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One-Dimensional Solute Transport in a Heterogeneous Porous Media with Pulse Type Input Source

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ABSTRACT. An advection-dispersion equation subjected to temporally and spatially dependent groundwater velocity, dispersion coefficient along with a time dependent pulse type input of time varying nature is solved for one dimensional semi-infinite porous medium. Input concentration is any continuous smooth function of time acts up to some finite time and then eliminated. Concentration gradient at other boundary domain is considered zero. Dispersion is linearly and squarely proportional to groundwater velocity in temporal and spatial measurement respectively. Initially, medium is uniformly polluted. Interpolation method is applied to reduce the input function into a polynomial. Using certain transformations the advection-dispersion equation is reduced to constant coefficient and freed from convective part then Laplace transform technique is applied to get the solution of advection-dispersion equation. Two different functions of input are discussed to understand the utility of the present study. Obtained result is demonstrated graphically with the help of numerical example.

1 Introduction

Mathematical models are useful tool to realize level of contaminant presence in a geological formation like aquifers including site/source of contaminant and also to predict the future possibilities about toxicity of water bodies due to presence of harmful contaminant introduced to water body by some natural phenomena or manmade activity. With growing population and heavy congestion of industries, challenges to meet the demand of harmless/ non

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toxic water have gone tough. Several analytical / numerical solutions are present in literatures up to the date for dealing such challenges with help of science. For finite/semi-infinite porous medium of type homogeneous, heterogeneous, isotropic and anisotropic , a plethora of analytical solutions for one, two & three-dimensional solute transport including a variety of input and boundary conditions have been presented. [1] observed the concentration distribution of pollutant arising from instantaneous point source in a two-dimensional water channel with non-uniform velocity distribution. Analytical solutions of the general one-dimensional solute transport model including distributed first-order decay, linear equilibrium sorption and a finite length input source for confined aquifers is presented by[2] . For hydrodynamic dispersion including longitudinal and transverse dispersion along with time and space-dependent sources consisting of the first-type boundary condition at the source, A generalized two-dimensional analytical solution is developed by using Laplace transform and Fourier analysis techniques simultaneously. The effect of scale dependency is studied with asymptotically scale dependent dispersion [3]. For a semi-infinite aquifer having uniform and spatially variable permeability, analytical and numerical solutions for unsteady ground water velocity is obtained by [4]. Analytical solution for space dependent dispersion coefficient and velocity presented by [5, 6]. Dispersion is taken proportional to groundwater velocity while groundwater velocity is linear to space. In latter one instantaneous input is considered to demonstrate the problem. [7] proposed the analytical and numerical models for the groundwater flow in the multi aquifer system. For a two-dimensional non-axis symmetric solute transport in a radially convergent flow field an analytical power series solution which is useful for quantitative hydro-geologic problems was attained using Laplace-transformed power series (LTPS) technique [8]. A study of solute transport phenomena in aquifers consisting of analytical and numerical solutions was presented by [9]. [10] studied solute transport phenomena under an imposed standing water surface wave. They obtained two close solutions of the advection diffusion equation by solute transport into a sub-aqueous sediment bed there. Effect of time dependent source and velocity are studied for solute transport phenomena in porous media in [11] and [12] respectively. Two dimensional solutions of advection-dispersion equation were proposed [13-15] and later established the solutions using cylindrical coordinate system. [16] presented an analytical approach to solute dispersion along and against transient groundwater flow in a homogeneous finite aquifer with pulse type boundary conditions.. An analytical solution of advection dispersion equation (ADE) along with decay of first order for multi-layered media using the classical integral transform technique (CITT) method is obtained [17]. [18] studied finite domain solute transport phenomena considering sinusoidal dispersion and ground water velocity for homogeneous porous media along with sinusoidal input source. An analytical solution for calculating the space-time variation of contaminant concentration in one-dimensional transient groundwater flow in a homogenous semi-infinite aquifer along with a dispersion proportional to the square of the velocity and a uniform and time varying input along transient groundwater flow is investigated by [19] . Solutions like pulse type including different geological situations are developed to measure the concentration level in presence and absence of the source. A pulse type input source concentration is considered at the intermediate portion in the aquifer different from origin for two-dimensional solute dispersion in saturated porous media and this model of dispersion problem was solved analytically by using the Laplace Transform Technique (LTT) and numerically with the help of Explicit Finite Difference (EFD) method [20]. Greens Function Method and pertinent coordinate transformation method are employed to get Analytical solutions for solute transport in groundwater and riverine flow [21] .

In this paper, a theoretical model is developed for the dispersion problem for one-dimensional heterogeneous porous medium Dispersion coefficient and seepage velocity are function of time and space. Dependency of dispersion coefficient and ground water velocity on space enhances the proximity of the present problem to real world problem. Initially, medium is not solute free which means it is laden with some concentration. Application of the present problem is broadened with an input that is of varying nature and is any smooth function of time acts up to certain time and then eliminated. Laplace transform technique is used to get the solution of present study. Utility of the solution is illustrated with two cases of inputs.

2 Mathematical Formulation of the Problem

The one-dimensional advection-dispersion equation which is second order parabolic equation and derived on basis of mass conservation and Ficks law of diffusion may be written as follows [22],

$$R\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}(D(x,t)\frac{\partial c}{\partial x} - u(x,t)c) - \mu(x,t)c + \gamma(x,t) \tag{1}$$

In which, $c[ML^{-3}]$ is the solute concentration, $u[LT^{-1}]$ refers to groundwater velocity, $D[L^2T^{-1}]$ denotes dispersion coefficient and R is retardation factor , which is a dimensionless quantity. The dispersion coefficient $D(x,t)$ is proportional to flow velocity $u(x,t)$ while only temporal nature is considered and square of it for spatial nature and assumed respectively as; $u(x,t) = u_0(1+ax)(1+mt)^{-1}$, $D(x,t) = D_0(1+ax)^2(1+mt)^{-1}$. Zero order production $\gamma(x,t)[ML^{-3}T^{-1}]$ and first order decay $\mu(x,t)[T^{-1}]$ are defined respectively as: $\gamma(x,t) = \gamma_0(1+mt)^{-1}$ and $\mu(x,t) = \mu_0(1+mt)^{-1}$, where $D_0[L^2T^{-1}], u_0[LT^{-1}], \mu_0[T^{-1}], \gamma_0[ML^{-3}T^{-1}]$ and R are constants. $m[T^{-1}]$ and $a[L^{-1}]$ are the parameters which regulate the time and space dependency of the dispersion and the groundwater velocity respectively.

