One-Dimensional Solute Transport in a Heterogeneous Porous Media with Pulse Type Input Source

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ABSTRACT. An advection-dispersion equation subjected to temporally and spatially dependent groundwater velocity, dispersion coefficient along with a time dependent pulse type input of time varying nature is solved for one dimensional semi-infinite porous medium. Input concentration is any continuous smooth function of time acts up to some finite time and then eliminated. Concentration gradient at other boundary domain is considered zero. Dispersion is linearly and squarely proportional to groundwater velocity in temporal and spatial measurement respectively. Initially, medium is uniformly polluted. Interpolation method is applied to reduce the input function into a polynomial. Using certain transformations the advection-dispersion equation is reduced to constant coefficient and freed from convective part then Laplace transform technique is applied to get the solution of advection-dispersion equation. Two different functions of input are discussed to understand the utility of the present study. Obtained result is demonstrated graphically with the help of numerical example.

1 Introduction

Mathematical models are useful tool to realize level of contaminant presence in a geological formation like aquifers including site/source of contaminant and also to predict the future possibilities about toxicity of water bodies due to presence of harmful contaminant introduced to water body by some natural phenomena or manmade activity. With growing population and heavy congestion of industries, challenges to meet the demand of harmless/ non

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toxic water have gone tough. Several analytical / numerical solutions are present in literatures up to the date for dealing such challenges with help of science. For finite/semi-infinite porous medium of type homogeneous, heterogeneous, isotropic and anisotropic, a plethora of analytical solutions for one, two & three-dimensional solute transport including a variety of input and boundary conditions have been presented. [1] observed the concentration distribution of pollutant arising from instantaneous point source in a two-dimensional water channel with non-uniform velocity distribution. Analytical solutions of the general one-dimensional solute transport model including distributed first-order decay, linear equilibrium sorption and a finite length input source for confined aquifers is presented by[2]. For hydrodynamic dispersion including longitudinal and transverse dispersion along with time and space-dependent sources consisting of the first-type boundary condition at the source, A generalized two-dimensional analytical solution is developed by using Laplace transform and Fourier analysis techniques simultaneously. The effect of scale dependency is studied with asymptotically scale dependent dispersion [3]. For a semi-infinite aquifer having uniform and spatially variable permeability, analytical and numerical solutions for unsteady ground water velocity is obtained by [4]. Analytical solution for space dependent dispersion coefficient and velocity presented by [5, 6]. Dispersion is taken proportional to groundwater velocity while groundwater velocity is linear to space. In latter one instantaneous input is considered to demonstrate the problem. [7] proposed the analytical and numerical models for the groundwater flow in the multi aquifer system. For a two-dimensional non-axis symmetric solute transport in a radially convergent flow field an analytical power series solution which is useful for quantitative hydro-geologic problems was attained using Laplace-transformed power series (LTPS) technique [8]. A study of solute transport phenomena in aquifers consisting of analytical and numerical solutions was presented by [9]. [10] studied solute transport phenomena under an imposed standing water surface wave. They obtained two close solutions of the advection diffusion equation by solute transport into a sub-aqueous sediment bed there. Effect of time dependent source and velocity are studied for solute transport phenomena in porous media in [11] and [12] respectively. Two dimensional solutions of advection-dispersion equation were proposed [13-15] and later established the solutions using cylindrical coordinate system. [16] presented an analytical approach to solute dispersion along and against transient groundwater flow in a homogeneous finite aquifer with pulse type boundary conditions. An analytical solution of advection dispersion equation (ADE) along with decay of first order for multi-layered media using the classical integral transform technique (CITT) method is obtained [17]. [18] studied finite domain solute transport phenomena considering sinusoidal dispersion and ground water velocity for homogeneous porous media along with sinusoidal input source. An analytical solution for calculating the space-time variation of contaminant concentration in one-dimensional transient groundwater flow in a homogenous semi-infinite aquifer along with a dispersion proportional to the square of the velocity and a uniform and time varying input along transient groundwater flow is investigated by [19]. Solutions like pulse type including different geological situations are developed to measure the concentration level in presence and absence of the source. A pulse type input source concentration is considered at the intermediate portion in the aquifer different from origin for two-dimensional solute dispersion in saturated porous media and this model of dispersion problem was solved analytically by using the Laplace Transform Technique (LTT) and numerically with the help of Explicit Finite Difference (EFD) method [20]. Greens Function Method and pertinent coordinate transformation method are employed to get Analytical solutions for solute transport in
groundwater and riverine flow [21].

In this paper, a theoretical model is developed for the dispersion problem for one-dimensional heterogeneous porous medium. Dispersion coefficient and seepage velocity are function of time and space. Dependency of dispersion coefficient and ground water velocity on space enhances the proximity of the present problem to real world problem. Initially, medium is not solute free which means it is laden with some concentration. Application of the present problem is broadened with an input that is of varying nature and is any smooth function of time acts up to certain time and then eliminated. Laplace transform technique is used to get the solution of present study. Utility of the solution is illustrated with two cases of inputs.

