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Further notions related to new operators and compactness via supra soft topological spaces

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ABSTRACT. The first objective of this paper, is to study the properties of supra soft limit points with the help of a soft point notion and to introduce the notions of supra soft boundary and supra soft closure operators. With regard to these notions, we describe a supra soft closure operator in terms of supra soft limit points; and generate a unique supra soft topology by using a specific soft mapping. Also, we define relative supra soft topologies and deduce some results which connect supra soft closure and supra soft interior operators via the original supra soft topologies and their relative supra soft topologies. The second objective, is to present the concepts of supra soft compact (supra soft Lindelöf), almost supra soft compact (almost supra soft Lindelöf) and mildly supra soft compact (mildly supra soft Lindelöf) spaces in supra soft topological spaces; and to investigate their main features. Throughout this investigation, we point out the relationships among these concepts and characterize each one of them. Also, we explore the notions of soft $S^{\star}$-continuous mappings and prove that the image of the kinds of supra soft compact and supra soft Lindelöf spaces are preserved under soft $S^{\star}$-continuous mappings.

1. Introduction

Molodtsov [21] introduced a concept of soft set as a new approach for dealing with problems containing uncertainties; and discussed its applications in different directions like game theory, smoothness of functions, Riemann Integration, and so on. In 2011, Shabir and Naz [25] utilizing a notion of soft sets to initiate soft topological spaces. They defined and studied basic notions such as soft interior and closure operators and soft separation

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axioms. Aygünoğlu and Aygün [13] defined soft compactness and pointed out that the soft continuous image of soft compact set is soft compact. Zorlutuna et al. [27] introduced the first version of a soft point notion. They also described soft compact spaces in terms of the finite intersection property. Together the authors of [14, 23] gave another version of a soft point notion. This version eased studying some topics via soft topological spaces such as soft metric spaces, soft neighborhood systems and soft limit points. The authors of [8, 10, 16] carried out some studies in order to correct some alleged results related to soft separation axioms. Recently, Al-shami et al.[11] defined new types of soft compact spaces, namely almost soft compact (almost soft Lindelöf), approximately soft Lindelöf and mildly soft compact (mildly soft Lindelöf) spaces.

To study a supra topology concept [20] via the soft set theory, El-Sheikh and Abd El-Latif [18], in 2014, initiated a concept of supra soft topological spaces which is wider and more general than the class of soft topological spaces. They defined γ-operator to introduce several types of generalized supra soft open sets. Abd El-Latif [1] presented and studied supra soft β-separation axioms; and Abd El-Latif and Hosny [2] defined new supra soft separation axioms by using a notion of soft points. Al-shami [4, 6, 7] formulated new kinds of supra compact, supra α-compact and supra semi-compact spaces.

As a continuation of study supra soft topological spaces, it is natural to investigate certain of main concepts in supra soft topological spaces. To this end, we study in the present work, the concepts of supra soft limit, supra soft boundary and supra soft closure operator. We investigate several results which connect among them and show that a supra soft closure operator generate a unique supra soft topology by using a specific soft mapping. In the last section of this work, we establish three sorts of compactness via supra soft topological spaces, namely supra soft compact, almost supra soft compact and mildly supra soft compact; and establish three sorts of Lindelöfness via supra soft topological spaces, namely supra soft Lindelöf, almost supra soft Lindelöf and mildly supra soft Lindelöf spaces. With the help of examples, we elucidate the relationships among these concepts. In addition, we deduce some findings which connect these sorts of compactness and Lindelöfness via supra topological spaces and their parametric supra topological spaces.

The paper is organized as follows: In section 2, some definitions and results on the soft sets theory and supra soft topologies are given. The properties of supra soft limit, supra soft boundary and supra soft closure operators are investigated in section 3. In section 4, The concepts of supra soft compact (supra soft Lindelöf), supra almost soft compact (almost soft Lindelöf) and supra mildly soft compact (supra mildly soft Lindelöf) spaces are introduced and many properties of them are established. Section 5, concludes the paper.

2. Preliminaries

We present in this section some definitions and results which will be needed in the sequels.

Definition 2.1. [21] A pair (G, E) is called a soft set over X if G is a mapping of E into 2^X.

Definition 2.2. [21] Let (G, E) be a soft set over X. Then we say that:

(i) $x \in (G, E)$ if $x \in G(e)$, for each $e \in E$.

(ii) $x \notin (G, E)$ if $x \notin G(e)$, for some $e \in E$.

Definition 2.3. [17] Let (G, E) be a soft set over X. Then we say that:

(i) $x \Subset (G, E)$ if $x \in G(e)$, for some $e \in E$.

(ii) $x \notin\!\!\Subset (G, E)$ if $x \notin G(e)$, for each $e \in E$.

Definition 2.4. [19] Let (G, A) and (H, B) be soft sets over X. We say that (G, A) is a soft subset of (H, B), denoted by $(G, A)\widetilde{\subseteq}(H, B)$, provided that:

(i) $A \subseteq B$;

(ii) $G(a) \subseteq H(a)$, for each $a \in A$.

Definition 2.5.[3] The relative complement of a soft set (G, E), denoted by $(G, E)^c$, is defined by $(G, E)^c = (G^c, E)$, where $G^c : E \to 2^X$ is a mapping given by $G^c(e) = X - G(e)$, for each $e \in E$.

Definition 2.6. [18] Let μ be a collection of soft sets over X and E be a set of parameters. Then μ is called a supra soft topology on X if the following two axioms are satisfied:

(i) $\widetilde{\Phi}$ and $\widetilde{X}$ belong to μ.

(ii) The union of any number of soft sets in μ belongs to μ.

The triple (X, μ, E) is called a supra soft topological space.
Each member of μ is called a supra soft open set and its relative complement is called a supra soft closed set.
Henceforth (X, μ, E) is indicates to a supra soft topological space.

Definition 2.7.[18] Let (G, E) be a soft subset of a supra soft topological space (X, μ, E). Then P_e^x is called supra soft limit point of (G, E) if $[(F, E)\backslash P_e^x]\widetilde{\bigcap}(G, E) \neq \widetilde{\Phi}$, for each supra soft open set (F, E) containing P_e^x.
All supra soft limit points of (G, E) is said to be supra derived soft set of (G, E) and denoted by $(G, E)^{ss\prime}$.

Definition 2.8.[18] The supra soft closure and supra soft interior of a soft subset of (G, E) of (X, μ, E) are defined, respectively, as follows:

(i) The supra soft closure of (G, E), denoted by $sscl(G, E)$, is the intersection of all supra soft closed sets containing (G, E).

(ii) The supra soft interior of (G, E), denoted by $ssint(G, E)$, is the union of all supra soft open sets contained in (G, E).

Definition 2.9. A soft set (P, E) over X is said to be:

(i) Pseudo constant soft set [24] if $P(e) = X$ or $\emptyset$, for each $e \in E$. The set of all pseudo constant soft sets is denoted by $CS(X, E)$.

(ii) Soft point [14, 23] if there exists $e \in E$ and $x \in X$ such that $P(e) = \{x\}$ and $P(\alpha) = \emptyset$, for each $\alpha \in E \setminus \{e\}$. A soft point will be shortly denoted by P_e^x.

(iii) Countable soft set [14] if $P(e)$ is countable, for each $e \in E$.

(iv) Finite soft set [14] if $P(e)$ is finite, for each $e \in E$..

Definition 2.10.[25] Let $x \in X$. A soft set (x, E) over X is defined as $x(e) = x$, for each $e \in E$.

Definition 2.11. [4] A supra topological spaces (X, μ) is called supra compact (resp. supra Lindelöf) provided that every supra open cover of X has a finite (resp. countable) subcover.