Therefore, Eq.(1) may be re-written as;

$$R\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}(D_0(1+mt)^{-1}(1+ax)^2\frac{\partial c}{\partial x} - u_0(1+mt)^{-1}(1+ax)c) - \mu_0(1+mt)^{-1}c + \gamma_0(1+mt)^{-1} \tag{2}$$

The geological formation is assumed of semi-infinite length along horizontal direction. It is initially uniformly polluted . The input source at one end is of varying nature and defined by any smooth function of time applied in the direction of the flow and concentration gradient at other end of domain is considered zero, which is also called flux type boundary condition. In order to formulate the proposed problem mathematically, the initial and boundary conditions may be written as:

$$c(x,t) = c_i; \quad t = 0, x \geq 0 \tag{3}$$

$$-D(x,t)\frac{\partial c(x,t)}{\partial x} + u(x,t)c = \begin{cases} u_0c_0F(m't),\ 0 < t \leq t_0 \\ x = 0 \\ 0,\ t > t_0 \end{cases} \tag{4}$$

$$\frac{\partial c(x,t)}{\partial x} = 0; \quad \text{as } x \to \infty, t \geq 0; \tag{5}$$

where, c_0 and c_i represent reference concentrations and $m'[T^{-1}]$ is an unsteady parameter. Dimensionless $F(m't)$ is smooth continuous and bounded function in a time domain $[0, t_0]$. Weirstrass approximation theorem says that any continuous function on a bounded interval can be uniformly approximated by polynomial function. Since $F(m't)$ is continuous and bounded in domain $[0, t_0]$, Following Weirstrass approximation theorem, $F(m't)$ may be written as a polynomial in t of degree n. Hence the proposed initial and boundary conditions Eqs. (3-5) may be written as:

$$c(x,t) = c_i; \quad t = 0, x \geq 0 \tag{6}$$

$$-D(x,t)\frac{\partial c(x,t)}{\partial x} + u(x,t)c = \begin{cases} u_0 c_0 G_n(t),\ 0 < t \leq t_0 \\ x = 0 \\ 0,\ t > t_0 \end{cases} \tag{7}$$

$$\frac{\partial c(x,t)}{\partial x} = 0; \quad \text{as } x \to \infty, t \geq 0; \tag{8}$$

where,

$$G_n(t) = a_0 + a_1 t + a_2 t^2 + ... + a_n t^n \tag{9}$$

Here $G_n(t)$ is dimensionless and a_i's are constant.

In order to eliminate the time dependent factor from the coefficient of advection-dispersion equation. A new time variable T is introduced as; [23],

$$T = \int_0^1 (1+mt)^{-1} dt \quad \text{or} \quad T = \frac{1}{m} log(1+mt) \quad \text{and} \quad t = \frac{1}{m}(e^{mT} - 1) \tag{10}$$

With this transformation Eq. (10), Eq. (2) and Eqs. (6-8) are reduced into new time variable T.

$$R\frac{\partial c}{\partial T} = \frac{\partial}{\partial x}(D_0(1+ax)^2\frac{\partial c}{\partial x} - u_0(1+ax)c) - \mu_0 c + \gamma_0 \tag{11}$$

$$c(x,T) = c_i; \ T = 0, \ x \geq 0 \tag{12}$$

$$-D_0(1+ax)^2\frac{\partial c(x,t)}{\partial x} + u_0(1+ax)c = \begin{cases} u_0 c_0 H(t) exp(mT),\ 0 < T \leq T_0 \\ x = 0 \\ 0,\ T > T_0 \end{cases} \tag{13}$$

$$\frac{\partial c(x,t)}{\partial x} = 0; \quad \text{as } x \to \infty, T \geq 0; \tag{14}$$

where $H(T) = b_0 e^{mT} + b_1 e^{2mT} + b_2 e^{3mT} + ... + b_n e^{(n+1)mT}$ b_i's are constants and $T_0 = \frac{1}{m} log(1+mt_0)$. In order to reduce advection-dispersion equation in to constant coefficient, we introduce a new space variable X [24].

$$X = \frac{1}{a} log(1+ax) \tag{15}$$

also,

$$x = \frac{1}{a}\{exp(aX) - 1\}$$

With this new space variable in Eq. (15) and Eqs. (11-14) may be written as:

$$R_0\frac{\partial c}{\partial T}=D_0\frac{\partial^2 c}{\partial X^2}-(u_0-D_0a)\frac{\partial c}{\partial X}-(\mu_0+\mu_0 a)c+\gamma_0 \tag{16}$$

or

$$R_0\frac{\partial c}{\partial T}=D_0\frac{\partial^2 c}{\partial X^2}-w_0\frac{\partial c}{\partial X}-\mu_1 c+\gamma_0 \tag{17}$$

$$c(X,T)=c_i;\ \ T=0,\ \ X\geq 0 \tag{18}$$

$$-D_0\frac{\partial c(X,T)}{\partial X}+u_0c=\begin{cases} u_0c_0H(T)exp(mT),\ 0<T\leq T_0\\ X=0\\ 0,\ T>T_0\end{cases} \tag{19}$$

$$\frac{\partial c(X,T)}{\partial X}=0;\quad \text{as }\ x\to\infty, T\geq 0; \tag{20}$$

where $w_0=u_0-D_0a$ and $\mu_1=\mu_0+u_0a$.

$H(T)\ =\ b_0+b_1e^{mT}+....+b_{n-1}e^{(n-1)mT}+b_ne^{nmT}$ Now using following transformation, convective term in advection-dispersion Eq. (17) is removed. Transformation is defined as:

$$c(X,T)=k(X,T)exp\left[\frac{w_0}{2D_0}X-\frac{1}{R}\left[\frac{w_0^2}{4D_0}+\mu_1\right]T\right]+\frac{\gamma_0}{\mu_1} \tag{21}$$

Where $k(X,T)$ is a new dependent variable. With this new transformation Eqs. (17-20) reduce into:

$$R\frac{\partial k}{\partial T}=D_0\frac{\partial^2 k}{\partial X^2} \tag{22}$$

$$k(X,T)=\left(c_i-\frac{\gamma_0}{\mu_1}\right)exp(-\beta X);\ \ T=0,\ \ X\geq 0 \tag{23}$$

$$-D_0\frac{\partial k(X,T)}{\partial X}+\left(u_0-\frac{w_0}{2}\right)k=\begin{cases} u_0c_0H(T)exp(\eta^2T),\ 0<T\leq T_0\\ X=0\\ -\frac{u_0\gamma_0}{\mu_1}exp(\eta^2T),\ T>T_0\end{cases} \tag{24}$$

$$\frac{\partial k(X,T)}{\partial X}+\frac{w_0}{2D_0}k(X,T)=0;\quad \text{as }\ X\to\infty,\ \ T\geq 0 \tag{25}$$

where $\beta=\frac{w_0}{2D_0}$ and $\eta=\sqrt{\frac{1}{R}\left(\frac{w_0^2}{4D_0}+\mu_1\right)}$.

