq f^+_{\gamma_I}(x)$ and $f^-_{\gamma_I}(xy) \leq f^-_{\gamma_I}(x)$. Hence γ_I is a bipolar fuzzy right ideal of R.
   ($\Leftarrow$) Let $y \in R, x \in I$. Since $x \in I$ then $f^+_{\gamma_I}(x) = m^+$ and $f^-_{\gamma_I}(x) = m^-$, so we have $f^+_{\gamma_I}(xy) \geq f^+_{\gamma_I}(x) = m^+$ and $f^-_{\gamma_I}(xy) \leq f^-_{\gamma_I}(x) = m^-$. Thus $f^+_{\gamma_I}(xy) = m^+$ and $f^-_{\gamma_I}(xy) = m^-$. Hence $xy \in I$.

4. ($\Rightarrow$) Since I is an ideal of R then I is left and right ideal of R. By point 2 and 3 it implies that γ_I is a bipolar fuzzy ideal of R.
   ($\Leftarrow$) Since γ_I is a bipolar fuzzy ideal of R then γ_I is a bipolar fuzzy left and right ideal of R. By point 2 and 3, we have I is an ideal of R.

Example 3.12. If $I = \{0, 2, 4\}$ is a non empty subset of R_2 and $\gamma_I = (f^+_{\gamma_I}, f^-_{\gamma_I})$ is a bipolar fuzzy ideal of R_2 that defined as follows, then by Theorem 3.11 we have I as an ideal of R_2.

$$f^+_{\gamma_I}(x) = \begin{cases} 0.5, \text{ if } x \in I \\ 0, \quad \text{if } x \notin I \end{cases}$$

and

$$f^-_{\gamma_I}(x) = \begin{cases} -0.8, \text{ if } x \in I \\ 0, \quad \text{if } x \notin I \end{cases}$$

Definition 3.13. If $M \neq \emptyset \subseteq R$ and $t' = (t^+, t^-) \in (0, 1] \times [-1, 0)$ then $t'A_M = (t^+A_M, t^-A_M)$ defined as follows is a bipolar fuzzy subset of R

$$t^+A_M(x) = \begin{cases} t^+ \in (0, 1], \text{ if } x \in M \\ 0, \quad \text{if } otherwise \end{cases}$$

$$t^-A_M(x) = \begin{cases} t^- \in [-1, 0) \text{ if } x \in M \\ 0, \quad \text{if } otherwise \end{cases}$$

Lemma 3.14. If M is a left ideal (right ideal, ideal) of R then $t'A_M$ is a bipolar fuzzy left ideal (right ideal, ideal) of R.

Proof. Let M be a left ideal and $x, y \in R$.

- If $x \in M$ and $y \in M$, then $t^+A_M(x - y) \geq 0 = min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^+A_M(xy) \geq 0 = \min\{t^+A_M(x), t^+A_M(y)\}$,

$t^-A_M(x-y) \leq 0 = \max\{t^-A_M(x), t^-A_M(y)\}$, and
$t^-A_M(xy) \leq 0 = \max\{t^-A_M(x), t^-A_M(y)\}$.
Since M is a left ideal then $xy \in M$, so we have $t^+A_M(xy) = t^+A_M(y)$ and $t^-A_M(xy) = t^-A_M(y)$.

- If $x \in M$ and $y \in M$, then $t^+A_M(x-y) \geq 0 = \min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^+A_M(xy) \geq 0 = \min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^-A_M(x-y) \leq 0 = \max\{t^-A_M(x), t^-A_M(y)\}$,
  $t^-A_M(xy) \leq 0 = \max\{t^-A_M(x), t^-A_M(y)\}, t^+A_M(xy) \geq 0 = t^+A_M(y)$, and $t^-A_M(xy) \leq 0 = t^-A_M(y)$.
- If $x, y \in M$, then we have the same results as if $x \in M$ and $y \in M$.
- If $x, y \in M$ then we have $t^+A_M(x-y) = t^+ = \min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^+A_M(xy) = t^+ = \min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^-A_M(x-y) = t^- = \max\{t^-A_M(x), t^-A_M(y)\}$,
  $t^-A_M(xy) = t^- = \max\{t^-A_M(x), t^-A_M(y)\}, t^+A_M(xy) = t^+ = t^+A_M(y)$, and $t^-A_M(xy) = t^- = t^-A_M(y)$.

Hence for all $x, y \in R, t'A_M$ is a bipolar fuzzy left ideal of R.
If M is a right ideal then we will proof that $t'A_M$ is a bipolar fuzzy right ideal of R. Take any $x, y \in R$.