Definition 2.12.[12, 27] A collection $\{(F_i, E) : i \in I\}$ of soft sets is said to have:

(i) The finite intersection property if every finite sub-collection $\{(F_{i_1}, E), (F_{i_2}, E), ..., (F_{i_n}, E) : n \in \mathcal{N}\}$ has a non-null soft intersection.

(ii) The countable intersection property if every countable sub-collection $\{(F_{i_n}, E) : n \in S$ and S is countable$\}$ has a non-null soft intersection.

3. Supra soft limit and supra soft boundary points

The properties of soft limit, soft boundary and soft closure operators are studied in [23], [26] and [25], respectively, but supra soft limit, supra soft boundary and supra soft closure operators are not investigated sufficiently until now. So we allocate this section to point out the main properties of these operators.

Proposition 3.1. *If the soft sets (G, E) and (H, E) are equal, then the following two statements hold.*

(i) $x \in (G, E)$ if and only if $x \in (H, E)$.

(ii) $x \Subset (G, E)$ if and only if $x \Subset (H, E)$.

Proof. Straightforward. □

To see that the converse of the above proposition fails, we construct the following example.

Example 3.2. *Assume that $X = \{x, y\}$ and $E = \{e_1, e_2\}$; and let the two soft sets (G, E) and (H, E) over X be given as follows:*

$(G, E) = \{(e_1, X), (e_2, \{x\})\}$ *and*

$(H, E) = \{(e_1, \{x\}), (e_2, X)\}$. *It can be seen that the two conditions mentioned in the above proposition holds, however $(G, E) \neq (H, E)$.*

Proposition 3.3. *The two soft sets (G, E) and (H, E) are equal if and only if $P_e^x \in (G, E) \Leftrightarrow P_e^x \in (H, E)$.*

Proof. If (G, E) and (H, E) are equal, then $(G, E)\widetilde{\subseteq}(H, E)$ and $(H, E)\widetilde{\subseteq}(G, E)$. So each $P_e^x \in (G, E)$ implies $P_e^x \in (H, E)$ and each $P_e^x \in (H, E)$ implies $P_e^x \in (G, E)$. This finishes the proof of this necessary part.
By taking the inverse steps above, the sufficient part is proved. □

For the sake of brevity, the proofs of the following two propositions have been omitted.

Proposition 3.4. *For two soft sets (G, E) and (H, E), the following statements hold.*

(i) If $P_e^x \in (G, E)$, then $x \notin (G^c, E)$.

(ii) If each $P_e^x \notin (G, E)$ implies that $P_e^x \notin (H, E)$, then $(H, E)\widetilde{\subseteq}(G, E)$.

Proposition 3.5. *If (G, E) and (H, E) are soft subsets of (X, μ, E), then the following statements hold.*

(i) $\widetilde{\Phi}^{ss\prime} = \widetilde{\Phi}$.

(ii) If $(G, E)\widetilde{\subseteq}(H, E)$, then $(G, E)^{ss\prime}\widetilde{\subseteq}(H, E)^{ss\prime}$.

(iii) $[(G, E)\widetilde{\bigcap}(H, E)]^{ss\prime}\widetilde{\subseteq}(G, E)^{ss\prime}\widetilde{\bigcap}(H, E)^{ss\prime}$.

(iv) $(G, E)^{ss\prime}\widetilde{\bigcup}(H, E)^{ss\prime}\widetilde{\subseteq}[(G, E)\widetilde{\bigcup}(H, E)]^{ss\prime}$.

In the following theorem, we formulate a supra soft closure operator in terms of a supra soft limit point.

Theorem 3.6. *Let (F, E) be a soft subset of (X, μ, E). Then:*

(i) (F, E) is supra soft closed if and only if $(F, E)^{ss\prime} \subseteq (F, E)$.

(ii) $(F, E)\widetilde{\bigcup}(F, E)^{ss\prime}$ is a supra soft closed set.

(iii) $sscl(F, E) = (F, E)\widetilde{\bigcup}(F, E)^{ss\prime}$.

Proof.

(i) To prove the "if" part, assume that (F, E) is supra soft closed and $P_e^x \notin (F, E)$. Then $P_e^x \in (F, E)^c \in \mu$. As $(F, E)^c\widetilde{\bigcap}(F, E) = \widetilde{\emptyset}$, then $P_e^x \notin (F, E)^{ss\prime}$. Therefore $(F, E)^{ss\prime}\widetilde{\subseteq}(F, E)$.
To prove the "only if" part, let $P_e^x \in (F, E)^c$ and $(F, E)^{ss\prime}\widetilde{\subseteq}(F, E)$. Then $P_e^x \notin (F, E)^{ss\prime}$. Therefore there exists $(G_x, E) \in \mu$ such that $[(G_x, E) \setminus P_e^x]\widetilde{\bigcap}(F, E) = \widetilde{\emptyset}$. As $P_e^x \in (F, E)^c$, then $(G_x, E)\widetilde{\bigcap}(F, E) = \widetilde{\emptyset}$. Now, $(G_x, E)\widetilde{\subseteq}(F, E)^c$. Therefore $(F, E)^c = \widetilde{\bigcup}\{(G_x, E) : \text{for each } (G_x, E) \in (F, E)^c\}$. Thus $(F, E)^c$ is supra soft open. This completes the proof.

(ii) Let $P_e^x \notin ((F, E)\widetilde{\bigcup}(F, E)^{ss\prime})$. Then $P_e^x \notin (F, E)$ and $P_e^x \notin (F, E)^{ss\prime}$. Therefore there exists $(G, E) \in \mu$ such that

$$(G, E)\widetilde{\bigcap}(F, E) = \widetilde{\emptyset} \tag{3.1}$$

On the other hand, for each $P_e^x \in (G, E)$, we have $P_e^x \notin (F, E)$. Therefore $[(G, E) \setminus P_e^x]\widetilde{\bigcap}(F, E) = \widetilde{\emptyset}$. Thus $P_e^x \notin (F, E)^{ss\prime}$ and this implies that

$$(G, E)\widetilde{\bigcap}(F, E)^{ss\prime} = \widetilde{\emptyset} \tag{3.2}$$

From (1) and (2), we get $(G, E)\widetilde{\bigcap}((F, E)\widetilde{\bigcup}(F, E)^{ss\prime}) = \widetilde{\emptyset}$ and this implies $P_e^x \notin ((F, E)\bigcup(F, E)^{ss\prime})^{ss\prime}$. Hence $((F, E)\widetilde{\bigcup}(F, E)^{ss\prime})^{ss\prime}\widetilde{\subseteq}((F, E)\widetilde{\bigcup}(F, E)^{ss\prime})$. By **(i)**, we obtain $(F, E)\widetilde{\bigcup}(F, E)^{ss\prime}$ is supra soft closed.

(iii) As $(F, E)\widetilde{\subseteq}sscl(F, E)$ and $(F, E)^{ss\prime}\widetilde{\subseteq}(sscl(F, E))^{ss\prime}\widetilde{\subseteq}sscl(F, E)$, then $(F, E)\widetilde{\bigcup}(F, E)^{ss\prime}\widetilde{\subseteq}sscl(F, E)$. Also, $sscl(F, E)$ is the smallest supra soft closed set containing (F, E) and $(F, E)\widetilde{\bigcup}(F, E)^{ss\prime}$ is supra soft closed set containing (F, E). Then $sscl(F, E)\widetilde{\subseteq}(F, E)\widetilde{\bigcup}(F, E)^{ss\prime}$. Therefore $sscl(F, E) = (F, E)\widetilde{\bigcup}(F, E)^{ss\prime}$.

□

Theorem 3.7. *Let (G, E) be a soft subset of (X, μ, E). Then:*

(i) $sscl(\widetilde{\emptyset}) = \widetilde{\emptyset}$.

(ii) $(G,E)\widetilde{\subseteq} sscl(G,E)$.

(iii) $sscl(sscl(G,E)) = sscl(G,E)$.