Applying Laplace transformations on Eqs. (22-25), mentioned equations reduce into following boundary problem of ordinary differential equation.

$$\frac{d^2\bar{k}}{dX^2}-\frac{pR}{D_0}\bar{k}=-\frac{R}{D_0}\left[\left[c_i-\frac{\gamma_0}{\mu_1}\right]exp(-\beta X)\right] \tag{26}$$

$$\begin{aligned}-D_0\frac{d\bar{k}}{dX}+(u_0-\frac{w_0}{2})\bar{k}=&-\frac{u_0\gamma_0}{\mu_1}\frac{[1-exp-(p-\eta^2)T_0]}{(p-\eta^2)}\\ &+u_0c_0b_0\frac{[1-exp-(p-(\eta^2+m))T_0]}{(p-(\eta^2+m)}\\ &+u_0c_0b_1\frac{[1-exp-(p-(\eta^2+2m))T_0]}{(p-(\eta^2+2m)}\\ &+.....+u_0c_0b_n\frac{[1-exp-(p-(\eta^2+(n+1)m))T_0]}{(p-(\eta^2+nm)}-\frac{u_0\gamma_0}{\mu_1}\frac{[-exp-(p-\eta^2)T_0]}{(p-\eta^2)},\ \ X=0\end{aligned} \tag{27}$$

$$\frac{d\bar{k}}{dX}+\frac{w_0}{2D_0}d\bar{k}=0, \quad \text{as} \quad X\to\infty \tag{28}$$

where, $\bar{f}(x,p)=L\{f(x,p)\}=\int_0^\infty e^{-pt}f(x,t)dt$, $p>0$ where P is a Laplace parameter.

Here, solution of Eq.(26) , by using Eq.(27) and Eq.(28) may be written as:

$$\begin{aligned}\bar{k}(X,p)=&-\frac{u_0\mu_0}{\sqrt{RD_0}\gamma_0}\frac{[1-exp-(p-\eta^2)T_0]}{(p-\eta^2)(\sqrt{p}+\sigma)}exp(-\sqrt{\frac{pR}{D_0}}X)\\&+\frac{u_0c_0b_0}{\sqrt{RD_0}}\frac{[1-exp[-\{p-(\eta^2+m)\}T_0]]}{\{p-(\eta^2+m)\}(\sqrt{p}+\sigma)}\times exp(-\sqrt{\frac{pR}{D_0}}X)\\&+\frac{u_0c_0b_1}{\sqrt{RD_0}}\frac{[1-exp[-\{p-(\eta^2+2m)\}T_0]]}{\{p-(\eta^2+2m)\}(\sqrt{p}+\sigma)}\times exp(-\sqrt{\frac{pR}{D_0}}X)\\&\vdots\\&+\frac{u_0c_0b_n}{\sqrt{RD_0}}\frac{[1-exp[-\{p-(\eta^2+(n+1)m)\}T_0]]}{\{p-(\eta^2+(n+1)m)\}(\sqrt{p}+\sigma)}\times exp(-\sqrt{\frac{pR}{D_0}}X)\\&-\frac{u_0\mu_0}{\sqrt{RD_0}\gamma_0}\frac{[exp[-\{p-(\eta^2)\}T_0]]}{\{p-(\eta^2)\}(\sqrt{p}+\sigma)}\times exp(-\sqrt{\frac{pR}{D_0}}X)\\&-\frac{u_0}{\sqrt{RD_0}}(c_i-\frac{\mu_0}{\gamma_0})\frac{1}{\{p-(\rho^2)\}(\sqrt{p}+\sigma)}\times exp(-\sqrt{\frac{pR}{D_0}}X)+(c_i-\frac{\gamma_0}{\mu_0})\frac{exp(-\beta X)}{p-\rho^2}\end{aligned} \tag{29}$$

where $\rho=\sqrt{\frac{D_0}{R}\beta^2}$ and $\sigma=\frac{1}{\sqrt{D_0R}}(u_0-\frac{w_0}{2})$.

Solution of present problem is generated by taking inverse Laplace Transform of Eq. (29) and then applying the transformation Eq. (21). Hence solution can be written as:

$$\begin{aligned}c(X,T)=&[-\frac{u_0\gamma_0}{\sqrt{RD_0\mu_1}}H_{n\sigma}(X,T)+\frac{u_0c_0b_0}{\sqrt{RD_0}}H_{(\sqrt{\eta^2+m})\sigma}(X,T)+...\\&+\frac{u_0c_0b_n}{\sqrt{RD_0}}H_{(\sqrt{\eta^2+(n+1)m})\sigma}(X,T)-\frac{u_0}{\sqrt{RD_0}}(c_i-\frac{\gamma_0}{\mu_0})H_{\rho\sigma}(X,T)+(c_i-\frac{\gamma_0}{\mu_0})exp(\rho^2T-\beta X)]\\&exp[\frac{w_0}{2D_0}X-\frac{1}{R}(\frac{w_0^2}{4D_0}+\mu_1)T]+\frac{\gamma_0}{\mu_1}\quad 0<T\le T_0\end{aligned} \tag{30a}$$

$$\begin{aligned}c(X,T)=&[-\frac{u_0\gamma_0}{\sqrt{RD_0\mu_1}}H_{n\sigma}(X,T)+\frac{u_0c_0b_0}{\sqrt{RD_0}}H_{(\sqrt{\eta^2+m})\sigma}(X,T)-H_{(\sqrt{\eta^2+m})\sigma}(X,T-T_0)exp(\eta^2+m)T_0\\&+\frac{u_0c_0b_0}{\sqrt{RD_0}}H_{(\sqrt{\eta^2+2m})\sigma}(X,T)-H_{(\sqrt{\eta^2+m})\sigma}(X,T-T_0)exp(\eta^2+2m)T_0\\&\vdots\\&+\frac{u_0c_0b_0}{\sqrt{RD_0}}H_{(\sqrt{\eta^2+(n+1)m})\sigma}(X,T)-H_{(\sqrt{\eta^2+(n+1)m})\sigma}(X,T-T_0)exp(\eta^2+m)T_0\\&-\frac{u_0}{\sqrt{RD_0}}(c_i-\frac{\gamma_0}{\mu_1})H_{\rho\sigma}(X,T)+(c_i-\frac{\gamma_0}{\mu_1})exp(\rho^2T-\beta X)]\\&exp[\frac{u_0}{2D_0}X-\frac{1}{R}(\frac{u_0^2}{4D_0}+\mu_1)T]+\frac{\gamma_0}{\mu_1}\quad T_0<T\end{aligned} \tag{30b}$$

where value of $H_{\phi\xi}(X,T)$ is given as [26]:

$H_{\phi\xi}(X,T)=\frac{1}{2(\phi+\xi)}exp(\phi^2T-\frac{\phi\sqrt{R}}{\sqrt{D_0}}X)erfc(\frac{X\sqrt{R}}{2\sqrt{D_0T}}-\phi\sqrt{T})-\frac{1}{2(\phi-\xi)}exp(\phi^2T+\frac{\phi\sqrt{R}}{\sqrt{D_0}}X)erfc(\frac{X\sqrt{R}}{2\sqrt{D_0T}}+\phi\sqrt{T})+\frac{\xi}{(\phi^2-\xi^2)}exp(\frac{\xi\sqrt{R}}{\sqrt{D_0}}X+\xi^2T)erfc(\frac{X\sqrt{R}}{2\sqrt{D_0T}}+\xi\sqrt{T})$, where $\rho = \sqrt{\frac{D_0}{R}\beta^2}$.

3 Results and discussions

The obtained solution Eqs. (30a, b) are elaborated with two forms of input function, namely $F(m't) = c_0(l + sin(m't))$(sinusoidal) and $F(m't) = c_0 exp(m't)$(exponential) for a chosen set of data taken from the experimental and theoretical published literature. In former $F(m't), l$ is a constant and can be chosen such that input concentration never takes negative value. In order to evaluate the concentration values c/c_0 reference concentrations c_0, c_i, are assumed 1.0 , 0.01 respectively. The figures are plotted for finite region $0 < x(km) \leq 5$ of semi-infinite domain. Presence of Source is assumed up to time t_0(year)=9. For figures 1,2,3,4,9,10,11,12 the values of initial parameter and dispersion coefficient D_0 taken 0.085(km^{-1}) and 1.2(km^{2}year^{-1}) respectively. Groundwater velocity depending upon geometrical conditions of porous medium is taken u_0=0.03(km year^{-1}) lying between 2m/day to 2m/year [25].Values of other common parameters described above are taken as: $R_0 = 1.25, \mu(year^{-1}) = 0.05$, and $\gamma_0 = 0.0007$. Hermite interpolation method is used to get the polynomial $G_n(t)$ as mentioned in Eq. (9) in following two cases.

3.1 Case I -When input is in the form $F(m't) = (l + sin(m't))$

For the present case, the value of unsteady parameter m', m and parameter l are taken 0.8,0.8 and 2 respectively. Concentration value are computed at times $t(year)$ =3,6 and 9 in the time domain $0 \leq t(year) \leq 9$ .While for $t(year) > 9(t_0)$ the same is computed at times $t(year) = 9.5, 12$ and 14.5. Fig.1 depicts the concentration pattern in presence of the source. At $x(km) = 0$ concentration level increases with different rates. It is because of the periodic nature of $F(m't)$. At all the three levels of time concentration attenuates up to the distance $x(km) = 5$. Fig. 2 illustrates the concentration profiles obtained at $t(year) = 9.5, 12$ and 14.5 from the solution in Eq. (30b), i.e., in the time domain $t > 9$. Unlike uniform nature input, presence of concentration at $x(km) = 0$ even after elimination of the source ascertains the behaviour of varying nature input. A steep decrement in level of concentration recorded in duration 9.5 (years) to 12(years) in comparison to duration 12 (years) to 14.5(years) at $x(km) = 0$.

Surface plots are presented to gauge the concentration pattern in every time and space combination of the proposed time and space domain in presence and absence of the source for fixed dispersion coefficient $D_0 = 1.2(km^2 year^{-1})$ and parameter $a = 0.085(km^{-1})$. In Fig. 3 i.e. in presence of the source a smooth wave like phenomena at $x = 0$ with varying time may be observed . It is due to sinusoidal nature of input. Also throughout the space domain concentration increases with increase of time. In Fig. 4 at $x(km) = 0$ concentration drops fast in very first quarter of year just after elimination of source then after it decreases slowly in coming years.

Fig.5 and Fig.6 are drawn to study the effect of dispersion coefficient in presence and absence of the source respectively for fixed $a = 0.085(km^{-1})$ and Concentration attenuates slight fast as the dispersion coefficient increases in both presence and absence of source. Trend of attenuation with space remains same in both cases for different values of dispersion.

The effect of parameter a is demonstrated from Fig.7 and Fig.8 in presence and absence of the source respectively for fixed dispersion coefficient $D_0 = 1.2(km^2year^{-1})$. As the value of the parameter a increases the dispersion at each point of the space domain increases, the net concentration injected in to the domain in the varying input for a particular time decreases in presence of the source. And in absence of the source, the concentration level decreases almost in first half of the space domain with increases of the value parameter a. Concentration pattern remains same for all the three values of a in both presence and absence of the source.

3.2 Case II- When value of input is $F(m't) = exp(m't)$

The unsteady parameters m' and m are 0.2 and 0.1 respectively for the present case.
Concentration value in the time domain $0 \leq t(year) \leq 9$ are computed at times $t(year) = 5.7$ and 9. While for $t(year) > 9(= t_0)$ the same is computed at times $t(year) = 9.5, 12$ and 14.5. Fig.9 displays the solute concentration in the presence of the source at times $t(year) = 5, 7$ and 9 in the time domain $(0 \leq t \leq t_0)$. The input concentration, c/c_0 at the origin, $x(km) = 0$ are 0.14,0.24 and 0.38 respectively. Almost equal increment in injected input at $x(km) = 0$ may be recorded. But concentration attenuates almost up to same level for all three values of times. Concentration pattern in absence of the source in depicted from Fig. 10. at three values time $t(year) = 9.5, 12$ and 14.5. It is recorded that concentration at near end decreases sharply as time elapses. And it spreads fast away from the source end.

Surface plots of concentration are obtained in presence and absence of the source in Fig. 11 and Fig. 12 respectively for fixed dispersion coefficient $D_0 = 1.2(km^2year^{-1})$ and parameter $a = 0.085(km^{-1})$. Unlike Fig 3., a continuous increase in concentration with time may be observed from Fig. 11 in presence of the source at each point of the space. As the source eliminated concentration at near end drops fast but rate of diminution slows down with time. In contrast at the far end of our study domain concentration level increase with time.
Effect of dispersion for current case of $F(m't)$ has shown from Fig. 13 and Fig. 14 in presence and absence of the source respectively for fixed $a = 0.085(km^{-1})$. Pattern of variations in concentration level with dispersion are alike the same of previous case of $F(m't)$ in both presence and absence of the source.