- If $x \in M$ and $y \in M$, then $t^+A_M(x-y) \geq 0 = \min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^+A_M(xy) \geq 0 = \min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^-A_M(x-y) \leq 0 = \max\{t^-A_M(x), t^-A_M(y)\}$,
  $t^-A_M(xy) \leq 0 = \max\{t^-A_M(x), t^-A_M(y)\}, t^+A_M(xy) \geq 0 = t^+A_M(x)$, and $t^-A_M(xy) \leq 0 = t^-A_M(x)$.
- If $x \in M$ and $y \in M$, then $t^+A_M(x-y) \geq 0 = \min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^+A_M(xy) \geq 0 = \min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^-A_M(x-y) \leq 0 = \max\{t^-A_M(x), t^-A_M(y)\}$,
  $t^-A_M(xy) \leq 0 = \max\{t^-A_M(x), t^-A_M(y)\}$.
  Since M is a right ideal then $xy \in M$. Thus $t^+A_M(xy) = t^+ = t^+A_M(x)$ and $t^-A_M(xy) = t^- = t^-A_M(x)$.
- If $x, y \in M$, then we have the same results as if $x \in M$ and $y \in M$.
- If $x, y \in M$ then we have $t^+A_M(x-y) = t^+ = \min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^+A_M(xy) = t^+ = \min\{t^+A_M(x), t^+A_M(y)\}$,
  $t^-A_M(x-y) = t^- = \max\{t^-A_M(x), t^-A_M(y)\}$,
  $t^-A_M(xy) = t^- = \max\{t^-A_M(x), t^-A_M(y)\}, t^+A_M(xy) = t^+ = t^+A_M(x)$, and $t^-A_M(xy) = t^- = t^-A_M(x)$.

Thus for all $x, y \in R, t'A_M$ is a bipolar fuzzy right ideal of R.
If M is an ideal of R then M is left and right ideal thus implies that $t'A_M$ is a bipolar fuzzy ideal of R.

Definition 3.16. A bipolar fuzzy subset $A = (f_A^+, f_A^-)$ of R is called a bipolar fuzzy completely prime ideal if for all $x, y \in R$

1. $\max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(xy)$
2. $\min\{f_A^-(x), f_A^-(y)\} \leq f_A^-(xy)$

3. $\max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(x-y)$
4. $\min\{f_A^-(x), f_A^-(y)\} \leq f_A^-(x-y)$

Definition 3.17. A bipolar fuzzy subset $A = (f_A^+, f_A^-)$ of R is called a weakly bipolar fuzzy completely prime ideal if for all $x, y \in R \ni xy \neq 0$

1. $\max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(xy)$
2. $\min\{f_A^-(x), f_A^-(y)\} \leq f_A^-(xy)$
3. $\max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(x-y)$
4. $\min\{f_A^-(x), f_A^-(y)\} \leq f_A^-(x-y)$

Example 3.18. A bipolar fuzzy subset $G = (f_G^+, f_G^-)$ defined as follows is a bipolar fuzzy completely prime ideal and weakly bipolar fuzzy completely prime ideal of R_1

$$f_G^+(x) = \begin{cases} 0.4, \text{ if } x = 0 \\ 0.7, \text{ if } x \in \{1,2,3,4\} \end{cases}$$

$$f_G^-(x) = \begin{cases} -0.1, \text{ if } x = 0 \\ -0.5, \text{ if } x \in \{1,2,3,4\} \end{cases}$$

Lemma 3.19. A bipolar fuzzy ideal $A = (f_A^+, f_A^-)$ of R is called a bipolar fuzzy completely prime ideal if and only if the following properties hold for all $x, y \in R$

1. $\max\{f_A^+(x), f_A^+(y)\} = f_A^+(xy)$
2. $\min\{f_A^-(x), f_A^-(y)\} = f_A^-(xy)$
3. $\max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(x-y)$
4. $\min\{f_A^-(x), f_A^-(y)\} \leq f_A^-(x-y)$

Proof. ($\Rightarrow$) Since A is a bipolar fuzzy ideal of R, then we have

$$f_A^+(xy) \geq \max\{f_A^+(x), f_A^+(y)\} \text{ and } f_A^-(xy) \leq \min\{f_A^-(x), f_A^-(y)\} \tag{3.1}$$

If A is a bipolar fuzzy completely prime ideal, then we have

$$\max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(xy) \text{ and } \min\{f_A^-(x), f_A^-(y)\} \leq f_A^-(xy) \tag{3.2}$$

$$\max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(x-y) \text{ and } \min\{f_A^-(x), f_A^-(y)\} \leq f_A^-(x-y) \tag{3.3}$$

From (3.1), (3.2), and (3.3) we have $\max\{f_A^+(x), f_A^+(y)\} = f_A^+(xy), \min\{f_A^-(x), f_A^-(y)\} = f_A^-(xy), \max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(x-y)$, and $\min\{f_A^-(x), f_A^-(y)\} \leq f_A^-(x-y)$.
($\Leftarrow$) Since $\max\{f_A^+(x), f_A^+(y)\} = f_A^+(xy)$ and $\min\{f_A^-(x), f_A^-(y)\} = f_A^-(xy)$, then
$max\{f_A^+(x), f_A^+(y)\} \leq f_A^+(xy), \max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(xy), \min\{f_A^-(x), f_A^-(y)\} \geq f_A^-(xy)$,
and $\min\{f_A^-(x), f_A^-(y)\} \geq f_A^-(xy)$.
Since $\max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(xy), \min\{f_A^-(x), f_A^-(y)\} \leq f_A^-(xy), \max\{f_A^+(x), f_A^+(y)\} \geq f_A^+(x-y)$, and $\min\{f_A^-(x), f_A^-(y)\} \leq$

$f_A^-(x-y)$ then A is a bipolar fuzzy completely prime ideal.