Proof. The proof is straightforward. □

Theorem 3.8. *Let (H,E) be a soft subset of (X,μ,E) and $P_e^x \in \widetilde{X}$. Then $P_e^x \in sscl(H,E)$ if and only if $(G,E)\widetilde{\bigcap}(H,E) \neq \widetilde{\Phi}$, for each supra soft open set (G,E) containing P_e^x.*

Proof. $[\Rightarrow]$: Let $P_e^x \in sscl(H,E)$. Suppose that there exists a supra soft open set (G,E) containing P_e^x such that $(G,E)\widetilde{\bigcap}(H,E) = \widetilde{\Phi}$. Then $(H,E)\widetilde{\subseteq}(G^c,E)$. So $sscl(H,E)\widetilde{\subseteq}(G^c,E)$. Thus $P_e^x \notin sscl(H,E)$. But this contradicts that $P_e^x \in sscl(H,E)$. This shows that the necessary condition holds.
$[\Leftarrow]$: Let $(G,E)\widetilde{\bigcap}(H,E) \neq \widetilde{\Phi}$, for each supra soft open set (G,E) containing P_e^x. Suppose that $P_e^x \notin sscl(H,E)$. Then there exists a supra soft closed set (F,E) containing (H,E) such that $P_e^x \notin (F,E)$. So $P_e^x \in (F^c,E)$ and $(F^c,E)\widetilde{\bigcap}(H,E) = \widetilde{\Phi}$. But this contradicts the given condition. This shows that the sufficient condition holds. □

Theorem 3.9. *Let T be a map of $SS(X)_E$ into $SS(X)_E$ satisfies the following axioms*

(i) $T(\widetilde{\emptyset}) = \widetilde{\emptyset}$.

(ii) $(A,E)\widetilde{\subseteq}T(A,E)$, *for each* $(A,E)\widetilde{\subseteq}\widetilde{X}$.

(iii) $T(T(A,E)) = T(A,E)$, *for each* $(A,E)\widetilde{\subseteq}\widetilde{X}$.

Then there exists one and only one supra soft topology μ on X such that T will be the supra soft closure of a subset (A,E) of $\widetilde{X}$.

Proof. Let $\xi = \{(F_i,E)\widetilde{\subseteq}\widetilde{X} : T(F_i,E) = (F_i,E)\}$ and $\mu = \{(G_i,E)\widetilde{\subseteq}\widetilde{X} : (G_i,E) = (F_i^c,E)\}$.
First of all, we prove that μ forms a supra soft topology on X as follows: As $T(\widetilde{\emptyset}) = \widetilde{\emptyset}$, then $\widetilde{X} \in \mu$. Also, $\widetilde{X}\widetilde{\subseteq}T(\widetilde{X})\widetilde{\subseteq}\widetilde{X}$ implies that $T(\widetilde{X}) = \widetilde{X}$. Therefore $\widetilde{\emptyset} \in \mu$. Suppose $(G_i,E) \in \mu$, for each $i \in I$, then $T(G_i^c,E) = (G_i^c,E)$, for each $i \in I$. Now, T is a map and $\widetilde{\bigcap}(G_i^c,E)\widetilde{\subseteq}(G_i^c,E)$ implies that $T(\widetilde{\bigcap}(G_i^c,E))\widetilde{\subseteq}T(G_i^c,E) = (G_i^c,E)$. It follows that $T(\widetilde{\bigcap}(G_i^c,E))\widetilde{\subseteq}\widetilde{\bigcap}T(G_i^c,E) = \widetilde{\bigcap}(G_i^c,E) = (\widetilde{\bigcup}(G_i,E))^c$. Consequently, $T(\widetilde{\bigcap}(G_i^c,E))\widetilde{\subseteq}(\widetilde{\bigcup}(G_i,E))^c$. On the other hand, from the second property of T, we get $(\widetilde{\bigcup}(G_i,E))^c = \widetilde{\bigcap}(G_i^c,E)\widetilde{\subseteq}T(\widetilde{\bigcap}(G_i^c,E))$. Then $T((\widetilde{\bigcup}(G_i,E))^c) = (\widetilde{\bigcup}(G_i,E))^c$. Therefore $\widetilde{\bigcup}(G_i,E) \in \mu$. Thus μ is a supra soft topology on X.
Now, we prove that T is a supra soft closure operator with respect to μ as follows:
On the one hand, $T(T(A,E)) = T(A,E)$ implies that $T(A,E)$ is a supra soft closed set containing (A,E). Since $sscl(A,E)$ is the smallest supra soft closed set containing (A,E), then $sscl(A,E)\widetilde{\subseteq}T(A,E)$.
On the other hand, $(A,E)\widetilde{\subseteq}sscl(A,E)$ implies that $T(A,E)\widetilde{\subseteq}T(sscl(A,E))$. Since $sscl(A,E)$ is a supra soft closed set in (X,μ,E), then $T(sscl(A,E)) = sscl(A,E)$. Therefore $T(A,E)\widetilde{\subseteq}sscl(A,E)$. Thus $T(A,E) = sscl(A,E)$. Hence T is the supra soft closure operator with respect to (X,μ,E).
The proof of the uniqueness of μ follows from the corresponding between the supra soft closed sets in (X,μ,E) and the supra soft closed sets in any supra soft topology satisfy the conditions of theorem. □

Definition 3.10. Let (G,E) be a soft subset of (X,μ,E). The supra soft boundary of (G,E), denoted by $ssb(G,E)$, is the set of all soft points sets which belong to $[ssint(G,E)\widetilde{\bigcup}ssint(G^c,E)]^c$.

Theorem 3.11. *Let (G,E) be a soft subset of (X,μ,E). Then:*

(i) $(ssint(G,E))^c = sscl(G^c,E)$.

(ii) $(sscl(G,E))^c = ssint(G^c,E)$.

Proof. **(i)** $(ssint(G,E))^c = \{\bigcup(H,E) : (H,E)$ is a supra soft open set included in $(G,E)\}^c = \bigcap\{(H,E)^c : (H,E)^c$ is a soft closed set including $(G,E)^c\} = scl(G^c,E)$.
By analogy with **(i)**, one can prove **(ii)**. □

Proposition 3.12. *Let (G,E) be a soft subset of (X,μ,E). Then:*

(i) $ssb(G,E) = sscl(G,E) \bigcap sscl(G^c,E)$.

(ii) $ssb(G,E) = sscl(G,E) \setminus ssint(G,E)$.

Proof.

(i)
$$\begin{aligned} ssb(G,E) &= \{P_e^x \in (X,E) : P_e^x \notin ssint(G,E) \text{ and } P_\lambda^x \notin ssint(G^c,E)\} \\ &= \{P_e^x \in (X,E) : P_e^x \notin (sscl(G^c,E))^c \text{ and } P_e^x \notin (sscl(G,E))^c\} \\ &= \{P_e^x \in X : P_e^x \in sscl(G^c,E) \text{ and } P_e^x \in sscl(G,E)\} \\ &= sscl(G,E)\widetilde{\bigcap} sscl(G^c,E) \end{aligned}$$

(ii)
$$\begin{aligned} ssb(G,E) &= sscl(G,E)\widetilde{\bigcap} sscl(G^c,E) \\ &= sscl(G,E)\widetilde{\bigcap}(ssint(G,E))^c \\ &= sscl(G,E) \setminus ssint(G,E). \end{aligned}$$

□

Corollary 3.14. *If (G,E) is a soft subset of (X,μ,E), then*

(i) $ssb(G,E) = ssb(G^c,E)$.

(ii) $ssb(G,E)$ is supra soft closed.

Proposition 3.15. *Let (G,E) be a soft subset of (X,μ,E). Then:*

(i) (G,E) is supra soft open if and only if $ssb(G,E)\widetilde{\bigcap}(G,E) = \widetilde{\emptyset}$.