Fig.15 and Fig.16 are drawn to study the effect parameter a on the concentration pattern in presence and absence of the source respectively for fixed dispersion coefficient $D_0 = 1.2(km^2year^{-1})$. The dispersion increases as the value of the parameter a increases at each point of the space domain. It is noticed that net concentration injected in presence of the source decreases with increase of parameter a . Moreover a drop in concentration is recorded for higher value of parameter a at the near end in absence of the source.

3.3 Verification of solution

In order to verify the solution we may consider the input function $F(m't) = \frac{1}{1+m't}$ in Eq. (4). This particular input may be solved for solution of advection-dispersion equation by method used in the present paper discussed above as well as the way given below:
The input boundary conditions with $F(m't) = \frac{1}{1+m't}$ may be described as

$$c(x,t) = c_i; \;\; t = 0, \; x \geq 0 \tag{A1}$$

$$-D(x,t)\frac{\partial c(x,t)}{\partial x}+u(x,t)c=\begin{cases}\frac{u_0c_0}{1+mt},\ 0<t\leq t_0\\ x=0\\ 0,\ t>t_0 \qquad\qquad (A2b)\end{cases} \tag{A2a}$$

$$\frac{\partial c(x,t)}{\partial x}=0;\ \text{ as }\ x\to\infty,\ \ t\geq 0. \tag{A3}$$

Here, $m=m'$

With transformation Eq. (10), Eqs. (A1-A3) may written as:

$$c(x,T)=c_i;\ \ T=0,\ x\geq 0 \tag{A4}$$

$$-D_0(1+ax)^2\frac{\partial c(x,T)}{\partial x}+u_0(1+ax)c=\begin{cases}u_0c_0,\ 0<T\leq T_0\\ x=0\\ 0,\ T>T_0 \qquad\qquad (A5b)\end{cases} \tag{A5a}$$

$$\frac{\partial c(x,T)}{\partial x}=0;\ \text{ as }\ x\to\infty,\ \ T\geq 0. \tag{A6}$$

With transformation Eq. (15), Eq.(A4-A6)may written as:

$$c(X,T)=c_i;\ \ T=0,\ X\geq 0 \tag{A7}$$

$$-D_0\frac{\partial c(X,T)}{\partial X}+u_0c=\begin{cases}u_0c_0,\ 0<T\leq T_0\\ X=0\\ 0,\ T>T_0 \qquad\qquad (A8b)\end{cases} \tag{A8a}$$

$$\frac{\partial c(X,T)}{\partial X}=0;\ \text{ as }\ X\to\infty,\ \ T\geq 0. \tag{A9}$$

Now, After using transformation Eq.(21) and then applying Laplace Transformation Technique, solution may be written as:

$$c(x,T)=[(c_0-\frac{\gamma_0}{\mu_1})H_{\rho\sigma}(X,T)-(c_i-\frac{\gamma_0}{\mu_1})H_{\rho\sigma}(X,T)+(c_i-\frac{\gamma_0}{\mu_1})exp(\rho^2T-\beta X)]\times$$
$$exp[\frac{w_0}{2D_0}X-\frac{1}{R}(\frac{w_0^2}{4D_0}+\mu_1)T]+\frac{\gamma_0}{\mu_1}\quad 0<T\leq T_0 \qquad\qquad (A10a)$$

$$c(X,T)=[(c_0[H_{\rho\sigma}(X,T)-H_{\rho\sigma}(X,T-T_0)exp(\eta^2T_0)])-\frac{\gamma_0}{\mu_1}H_{\rho\sigma}-(c_i-\frac{\gamma_0}{\mu_1})H_{\rho\sigma}(X,T)+(c_i-\frac{\gamma_0}{\mu_1})exp(\rho^2T-\beta X)]\times$$
$$exp[\frac{w_0}{2D_0}X-\frac{1}{R}(\frac{w_0^2}{4D_0}+\mu_1)T]+\frac{\gamma_0}{\mu_1}\quad 0<T\leq T_0 \qquad\qquad (A10b)$$

In order to compare the results obtained from solutions Eqs. $(30a\&30b)$ and Eqs. $(A10a\&A10b)$ with help of figures and tables, the values of constants and parameters are taken as: $c_0=1.0$,$c_i=0.01$, $u_0=0.03(km\ year^{-1})$ and $D_0=1.2(km^2year^{-1})$, $R=1.25, \mu_0(year^{-1})=0.05$, $\gamma_0=0.0007$, $a(km-1)=0.085$ and $m=m'=0.1$ and time of elimination of source $t_0(year)=9$.

Concentration patterns obtained from solution Eq.$(A10a\&A10b)$ are in good agreement with same obtained from the Eq.$(30a\&30b)$ as shown with help of figures 4(17&18). It verifies the authenticity of the obtained solution Eq.$(30a\&30b)$. Table 1 represents the error occurring in measurement of the concentration from solution Eqs.

(30a) in comparison to Eqs. (A10a)(i.e. Eqs. (30a)- Eqs. (A10a)) While Table 2 represents the same from solution Eqs. (30b) in comparison to Eqs. (A10b) (i.e. Eqs. (30b)-Eqs. (A10b)).

Concentration pattern obtained from solution Eq.(A10a& A10b) in appendix has been in good agreement with same obtained from the Eq.(30a &30b) as shown with help of figure (5&6). It verifies the authenticity of the obtained solution Eq.(30a &30b).

Table 1. Representing Error in Measurement in solute concentration from Solution Eq. (30a) Comparison to Solution Eq. (A10a) in Presence of Source.

Table 2. Representing Error in Measurement in solute concentration from Solution Eq. (30b) Comparison to Solution Eq. (A10b)

Errors measured in table 1 (presence of source) and table 2 (absence of source) are nominal in all the combination of time and space. Tables 1 &2 validates Eq. (30a) and Eq. (30b) at large extent.

4 Conclusion

In the present study, solute transport phenomena in one-dimensional semi-infinite porous media has been studied for a varying pulse type input source. Solution maintains the trends of such varying nature inputs in which concentration of the solute does not reduce to zero at origin as the source eliminated rather the same decreases gradually. Laplace transformation technique is employed to get the solution of the present problem. Two different time dependent functions taken as input concentration in numerical examples are feasible.

- The obtained result predicts the concentration profile for non-reactive contaminants and an appropriate interpolation method provides optimum result.
- Authenticity of the solution has been provided with help of a particular input.
- The solution may help to determine minimum/maximum or harmless concentration at any position and time. Generalization of the input source as any function smooth function that acts up to certain duration has enhanced utility of the solution.
- Model may be helpful to validate the some other solute transport problem.
- The concentration pattern depicting the solute transport with varying input phenomena provides the different trend in absence and presence of input comparison to uniform input source. Concentration presence in domain may be noticed less in comparison to uniform input.