Theorem 3.20. Let R be an LA-ring and β be an index. If bipolar fuzzy subsets $P_i = (f^+_{i_{(P_i)}}, f^-_{i_{(P_i)}}), i \in \beta$ are bipolar fuzzy completely prime (weakly bipolar fuzzy completely prime) ideals of R then $\bigcup_{i\in\beta} P_i = (\bigcup_{i\in\beta} f^+_{i_{(P_i)}}, \bigcap_{i\in\beta} f^-_{i_{(P_i)}})$ is a bipolar fuzzy completely prime (weakly bipolar fuzzy completely prime) ideal of R.

Proof. Let bipolar fuzzy subsets $P_i = (f^+_{i_{(P_i)}}, f^-_{i_{(P_i)}}), i \in \beta$ are bipolar fuzzy completely prime (weakly bipolar fuzzy completely prime) ideals of R then for all $x, y \in R, i \in \beta$ ($xy \neq 0$ for weakly bipolar fuzzy completely prime ideal) we have $\max\{f^+_{i_{(P_i)}}(x), f^+_{i_{(P_i)}}(y)\} \geq f^+_{i_{(P_i)}}(xy), \min\{f^-_{i_{(P_i)}}(x), f^-_{i_{(P_i)}}(y)\} \leq f^-_{i_{(P_i)}}(xy)$,
$\max\{f^+_{i_{(P_i)}}(x), f^+_{i_{(P_i)}}(y)\} \geq f^+_{i_{(P_i)}}(x-y)$, and $\min\{f^-_{i_{(P_i)}}(x), f^-_{i_{(P_i)}}(y)\} \leq f^-_{i_{(P_i)}}(x-y)$.
If $\max\{f^+_{i_{(P_i)}}(x), f^+_{i_{(P_i)}}(y)\} \geq f^+_{i_{(P_i)}}(xy)$ and $\min\{f^-_{i_{(P_i)}}(x), f^-_{i_{(P_i)}}(y)\} \leq f^-_{i_{(P_i)}}(xy)$ then $\max\{\bigcup_{i\in\beta} f^+_{i_{(P_i)}}(x), \bigcup_{i\in\beta} f^+_{i_{(P_i)}}(y)\} \geq f^+_{i_{(P_i)}}(xy)$, and $\min\{\bigcap_{i\in\beta} f^-_{i_{(P_i)}}(x), \bigcap_{i\in\beta} f^-_{i_{(P_i)}}(y)\} \leq f^-_{i_{(P_i)}}(xy)$
Thus for all $i \in \beta$ we have
$\max\{\bigcup_{i\in\beta} f^+_{i_{(P_i)}}(x), \bigcup_{i\in\beta} f^+_{i_{(P_i)}}(y)\} \geq \bigcup_{i\in\beta} f^+_{i_{(P_i)}}(xy)$ and $\min\{\bigcap_{i\in\beta} f^-_{i_{(P_i)}}(x), \bigcap_{i\in\beta} f^-_{i_{(P_i)}}(y)\} \leq \bigcap_{i\in\beta} f^-_{i_{(P_i)}}(xy)$.
Since $\max\{f^+_{i_{(P_i)}}(x), f^+_{i_{(P_i)}}(y)\} \geq f^+_{i_{(P_i)}}(x-y)$ and $\min\{f^-_{i_{(P_i)}}(x), f^-_{i_{(P_i)}}(y)\} \leq f^-_{i_{(P_i)}}(x-y)$ then $\max\{\bigcup_{i\in\beta} f^+_{i_{(P_i)}}(x), \bigcup_{i\in\beta} f^+_{i_{(P_i)}}(y)\} \geq f^+_{i_{(P_i)}}(x-y)$ and $\min\{\bigcap_{i\in\beta} f^-_{i_{(P_i)}}(x), \bigcap_{i\in\beta} f^-_{i_{(P_i)}}(y)\} \leq f^-_{i_{(P_i)}}(x-y)$.
Thus, $\max\{\bigcup_{i\in\beta} f^+_{i_{(P_i)}}(x), \bigcup_{i\in\beta} f^+_{i_{(P_i)}}(y)\} \geq \bigcup_{i\in\beta} f^+_{i_{(P_i)}}(x-y)$ and $\min\{\bigcap_{i\in\beta} f^-_{i_{(P_i)}}(x), \bigcap_{i\in\beta} f^-_{i_{(P_i)}}(y)\} \leq \bigcap_{i\in\beta} f^-_{i_{(P_i)}}(x-y)$.
So it prove that $\bigcup_{i\in\beta} f_{i_{(P_i)}}$ is a bipolar fuzzy completely prime (weakly bipolar fuzzy completely prime) ideal of R.

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