(ii) (G,E) is supra soft closed if and only if $ssb(G,E)\widetilde{\subseteq}(G,E)$.

Proof.

(i) Necessity: $ssb(G,E)\widetilde{\bigcap}(G,E) = ssb(G,E)\widetilde{\bigcap} ssint(G,E) = \widetilde{\emptyset}$.
Sufficiency: Let $P_e^x \in (G,E)$. Then $P_e^x \in ssint(G,E)$ or $P_e^x \in ssb(G,E)$. As $ssb(G,E)\widetilde{\bigcap}(G,E) = \widetilde{\emptyset}$, then $P_e^x \in ssint(G,E)$. Therefore $(G,E)\widetilde{\subseteq} ssint(G,E)$. Thus (G,E) is supra soft open.

(ii) Necessity: As $ssb(G,E) = sscl(G,E) \setminus ssint(G,E) = (G,E) \setminus ssint(G,E)$, then $ssb(G,E)\widetilde{\subseteq}(G,E)$.
Sufficiency: Let $ssb(G,E)\widetilde{\subseteq}(G,E)$ and $P_e^x \in (G^c,E)$. Then $P_e^x \notin ssb(G,E)$. For each supra soft open set (W,E) with $P_e^x \in (W,E)$, we obtain $(W,E)\widetilde{\bigcap}(G,E) = \widetilde{\emptyset}$. Therefore $P_e^x \in (W,E)\widetilde{\subseteq}(G^c,E)$. Thus (G^c,E) is a union of all supra soft open sets which contain in (G^c,E). Hence (G^c,E) is supra soft open. This completes the proof.

□

Corollary 3.16. *A soft subset (G, E) of (X, μ, E) is both supra soft open and supra soft closed if and only if $ssb(G, E) = \widetilde{\emptyset}$.*

Definition 3.17. Let (X, τ, K) be a supra soft topological space and (A, K) be a non-null soft subset of $\widetilde{X}$. Then $\tau_{(A,K)} = \{(A, K)\widetilde{\bigcap}(G, K) : (G, K) \in \tau\}$ is said to be a relative soft topology on (A, K) and $((A, K), \tau_{(A,K)}, K)$ is called a soft subspace of (X, τ, K).

Proposition 3.18. *Consider $((A, K), \mu_{(A,K)}, K)$ is a soft subspace of (X, μ, K) and let $sscl_A(G, K)$ and $ssint_A(G, K)$ stand for the supra soft closure and supra soft interior operators, respectively, in $((A, K), \mu_{(A,K)}, K)$. Then:*

(i) $sscl_A(G, K) = sscl(G, K)\widetilde{\bigcap}(A, K)$, for each $(G, K)\widetilde{\subseteq}(A, K)$.

(ii) $ssint(G, K)\widetilde{\subseteq}ssint_A(G, K)\widetilde{\bigcap}ssint(A, K)$, for each $(G, K)\widetilde{\subseteq}(A, K)$.

Proof.

(i) Let $P_k^x \in sscl_A(G, K)$. Then for any supra soft open set (D, K) in $\tau_{(A,K)}$ such that $P_k^x \in (D, K)$, we have $(D, K)\widetilde{\bigcap}(G, K) \neq \widetilde{\emptyset}$. If (H, K) is a supra soft open set in τ such that $P_k^x \in (H, K)$, then $(L, K) = (H, K)\widetilde{\bigcap}(A, K)$ is a supra soft open set in $\tau_{(A,K)}$ containing P_k^x and $(L, K)\widetilde{\bigcap}(G, K) \neq \widetilde{\emptyset}$. Obviously, we find that $(H, K)\widetilde{\bigcap}(G, K) \neq \widetilde{\emptyset}$. So $P_k^x \in sscl(G, K)$. Thus $sscl_A(G, K)\widetilde{\subseteq}sscl(G, K)\widetilde{\bigcap}(A, K)$. On the other hand, let $P_k^x \in [sscl(G, K)\widetilde{\bigcap}(A, K)]$ and consider (D, K) is a supra soft open set in $\tau_{(A,K)}$ such that $P_k^x \in (D, K)$. Then $(D, K) = (F, K)\widetilde{\bigcap}(A, K)$, where (F, K) is a supra soft open set in τ. Since $P_k^x \in sscl(G, K)$, then $(F, K)\widetilde{\bigcap}(G, K) \neq \widetilde{\emptyset}$. So $P_k^x \in sscl_A(G, K)$. Thus $sscl(G, K)\widetilde{\bigcap}(A, K)\widetilde{\subseteq}sscl_A(G, K)$. Hence $sscl_A(G, K) = sscl(G, K)\widetilde{\bigcap}(A, K)$.

(ii) Let $P_k^x \in ssint(G, K)\widetilde{\subseteq}ssint(A, K)$. Then there exists a supra soft open set (F, K) in τ such that $P_k^x \in (F, K)\widetilde{\subseteq}(G, K)$. So we find that $(F, K)\widetilde{\bigcap}(G, K)\widetilde{\subseteq}(G, K)$ is a supra soft open set in $\tau_{(A,K)}$. This implies that $P_k^x \in ssint_A(G, K)$. Thus $ssint(G, K)\widetilde{\subseteq}ssint_A(G, K)\widetilde{\bigcap}ssint(A, K)$.

□

Theorem 3.18. *Let $((Y, E), \mu_{(Y,E)}, E)$ be a supra soft subspace of (X, μ, E). Then (H, E) is a supra soft closed subset of $((Y, E), \mu_{(Y,E)}, E)$ if and only if there exists a supra soft closed subset (F, E) of (X, μ, E) such that $(H, E) = (F, E)\widetilde{\bigcap}(Y, E)$.*

Proof. Necessity: Let (H, E) be a supra soft closed subset of $((Y, E), \mu_{(Y,E)}, E)$. Then there is a supra soft open subset (W, E) of $((Y, E), \mu_{(Y,E)}, E)$ such that $(H, E) = (Y, E) \setminus (W, E)$. Now, there is a supra soft open subset (V, E) of (X, μ, E) such that $(W, E) = \widetilde{Y}\widetilde{\bigcap}(V, E)$. Therefore $(H, E) = (Y, E) \setminus ((Y, E)\widetilde{\bigcap}(V, E)) = (Y, E)\widetilde{\bigcap}(V, E)^c$. By taking $(F, E) = (V, E)^c$, the necessary proof is proved.
Sufficiency: Let $(H, E) = (Y, E)\widetilde{\bigcap}(F, E)$ such that (F, E) is a supra soft closed subset of (X, μ, E). Then $(Y, E) \setminus (H, E) = (Y, E) \setminus ((Y, E)\widetilde{\bigcap}(F, E)) = ((Y, E)\widetilde{\bigcap}\widetilde{X}) \setminus ((Y, E)\widetilde{\bigcap}(F, E)) = (Y, E)\widetilde{\bigcap}(\widetilde{X} \setminus (F, E))$. As $\widetilde{X} \setminus (F, E)$ is a supra soft open subset of (X, μ, E), then $(Y, E) \setminus (H, E)$ is a supra soft open subset of $((Y, E), \mu_{(Y,E)}, E)$. Therefore (H, E) is a supra soft closed subset of $((Y, E), \mu_{(Y,E)}, E)$. □

We introduce another definition of a supra soft subspace, which is a special case of Definition 3.17, as follows.

Definition 3.19. Let (X, μ, E) be a supra soft topological space over X and Y be a non-empty subset of X. Then $\mu_Y = \{\widetilde{Y}\widetilde{\bigcap}G_E : G_E \in \mu\}$ is called a supra soft relative topology on Y and (Y, μ_Y, E) is called a supra soft subspace of (X, μ, E).