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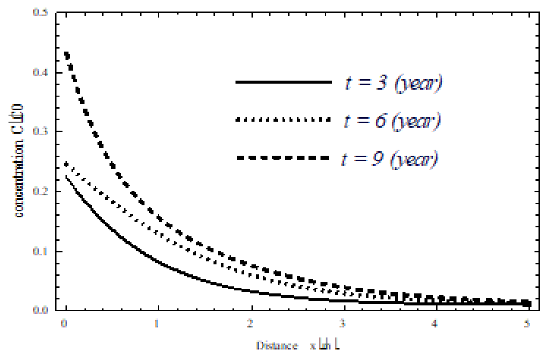

Fig.1. Distribution of dimensionless concentration for various time $(0 \le t \le t_0)$.

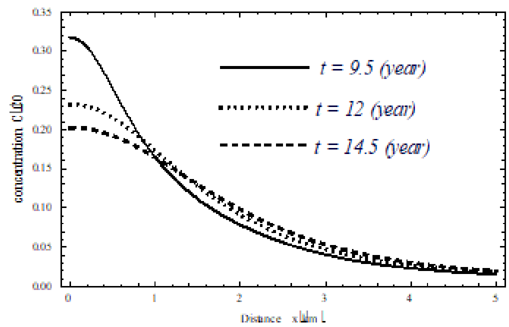

Fig.2. Distribution of dimensionless concentration for various time $(t_0 < t)$.

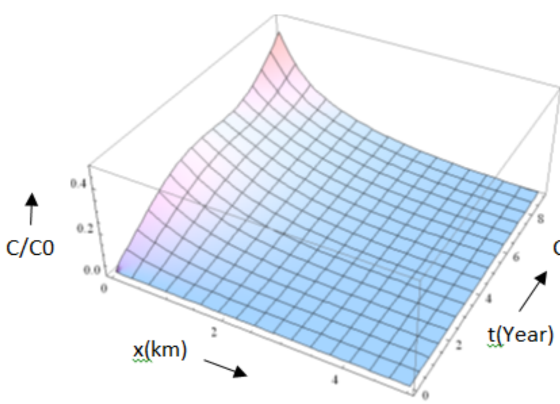

Fig.3. Surface Plot in presence of source i.e. $(0 \le t \le t_0)$.

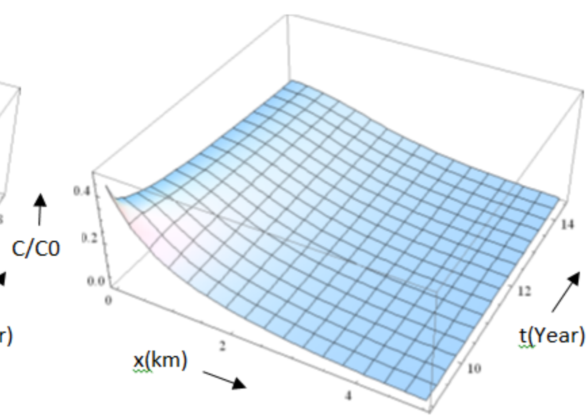

Fig.4. Surface plot in absence of source i.e. $(t_0 < t)$.

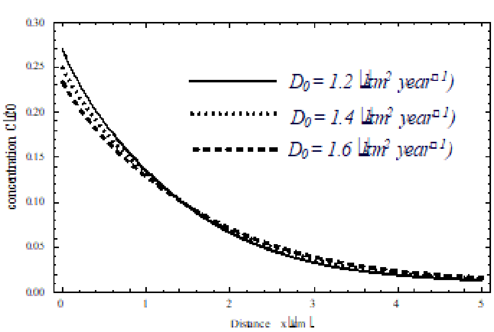

Fig.5. Distribution of dimensionless concentration for various dispersion coefficient at time $t(year) = 7$ $(0 \le t \le t_0)$.

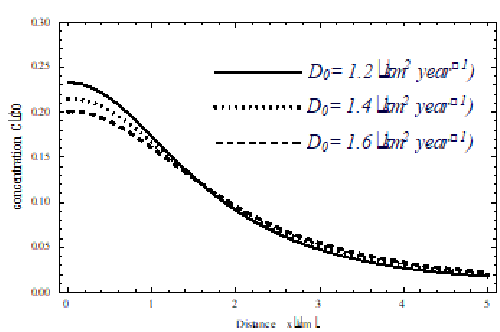

Fig.6. Distribution of dimensionless concentration for various dispersion coefficient at time $t(year) = 12$ $(t_0 < t)$.

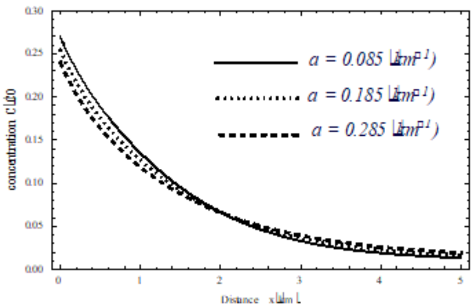

Fig.7. Distribution of dimensionless concentration for various value of parameter a at time $t(year) = 7$ $(0 \le t \le t_0)$.

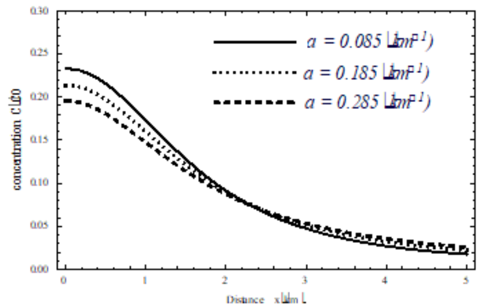

Fig.8. Distribution of dimensionless concentration for various value of parameter a at time $t(year) = 12$ $(t_0 < t)$.

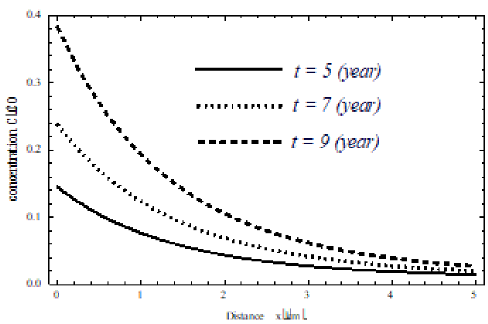

Fig.9. Distribution of dimensionless concentration for various time $(0 \le t \le t_0)$.