Remark 3.20. *In [9], the author pointed out that: If (G, E) is a soft open subset of a soft topology (X, τ, E), then $(G, E)\widetilde{\bigcap}scl(H, E)\widetilde{\subseteq}scl[(G, E)\widetilde{\bigcap}(H, E)]$. But in a supra soft topology, if (G, E) is a supra soft open set, then $(G, E) \bigcap sscl(H, E)$ need not be soft subset of $sscl[(G, E) \bigcap (H, E)]$ as the next example shows.*

Example 3.21. *Consider $X = \{x, y, z\}$, $E = \{e_1, e_2\}$ and $\mu = \{\widetilde{\emptyset}, \widetilde{X}, (G, E), (F, E)\}$ be a supra topology on X such that*

$(G, E) = \{(e_1, \{x\}), (e_2, \{x, z\})\}$ *and*

$(F, E) = \{(e_1, \{y, z\}), (e_2, \{x, y\})\}$. *Let a soft set (H, E) be defined as $(G, E) = \{(e_1, \{x\}), (e_2, \{y\})\}$. Then $(G, E)\widetilde{\bigcap}sscl(H, E) = (G, E)$ and $sscl[(G, E)\widetilde{\bigcap}(H, E)] = (F^c, E)$. Therefore $(G, E)\widetilde{\bigcap}sscl(H, E)\widetilde{\not\subseteq}sscl[(G, E)\widetilde{\bigcap}(H, E)]$.*

4. Some types of supra soft compact and supra soft Lindelöf spaces

We devote this section to introducing some notions via supra soft topological spaces, namely supra soft compact (supra soft Lindelöf), almost supra soft compact (almost supra soft Lindelöf) and mildly supra soft compact (mildly supra soft Lindelöf) spaces and deriving their main properties.

4.1. Supra soft compact and supra soft Lindelöf spaces

Definition 4.1.1. A collection $\{(H_i, E) : i \in I\}$ of supra soft open subsets of (X, μ, E) is called supra soft open cover of a soft subset (G, E) of $\widetilde{X}$ provided that $(G, E)\widetilde{\subseteq}\widetilde{\bigcup}\{(H_i, E) : i \in I\}$.

Definition 4.1.2. A supra soft topological spaces (X, μ, E) is called supra soft compact (resp. supra soft Lindelöf) provided that every supra soft open cover of $\widetilde{X}$ has a finite (resp. countable) subcover.

In the following, we give two examples, the first one satisfies a concept of supra soft compact and the second one does not satisfy.

Example 4.1.3. *Let $E = \{e_1, e_2\}$ be a set of parameters and $\mu = \{\widetilde{\emptyset}, (G_i, E)\widetilde{\subseteq}\widetilde{\mathcal{R}} : (G_i^c, E)$ is finite $\}$ be a supra soft topology on $\mathcal{R}$. Then $(\mathcal{R}, \mu, E)$ is supra soft compact.*

Example 4.1.4. *Let $E = \{e_1, e_2\}$ be a set of parameters and $\mu = \{\widetilde{\emptyset}, (G_i, E)\widetilde{\subseteq}\widetilde{\mathcal{Q}}$, where $1 \in G_i(e_1)$ or $2 \in G_i(e_2)\}$ be a supra soft topology on $\mathcal{Q}$. Then $(\mathcal{Q}, \mu, E)$ is not supra soft compact.*

For the sake of economy, the proofs of the following two propositions have been omitted.

Proposition 4.1.5. *Every supra soft compact space is supra soft Lindelöf.*

The converse of this proposition is not always true as shown in Example 4.1.4.

Proposition 4.1.6. *A finite (resp. countable) union of supra soft compact (supra soft Lindelöf) subsets of (X, μ, E) is supra soft compact (resp. supra soft Lindelöf).*

Definition 4.1.7. A soft subset (G, E) of (X, μ, E) is said to be supra soft compact (resp. supra soft Lindelöf) relative to $\widetilde{X}$ if every supra soft open cover of (G, E), reducible to a finite (resp. countable) subcover.

Proposition 4.1.8. *Every supra soft closed subset of a supra soft compact (resp. supra soft Lindelöf) space (X, μ, E) is supra soft compact (resp. supra soft Lindelöf).*

Proof. Let (X, μ, E) be supra soft Lindelöf. Consider (F, E) is a supra soft closed subset of $\widetilde{X}$ and $\{(H_i, E) : i \in I\}$ is a supra soft open cover of (F, E). Then (F^c, E) is supra soft open and $(F, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, E)$. Therefore $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(H_i, E)\widetilde{\bigcup}(F^c, E)$. Since $\widetilde{X}$ is supra soft Lindelöf, then $\widetilde{X} = \widetilde{\bigcup}_{i\in S}(H_i, E)\widetilde{\bigcup}(F^c, E)$ such that S is countable. Therefore $(F, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}(H_i, E)$. Thus (F, E) is supra soft Lindelöf.
The proof is made similarly in the case of (X, μ, E) is supra soft compact. □

Corollary 4.1.9. *If (G, E) is a supra soft Lindelöf (resp. supra soft compact) subset of $\widetilde{X}$ and (F, E) is a supra soft closed subset of $\widetilde{X}$, then $(G, E)\widetilde{\bigcap}(F, E)$ is supra soft Lindelöf (resp. supra soft compact).*

Proof. Let (G, E) be a supra soft Lindelöf set and consider $\Lambda = \{(H_i, E) : i \in I\}$ be a supra soft open cover of $(G, E)\widetilde{\bigcap}(F, E)$. Then $(G, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, E) \bigcup (F^c, E)$. Since (G, E) is supra soft Lindelöf, then there exists a countable set S such that $(G, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}(H_i, E)\widetilde{\bigcup}(F^c, E)$. Therefore $(G, E)\widetilde{\bigcap}(F, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in S}(H_i, E)$. Thus $(G, E)\widetilde{\bigcap}(F, E)$ is supra soft Lindelöf.
The proof is made similarly in the case of a supra soft compact set. □

Corollary 4.1.10. *Let (F, E) be a soft subset of a supra soft compact (resp. supra soft Lindelöf) space (X, μ, E). Then $(F, E)^{ss\prime}$ is supra soft compact (resp. supra soft Lindelöf).*

In Example 4.1.4, let (H, E) be a soft set such that $H(e_1) = H(e_2) = \mathbb{N}$. Then (H, E) is a supra soft Lindelöf subset of a supra soft Lindelöf space. On the other hand, (H, E) is not soft closed. This illuminate that the converse of the above theorem is not necessarily true.

Theorem 4.1.11. *A supra soft topological space (X, μ, E) is supra soft compact (resp. supra soft Lindelöf) if and only if every collection of supra soft closed subsets of $\widetilde{X}$, satisfying the finite (resp. countable) intersection property, has, itself, a non-null soft intersection.*

Proof. We prove the theorem in the case of supra soft Lindelöf spaces and the other case can be made similarly.
Necessity: Let $\Lambda = \{(F_i, E) : i \in I\}$ be a collection of supra soft closed subsets of $\widetilde{X}$ which has the countable intersection property. Assume that $\widetilde{\bigcap}_{i\in I}(F_i, E) = \widetilde{\emptyset}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c, E)$. As $\widetilde{X}$ is supra soft Lindelöf, then $\widetilde{X} = \widetilde{\bigcup}_{i\in S}(F_i^c, E)$ and S is countable. Therefore $\widetilde{\bigcap}_{i\in S}(F_i, E) = \widetilde{\emptyset}$ contradicts that Λ has the countable intersection

property. Thus Λ has, itself, a non-empty intersection.
Sufficiency: Let $\{(H_i, E) : i \in I\}$ be a supra soft open cover of $\widetilde{X}$. Suppose $\{(H_i, E) : i \in I\}$ has no countable soft sub-cover. Then $\widetilde{X} \backslash \widetilde{\bigcup}_{i \in S}(H_i, E) \neq \widetilde{\emptyset}$, for any countable set S. Now, $\widetilde{\bigcap}_{i \in S}(H_i^c, E) \neq \widetilde{\emptyset}$ implies that $\{(H_i^c, E) : i \in S\}$ is a collection of supra soft closed subsets of $\widetilde{X}$ which has the countable intersection property. Therefore $\widetilde{\bigcap}_{i \in I}(H_i^c, E) \neq \widetilde{\emptyset}$. Thus $\widetilde{X} \neq \widetilde{\bigcup}_{i \in I}(H_i, E)$. But this contradicts that $\{(H_i, E) : i \in I\}$ is a supra soft open cover of $\widetilde{X}$. Hence (X, μ, E) is supra soft Lindelöf. □

Definition 4.1.12. A soft mapping $g_\phi : X \to Y$ is called soft $S^\star$-continuous if the inverse image of each supra soft open subset of $\widetilde{Y}$ is a supra soft open subset of $\widetilde{X}$.

Theorem 4.1.13. *Let $g_\phi : X \to Y$ be a soft $S^\star$-continuous mapping. Then $g_\phi(D, E)$ is supra soft Lindelöf (resp. supra soft compact), for each supra soft Lindelöf (resp. supra soft compact) subset (D, E) of $\widetilde{X}$.*

Proof. We give a proof for the theorem in the case of (D, E) is a supra soft Lindelöf set and the other case achieved similarly.
Suppose that $\{(H_i, E) : i \in I\}$ is a supra soft open cover of $g_\phi(D, E)$. Then $g_\phi(D, E) \widetilde{\subseteq} \widetilde{\bigcup}_{i \in I}(H_i, E)$. Now, $(D, E) \widetilde{\subseteq} \widetilde{\bigcup}_{i \in I} g_\phi^{-1}(H_i, E)$. Since g_ϕ is soft $S^\star$-continuous, then $g_\phi^{-1}(H_i, E)$ is supra soft open, for all $i \in I$. By hypotheses, (D, E) is supra soft Lindelöf, then there exists a countable set S such that $(D, E) \widetilde{\subseteq} \widetilde{\bigcup}_{i \in S} g_\phi^{-1}(H_i, E)$. Therefore $g_\phi(D, E) \widetilde{\subseteq} \widetilde{\bigcup}_{i \in S}(H_i, E)$. Thus $g_\phi(D, E)$ is supra soft Lindelöf. □

Definition 4.1.14. A supra soft topology μ on X is said to be an enriched supra soft topology if **(i)** of Definition 2.6 is replaced by the following condition: $(G, E) \in \tau$, for all $(G, E) \in CS(X, E)$. The triplet (X, μ, E) is called an enriched supra soft topological space.

Proposition 4.1.15. *If (X, μ, E) is an enriched supra soft compact (resp. enriched supra soft Lindelöf) space, then (X, μ_e) is supra compact (resp. supra Lindelöf), for each $e \in E$.*

Proof. We prove the theorem in the case of enriched supra soft compact and the other case follows similar lines.
Let $\{H_j(e) : j \in J\}$ be a supra open cover for (X, μ_e). We construct a supra soft open cover for (X, μ, E) which consists of the following soft sets:

(i) All supra soft open sets (F_j, E) in which $F_j(e) = H_j(e)$, for each $j \in J$.

(ii) Since (X, μ, E) is enriched, then we take a supra soft open set (G_j, E) such that $G_j(e) = \emptyset$ and $G_j(e_i) = X$, for all $e_i \neq e$.

Obviously, $\{(F_j, E) \widetilde{\bigcup} (G, E) : j \in J\}$ is a supra soft open cover for (X, μ, E). As (X, μ, E) is supra soft compact, then $\widetilde{X} = \bigcup_{j=1}^{j=n} (F_j, E) \widetilde{\bigcup} (G, E)$. Therefore $X = \bigcup_{j=1}^{j=n} F_j(e) = \bigcup_{j=1}^{j=n} H_j(e)$. Thus (X, μ_e) is supra soft compact. □

Theorem 4.1.16. *If (X, μ_e) is supra compact (resp. supra Lindelöf) for each $e \in E$ and $\mid E \mid = m$ (resp. E is countable), then (X, μ, E) is supra soft compact (resp. supra soft Lindelöf).*

Proof. We prove the theorem in the case of supra compact and the other case follows similar lines.

Let $\{(G_j, E) : j \in J\}$ be a supra soft open cover for (X, μ, E). Then $X = \bigcup_{j\in J} G_j(e_i)$ for each $e_i \in E$. As (X, μ_e) is supra compact for each $e \in E$, then $X = \bigcup_{j=1}^{j=n_1} G_j(e_1)$, $X = \bigcup_{j=n_1+1}^{j=n_2} G_j(e_2)$,..., $X = \bigcup_{j=n_{m-1}+1}^{j=n_m} G_j(e_m)$. Therefore $\widetilde{X} = \widetilde{\bigcup}_{j=1}^{j=n_m}(G_j, E)$. Thus (X, μ, E) is supra compact. □

4.2. Almost supra soft compact and almost supra soft Lindelöf spaces

Definition 4.2.1. A supra soft topological spaces (X, μ, E) is called almost supra soft compact (resp. almost supra soft Lindelöf) provided that every supra soft open cover of $\widetilde{X}$ has a finite (resp. countable) sub-collection the supra soft closure of whose member cover $\widetilde{X}$.

In the following, we give two examples, the first one satisfies a concept of almost supra soft compact and the second one does not satisfy.

Example 4.2.2. *Let $E = \{e_1, e_2\}$ be a set of parameters and $\mu = \{\widetilde{\emptyset}, (G_i, E)\widetilde{\subseteq}\widetilde{\mathcal{R}}$, where $G_i(e_1)$ containing $\{1, 2\}$ and $G_i(e_2)$ containing $\{1, 3\}\}$ be a supra soft topology on $\mathcal{R}$. The supra soft closure of each supra soft open subset of $\mathcal{R}$ is $\widetilde{\mathcal{R}}$, and consequently $(\mathcal{R}, \mu, E)$ is almost supra soft compact.*

Example 4.2.3. *Let $E = \{e_1, e_2, ..., e_n, ...\}$ be a set of parameters and $\mu = \{\widetilde{\emptyset}, (G_i, E)\widetilde{\subseteq}\widetilde{\mathcal{Q}}$, where $1 \in (G_i, E)$ or $2 \in (G_i, E)\}$ be a supra soft topology on $\mathcal{Q}$. Let $\Lambda = \{(H_i, E) : H_i(e_j) = \{1, x\}$ and $x \neq 2\}\widetilde{\bigcup}\{(H, E) : H(e) = \{2\}$ for each $e \in E\}$ be a supra soft open cover of $\widetilde{\mathcal{Q}}$. As $sscl(H_i, E) = (H_i, E)$ and $sscl(H, E) = (H, E)$, then $(\mathcal{Q}, \mu, E)$ is not almost supra soft compact.*

The proofs of the following propositions are straightforward and will be omitted.

Proposition 4.2.4. *Every almost supra soft compact space is almost supra soft Lindelöf.*

Proposition 4.2.5. *Every supra soft compact (resp. supra soft Lindelöf) is almost supra soft compact (resp. almost supra soft Lindelöf).*

Proposition 4.2.6. *A finite (resp. countable) union of almost supra soft compact (resp. almost supra soft Lindelöf) subsets of (X, μ, E) is almost supra soft compact (resp. almost supra soft Lindelöf).*

The converse of Proposition 4.2.4 and Proposition 4.2.5 generally are not true as illustrated in Example 4.2.3 and Example 4.2.2, respectively.

Definition 4.2.7. A soft subset (D, E) of (X, μ, E) is called supra soft clopen provided that it is both supra soft open and supra soft closed.