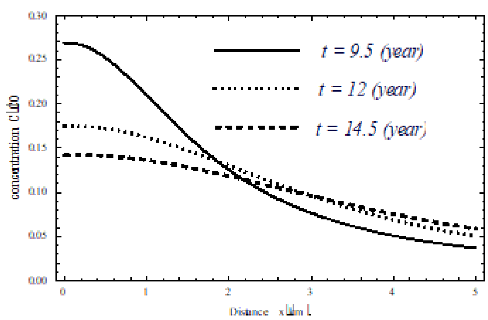

Fig.10. Distribution of dimensionless concentration for various time $(t_0 < t)$.

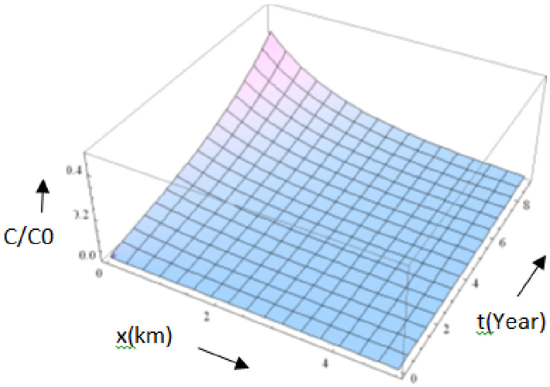

Fig.11. Surface Plot in presence of source i.e. $(0 \le t \le t_0)$.

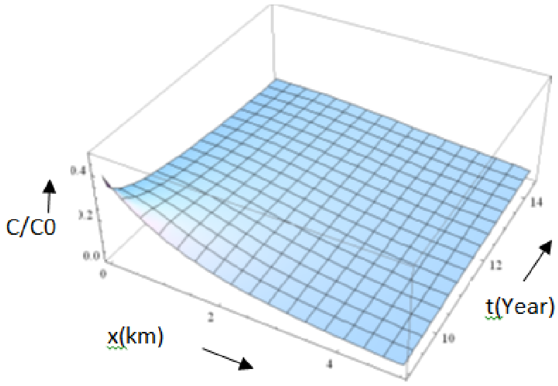

Fig.12. Surface plot in absence of source i.e. $(t_0 < t)$.

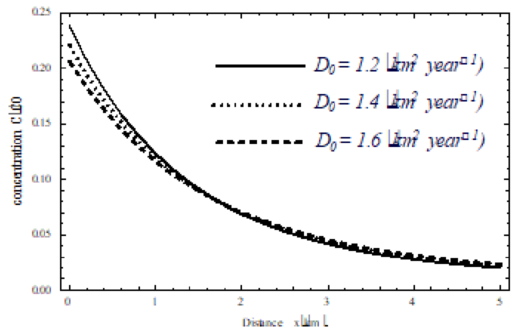

Fig.13. Distribution of dimensionless concentration for various dispersion coefficient at time $t(year) = 7\ (0 \le t \le t_0)$.

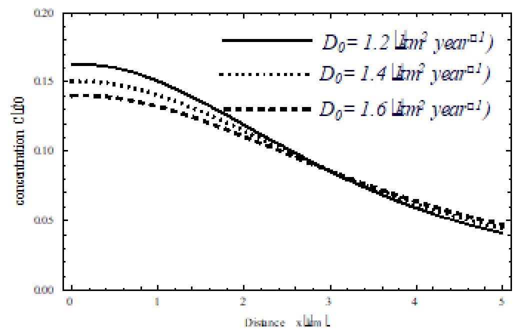

Fig.14. Distribution of dimensionless concentration for various dispersion coefficient at time $t(year) = 12\ (t_0 < t)$.

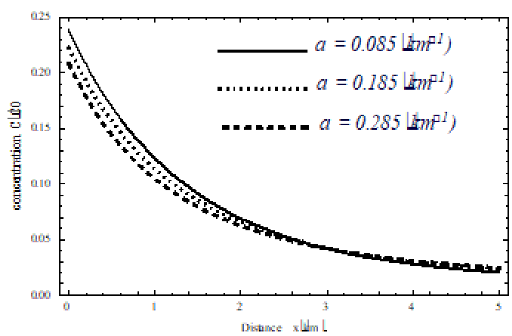

Fig.15. Distribution of dimensionless concentration for various value of parameter a at time $t(year) = 7\ (0 \le t \le t_0)$.

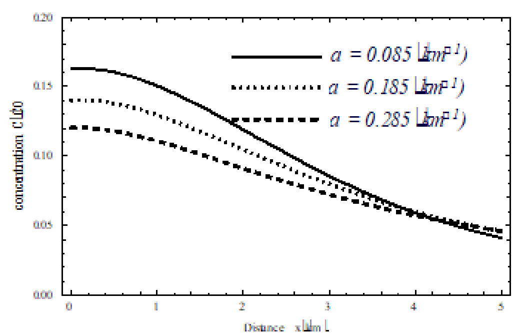

Fig.16. Distribution of dimensionless concentration for various value of parameter a at time $t(year) = 12\ (t_0 < t)$.

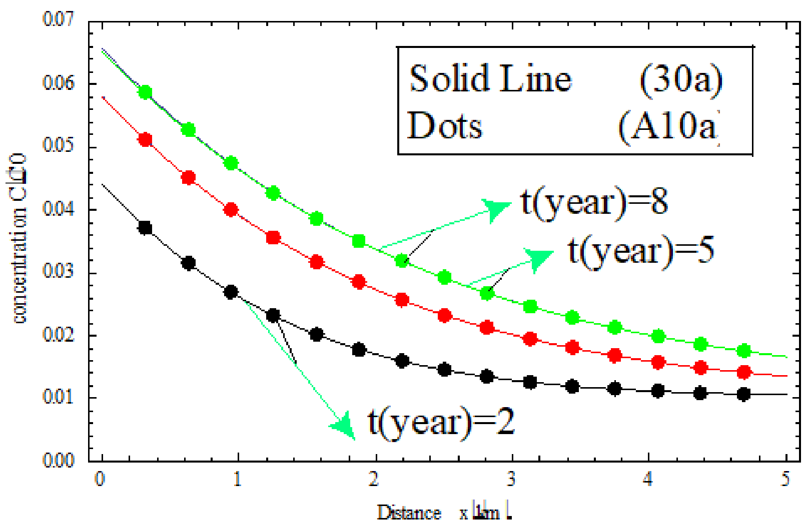

Fig.17. Distribution of dimensionless concentration for various times in presence of source