Proposition 4.2.8. *If (D, E) is a supra soft clopen subset of an almost supra soft compact (resp. almost supra soft Lindelöf)*

space (X, μ, E), then (D, E) is almost supra soft compact (resp. almost supra soft Lindelöf).

Proof. We prove the proposition in the case of (X, μ, E) is almost supra soft compact and the other case follows similar lines.
Let (D, E) be a supra soft clopen subset of $\widetilde{X}$ and $\{(H_i, E) : i \in I\}$ be a supra soft open cover of (D, E). Then (D^c, E) is supra soft open and $(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, E)$. Therefore $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(H_i, E)\widetilde{\bigcup}(D^c, E)$. Since $\widetilde{X}$ is almost supra soft compact, then $\widetilde{X} = \widetilde{\bigcup}_{i=1}^{i=n} sscl(H_i, E)\widetilde{\bigcup}(D^c, E)$. Thus $(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} sscl(H_i, E)$. Hence (D, E) is almost supra soft compact. □

Corollary 4.2.9. *If (G, E) is an almost supra soft compact (resp. almost supra soft Lindelöf) subset of $\widetilde{X}$ and (D, E) is a supra soft clopen subset of $\widetilde{X}$, then $(G, E)\widetilde{\bigcap}(D, E)$ is almost supra soft compact (resp. almost supra soft Lindelöf).*

Proof. Let $\Lambda = \{(H_i, E) : i \in I\}$ be a supra soft open cover of $(G, E)\widetilde{\bigcap}(D, E)$. Then $(G, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, E)\widetilde{\bigcup}(D^c, E)$. Since (G, E) is almost supra soft compact, then $(G, E) \subseteq \bigcup_{i=1}^{i=n} sscl(H_i, E) \bigcup (D^c, E)$. Therefore $(G, E)\widetilde{\bigcap}(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}$ $sscl(H_i, E)$. Thus $(G, E)\widetilde{\bigcap}(D, E)$ is almost supra soft compact.
The proof is similar in case of an almost supra soft Lindelöf set. □

The converse of the above proposition need not be true as shown in Example 4.2.2, let (H, E) be a soft subset of $(\mathcal{R}, \mu, E)$. Where $H(e_1) = \{1, 4\}$ and $H(e_2) = \{4, 5\}$. Then (H, E) is almost supra soft compact, but is not supra soft clopen subset of $(\mathcal{R}, \mu, E)$.

Lemma 4.2.10. *A soft mapping $g_\phi : X \to Y$ is soft $S^\star$-continuous if and only if $g_\phi(sscl(G, E))\widetilde{\subseteq}sscl(g_\phi(G, E))$, for each $(G, E)\widetilde{\subseteq}\widetilde{X}$.*

Proof. Necessity: Let (G, E) be a supra soft subset of $\widetilde{X}$. Then $(G, E)\widetilde{\subseteq}g_\phi^{-1}(g_\phi(G, E))$
$\widetilde{\subseteq}g_\phi^{-1}(sscl(g_\phi(G, E)))$. Since $sscl(g_\phi(G, E))$ is supra soft closed, then $g_\phi^{-1}(sscl(g_\phi(G, E)))$ is supra soft closed. Therefore $sscl(G, E)\widetilde{\subseteq}g_\phi^{-1}(sscl(g_\phi(G, E)))$. Thus $g_\phi(sscl(G, E))\widetilde{\subseteq}sscl(g_\phi(G, E))$.
Sufficiency: Suppose that (H, E) is a supra soft closed subset of $\widetilde{Y}$ such that $(G, E) = g_\phi^{-1}(H, E)$. Then $g_\phi(sscl(G, E))\widetilde{\subseteq}g_\phi(sscl(g_\phi^{-1}(H, E))$ (H, E). Therefore $sscl(G, E)\widetilde{\subseteq}g_\phi^{-1}(H, E) = (G, E)$. Thus $g_\phi^{-1}(H, E)$ is supra soft closed. Hence g_ϕ is soft $S^\star$-continuous. □

Theorem 4.2.11. *Let $g_\phi : X \to Y$ be a soft $S^\star$-continuous mapping. Then $g_\phi(D, E)$ is almost supra soft compact (resp. almost supra soft Lindelöf), for each almost supra soft compact (resp. almost supra soft Lindelöf) subset (D, E) of $\widetilde{X}$.*

Proof. Suppose that $\{(H_i, E) : i \in I\}$ be a supra soft open cover of $g_\phi(D, E)$. Then $g_\phi(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, E)$. Now, $(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}g_\phi^{-1}(H_i, E)$. Since g_ϕ is soft $S^\star$-continuous, then $g_\phi^{-1}(H_i, E)$ is supra soft open. By hypotheses, (D, E) is almost supra soft compact, then $(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} sscl(g_\phi^{-1}(H_i, E))$. Therefore $g_\phi(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n} g_\phi(sscl(g_\phi^{-1}(H_i, E)))$. Since g_ϕ is soft $S^\star$-continuous, then $g_\phi(sscl(g_\phi^{-1}(H_i, E)))\widetilde{\subseteq}sscl(g_\phi(g_\phi^{-1}(H_i, E)))\widetilde{\subseteq}sscl(H_i, E)$. So $g_\phi(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}$ $sscl(H_i, E)$. Hence $g_\phi(D, E)$ is almost supra soft compact.
A similar proof is given in case of an almost supra soft Lindelöf space. □

Definition 4.2.12. A collection $\Lambda = \{(F_i, K) : i \in I\}$ of soft sets is said to satisfy a condition C_1 (resp. condition

C_1') if $\widetilde{\bigcap}_{i=1}^{i=n} ssint(F_i, K) \neq \widetilde{\emptyset}$ (resp. $\widetilde{\bigcap}_{i\in S} ssint(F_i, K) \neq \widetilde{\emptyset}$).

It is clear that a collection satisfies a condition C_1 (resp. condition C_1'), it also satisfies the finite (resp. countable) intersection property.

Theorem 4.2.13. *A supra soft topological space (X, τ, K) is almost supra soft compact (resp. almost supra soft Lindelöf) if and only if every collection of supra soft closed subsets of (X, τ, K), satisfying a condition C_1 (resp. condition C_1'), has, itself, a non-null soft intersection.*

Proof. We will begin with the proof for almost supra soft compact spaces, because the proof for almost supra soft Lindelöf spaces is analogous.
Let $\Lambda = \{(F_i, K) : i \in I\}$ be a collection of supra soft closed subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i, K) = \widetilde{\emptyset}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c, K)$. Since (X, τ, K) is almost supra soft compact, then $\widetilde{X} = \widetilde{\bigcup}_{i=1}^{i=n} sscl(F_i^c, K)$. So $\widetilde{\emptyset} = (\widetilde{\bigcup}_{i=1}^{i=n} sscl(F_i^c, K))^c = \widetilde{\bigcap}_{i=1}^{i=n} ssint(F_i, K)$. Thus the necessary part holds.
Conversely, let Λ be a collection of supra soft closed subsets of $\widetilde{X}$ which satisfies a condition C_1. Then it also satisfies the finite intersection property. Since Λ has a non-null soft intersection, then (X, τ, K) is a supra soft compact space. It follows, by Proposition 4.2.5, that (X, τ, K) is almost supra soft compact. □

Remark 4.2.14. *Let the supra soft topological space $(\mathcal{R}, \mu, E)$ is the same as in Example 4.2.2 and let $\Lambda = \{(H_n, E) : n \geq 4\}$, where $H_n(e_1) = \{n, n+1, ...\}$ and $H_n(e_2) = \{n\}$ be a collection of supra soft closed subsets of $(\mathcal{R}, \mu, E)$. Now, $(\mathcal{R}, \mu, E)$ is almost supra soft compact, whereas $\widetilde{\bigcap}_{i=4}^{\infty}(H_n, E) = \widetilde{\emptyset}$.*

4.3. Mildly supra soft compact and mildly supra soft Lindelöf spaces

Definition 4.3.1. A supra soft topological space (X, μ, E) is called mildly supra soft compact (resp. mildly supra soft Lindelöf) provided that every supra soft clopen cover of $\widetilde{X}$ has a finite (resp. countable) subcover.

Proposition 4.3.2. *Every mildly supra soft compact space is mildly supra soft Lindelöf.*

Proof. Straightforward. □

The converse of the previous proposition need not be true in general as illustrated in Example 4.2.3.

Proposition 4.3.3. *A finite (resp. countable) union of mildly supra soft compact (resp. mildly supra soft compact Lindelöf) subsets of (X, μ, E) is mildly supra soft compact (resp. mildly supra soft Lindelöf).*

Proof. Straightforward. □

Proposition 4.3.4. *Every almost supra soft compact (resp. almost supra soft Lindelöf) space (X, μ, E) is mildly supra soft compact (resp. mildly supra soft Lindelöf).*

Proof. Let $\Lambda = \{(H_i, E) : i \in I\}$ be a supra soft clopen cover of (X, μ, E). Since (X, μ, E) is almost supra soft compact (resp. almost supra soft Lindelöf), then $\widetilde{X} = \widetilde{\bigcup}_{i=1}^{i=n} sscl(H_i, E)$ (resp. $\widetilde{X} = \widetilde{\bigcup}_{i\in S} sscl(H_i, E)$). It is clear that $sscl(H_i, E) = (H_i, E)$. So (X, μ, E) is mildly supra soft compact (resp. mildly supra soft Lindelöf). □

Corollary 4.3.5. *Every supra soft compact (resp. supra soft Lindelöf) space (X, μ, E) is mildly supra soft compact (resp. mildly supra soft Lindelöf).*

The converse of the above proposition need not be true in general as the next example shows.

Example 4.3.6. *Let $E = \{e_1, e_2\}$ be a set of parameters and $\mu = \{\widetilde{\emptyset}, (G_i, E)\widetilde{\subseteq}\widetilde{\mathcal{R}}$, where $G_i(e_1) = G_i(e_2) = (-\infty, a)$ or $G_i(e_1) = G_i(e_2) = (b, \infty)$ or $G_i(e_1) = G_i(e_2) = (-\infty, a) \bigcup (b, \infty) : a, b \in \mathcal{R}\}$ be a supra soft topology on $\mathcal{R}$. Then $(\mathcal{R}, \mu, E)$ is mildly supra soft compact, but is not almost supra soft compact.*

Proposition 4.3.7. *If (D, E) is a supra soft clopen subset of a mildly supra soft compact (resp. mildly supra soft Lindelöf) space (X, μ, E), then (D, E) is mildly supra soft compact (resp. mildly supra soft Lindelöf).*

Proof. The proof is given in the case of mildly supra soft compact spaces and the case between parenthesis can be made similarly.
Let (D, E) be a supra soft clopen subset of $\widetilde{X}$ and $\{(H_i, E) : i \in I\}$ be a supra soft clopen cover of (D, E). Then (D^c, E) is a supra soft clopen and $(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, E)$. So $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(H_i, E)\widetilde{\bigcup}(D^c, E)$. Since (X, μ, E) is mildly supra soft compact, then $\widetilde{X} = \widetilde{\bigcup}_{i=1}^{i=n}(H_i, E)\widetilde{\bigcup}(D^c, E)$. Thus $(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(H_i, E)$. Hence (D, E) is mildly supra soft compact. □

Corollary 4.3.8. *If (G, E) is a mildly supra soft compact (resp. mildly supra soft Lindelöf) subset of $\widetilde{X}$ and (D, E) is a supra soft clopen subset of $\widetilde{X}$, then $(G, E)\widetilde{\bigcap}(D, E)$ is mildly supra soft compact (resp. mildly supra soft Lindelöf).*

Proof. The proof is given in the case of mildly supra soft compact spaces and the case between parenthesis can be made similarly.
Let $\Lambda = \{(H_i, E) : i \in I\}$ be a supra soft clopen cover of $(G, E)\widetilde{\bigcap}(D, E)$. Then $(G, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i\in I}(H_i, E) \bigcup (D^c, E)$. By hypothesis, (G, E) is mildly supra soft compact and (D, E) is supra soft clopen, and consequently $(G, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(H_i, E)$ $\widetilde{\bigcup}(D^c, E)$. Therefore $(G, E)\widetilde{\bigcap}(D, E)\widetilde{\subseteq}\widetilde{\bigcup}_{i=1}^{i=n}(H_i, E)$. Thus $(G, E)\widetilde{\bigcap}(D, E)$ is mildly supra soft compact. □

Theorem 4.3.9. *A supra soft topological space (X, μ, K) is mildly supra soft compact (resp. mildly supra soft Lindelöf) if and only if every collection of supra soft clopen subsets of (X, τ, K), satisfying the finite (resp. countable) intersection property, has, itself, a non-null soft intersection.*

Proof. We only prove the theorem when (X, τ, K) is mildly soft compact, the other case can be made similarly.
Let $\Lambda = \{(H_i, K) : i \in I\}$ be a soft clopen cover of $\widetilde{X}$. Suppose that Λ has no finite soft sub-collection which cover $\widetilde{X}$. Then $\widetilde{X} \setminus \bigcup_{i=1}^{i=n}(H_i, K) \neq \widetilde{\emptyset}$, for any $n \in \mathcal{N}$. Now, $\widetilde{\bigcap}_{i=1}^{i=n}(H_i^c, K) \neq \widetilde{\emptyset}$ implies that $\{(H_i^c, K) : i \in I\}$ is a soft collection of supra soft clopen subsets of $\widetilde{X}$ which has the finite intersection property. Thus $\widetilde{\bigcap}_{i\in I}(H_i^c, K) \neq \widetilde{\emptyset}$ implies that $\widetilde{X} \neq \widetilde{\bigcup}_{i\in I}(H_i, K)$. But this contradicts that Λ is a supra soft clopen cover of $\widetilde{X}$. Hence (X, τ, K) is mildly supra soft compact.

Conversely, Let $\Lambda = \{(F_i, K) : i \in I\}$ be a supra soft clopen subsets of $\widetilde{X}$. Suppose that $\widetilde{\bigcap}_{i\in I}(F_i, K) = \widetilde{\emptyset}$. Then $\widetilde{X} = \widetilde{\bigcup}_{i\in I}(F_i^c, K)$. As (X, τ, K) is mildly supra soft compact, then $\widetilde{\bigcup}_{i=1}^{i=n}(F_i^c, K) = \widetilde{X}$. Therefore $\widetilde{\bigcap}_{i=1}^{i=n}(F_i, K) = \widetilde{\emptyset}$. This completes the proof. □

5. Conclusion

In this article, the concepts of supra soft limit, supra soft boundary and supra soft closure operators are given and their properties are studied in detail. Also, the notions of supra soft compact (supra soft Lindelöf), almost supra soft compact (almost supra soft Lindelöf) and mildly supra soft compact (mildly supra soft Lindelöf) are investigated with the help of examples. By using the finite intersection property for supra soft closed (supra soft clopen) sets the concepts of supra soft compact (mildly supra soft compact) spaces are characterized; and by using the countable intersection property for supra soft closed (supra soft clopen) sets the concepts of supra soft Lindelöf (mildly supra soft Lindelöf) spaces are characterized. We define two conditions, namely C_1 and C_1'; and then we utilize them to give a completely description for almost supra soft compact and almost supra soft Lindelöf spaces. In the imminent paper, we use the notions of soft ordered sets [12] to define a supra soft topological spaces. Eventually, the obtained results in this work form an introductory platform and open scopes for studying further important findings related to supra soft topological spaces.

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