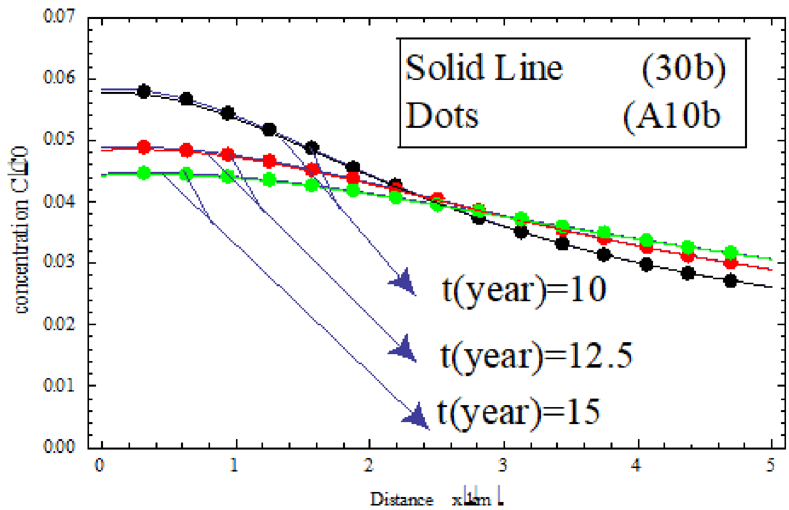

Fig. 18. Distribution of dimensionless concentration for various times in absence of source

Table 1. Representing Error in Measurement in solute concentration from Solution Eq. (30a) Comparison to Solution Eq. (A10a) in Presence of Source.

t(yr) \ x(km)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.001	-9.3570×10^{-15}	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
1.0	-6.9294×10^{-13}	-4.5831×10^{-13}	-2.9292×10^{-13}	-1.8038×10^{-13}	-1.0200×10^{-13}	-5.3118×10^{-14}	-2.5373×10^{-14}	-1.1221×10^{-14}	-4.6559×10^{-15}	-1.8336×10^{-15}	-6.8348×10^{-16}
2.0	-6.1167×10^{-12}	-1.1380×10^{-12}	-5.3846×10^{-13}	-3.3445×10^{-13}	-2.1103×10^{-13}	-1.3472×10^{-13}	-8.6730×10^{-14}	-5.5476×10^{-14}	-3.5076×10^{-14}	-2.1741×10^{-14}	-1.3222×10^{-14}
3.0	-1.1931×10^{-8}	-3.3327×10^{-9}	-9.3589×10^{-10}	-2.6390×10^{-10}	-7.4684×10^{-11}	-2.1234×10^{-11}	-6.0921×10^{-12}	-1.7845×10^{-12}	-5.4734×10^{-13}	-1.8455×10^{-13}	-7.2360×10^{-14}
4.0	-4.1932×10^{-7}	-1.4747×10^{-7}	-5.2709×10^{-8}	-1.9120×10^{-8}	-7.0317×10^{-9}	-2.6188×10^{-9}	-9.8687×10^{-10}	-3.7599×10^{-10}	-1.4474×10^{-10}	-5.6269×10^{-11}	-2.2083×10^{-11}
5.0	−0.0000 0479826	−0.0000 0189780	-7.6493×10^{-7}	-3.1376×10^{-7}	-1.3082×10^{-7}	-5.5379×10^{-8}	-2.3780×10^{-8}	-1.0349×10^{-8}	-4.5609×10^{-9}	-2.0341×10^{-9}	-9.1748×10^{-10}
6.0	−0.0000 3101066	−0.0000 1321027	−0.0000 0574062	−0.0000 0254117	−0.0000 0114443	-5.2377×10^{-7}	-2.4335×10^{-7}	-1.1468×10^{-7}	-5.4774×10^{-8}	-2.6493×10^{-8}	-1.2968×10^{-8}
7.0	−0.0001 1030795	−0.0000 3285397	−0.000 0289818	−0.0000 1352542	−0.0000 0642417	−0.0000 0310188	−0.0000 1520989	-7.5668×10^{-7}	-3.8161×10^{-7}	-1.9495×10^{-7}	-1.0081×10^{-7}
8.0	−0.0005 049	−0.0002 3554101	−0.0001 1215631	−0.0000 5443211	−0.0000 2689010	−0.0000 1350603	−0.0000 0688978	−0.0000 0356628	−0.0000 0187148	-9.9490×10^{-7}	-5.3542×10^{-7}
9.0	−0.0015 1943771	−0.0007 3054954	−0.0003 5855862	−0.0002 3782482	−0.0001 1349274	−0.0000 4729919	−0.0000 4874580	−0.0000 3273848	−0.0000 0718128	−0.0000 3935787	−0.0000 1836123

Table 2. Representing Error in Measurement in solute concentration from Solution Eq. (30b) Comparison to Solution Eq. (A10b)

t(yr) \ x(km)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
9.001	−0.0014 5532318	−0.0007 3606561	−0.0003 5121486	−0.0001 8056856	−0.0000 9205756	−0.0000 7715789	−0.0000 2511843	−0.0000 1341647	−0.0000 0726485	−0.0000 0398491	−0.0000 2212625
9.5	−0.0007 0369261	−0.0006 6115054	−0.0004 0114545	−0.0002 9955507	−0.0001 4927688	−0.0000 3155648	−0.0000 4458210	−0.0000 2775373	−0.0000 3243758	−0.0000 0735771	−0.0000 3766250
10.0	−0.0005 5718711	−0.0005 5385432	−0.0004 3034635	−0.0003 4022907	−0.0002 9393996	−0.0001 2286228	−0.0000 7139219	−0.0000 7200926	−0.0000 2950050	−0.0000 2999614	−0.0000 0741760
11.0	−0.0004 5208615	−0.0004 3453530	−0.0003 8053715	−0.0003 8025303	−0.0002 3332330	−0.0001 6721661	−0.0001 1451489	−0.0000 7561914	−0.0000 8546271	−0.0000 3052398	−0.0000 8919795
12.0	−0.0003 7640836	−0.0003 6665718	−0.0003 3211295	−0.0002 8541393	−0.0002 3236988	−0.0001 1120589	−0.0001 3598510	−0.0000 9890367	−0.0000 7002551	−0.0000 4847895	−0.0000 2949317
13.0	−0.0003 2750296	−0.0003 1371021	−0.0002 9809853	−0.0002 5274336	−0.0002 2350247	−0.0001 8286797	−0.0001 5028025	−0.0001 1194766	−0.0000 4411196	−0.0000 6237372	−0.0000 2965433
14.0	−0.0002 9253895	−0.0002 8841125	−0.0002 7105171	−0.0002 4452033	−0.0002 1292368	−0.0001 7979976	−0.0001 4781321	−0.0001 1870289	−0.0000 3391680	−0.0000 7217280	−0.0000 5490957