2 Mathematical Formulation of the Problem

The one-dimensional advection-dispersion equation which is second order parabolic equation and derived on basis of mass conservation and Ficks law of diffusion may be written as follows [22],

\[
R \frac{dc}{dt} = \frac{\partial}{\partial x} (D(x,t) \frac{\partial c}{\partial x}) - u(x,t)c - \mu(x,t)c + \gamma(x,t)
\]

In which, \( c \) [ML\(^{-3}\)] is the solute concentration, \( u[LT^{-1}] \) refers to groundwater velocity, \( D[L^2T^{-1}] \) denotes dispersion coefficient and \( R \) is retardation factor, which is a dimensionless quantity. The dispersion coefficient \( D(x,t) \) is proportional to flow velocity \( u(x,t) \) while only temporal nature is considered and square of it for spatial nature and assumed respectively as; \( u(x,t) = u_0(1 + ax)(1 + m t)^{-1}, D(x,t) = D_0(1 + ax)^2(1 + m t)^{-1} \). Zero order production \( \gamma(x,t)[ML^{-3}T^{-1}] \) and first order decay \( \mu(x,t)[T^{-1}] \) are defined respectively as: \( \gamma(x,t) = \gamma_0(1 + m t)^{-1} \) and \( \mu(x,t) = \mu_0(1 + m t)^{-1} \), where \( D_0[L^2T^{-1}], u_0[LT^{-1}], \mu_0[T^{-1}], \gamma_0[ML^{-3}T^{-1}] \) and \( R \) are constants. \( m[T^{-1}] \) and \( a[L^{-1}] \) are the parameters which regulate the time and space dependency of the dispersion and the groundwater velocity respectively.

Therefore, Eq.(1) may be re-written as;

\[
R \frac{dc}{dt} = \frac{\partial}{\partial x} (D_0(1 + m t)^{-1}(1 + ax)^2 \frac{\partial c}{\partial x}) - u_0(1 + m t)^{-1}(1 + ax)c - \mu_0(1 + m t)^{-1}c + \gamma_0(1 + m t)^{-1}
\]

(2)

The geological formation is assumed of semi-infinite length along horizontal direction. It is initially uniformly polluted. The input source at one end is of varying nature and defined by any smooth function of time applied in the direction of the flow and concentration gradient at other end of domain is considered zero, which is also called flux type boundary condition. In order to formulate the proposed problem mathematically, the initial and boundary conditions may be written as:

\[
c(x,t) = c_i; \quad t = 0, x \geq 0
\]

\[
-D(x,t) \frac{\partial c(x,t)}{\partial x} + u(x,t)c = \begin{cases} u_0c_0F(t), & 0 < t \leq t_0 \\ x = 0 \\ 0, & t > t_0 \end{cases}
\]

(4)
\[
\frac{\partial c(x, t)}{\partial x} = 0; \quad \text{as } x \to \infty, t \geq 0; \quad (5)
\]
where, \(c_0\) and \(c_i\) represent reference concentrations and \(m'(T^{-1})\) is an unsteady parameter. Dimensionless \(F(m't)\) is smooth continuous and bounded function in a time domain \([0, t_0]\). Weirstrass approximation theorem says that any continuous function on a bounded interval can be uniformly approximated by polynomial function. Since \(F(m't)\) is continuous and bounded in domain \([0, t_0]\), Following Weirstrass approximation theorem, \(F(m't)\) may be written as a polynomial in \(t\) of degree \(n\). Hence the proposed initial and boundary conditions Eqs. (3-5) may be written as:
\[
c(x, t) = c_i; \quad t = 0, x \geq 0 \quad (6)
\]
\[
-D(x, t) \frac{\partial c(x, t)}{\partial x} + u(x, t)c = \begin{cases} u_0c_0G_n(t), & 0 < t \leq t_0 \\ x = 0 \\ 0, \ t > t_0 \end{cases} \quad (7)
\]
\[
\frac{\partial c(x, t)}{\partial x} = 0; \quad \text{as } x \to \infty, t \geq 0; \quad (8)
\]
where, \(G_n(t) = a_0 + a_1t + a_2t^2 + ... + a_nt^n\) \( (9)\)

Here \(G_n(t)\) is dimensionless and \(a_i's\) are constant.

In order to eliminate the time dependent factor from the coefficient of advection-dispersion equation. A new time variable \(T\) is introduced as; \([23]\),
\[
T = \int_0^1 (1 + mt)^{-1} dt \quad \text{or} \quad T = \frac{1}{m} \log(1 + mt) \quad \text{and} \quad t = \frac{1}{m}(e^{mt} - 1) \quad (10)
\]
With this transformation Eq. (10), Eq. (2) and Eqs. (6-8) are reduced into new time variable \(T\).
\[
R \frac{\partial c}{\partial T} = \frac{\partial}{\partial x}(D_0(1 + ax)^2 \frac{\partial c}{\partial x} - u_0(1 + ax)c - \mu_0c + \gamma_0 \quad (11)
\]
\[
c(x, T) = c_i; \quad T = 0, \ x \geq 0 \quad (12)
\]
\[
-D_0(1 + ax)^2 \frac{\partial c(x, T)}{\partial x} + u_0(1 + ax)c = \begin{cases} u_0c_0H(t)exp(mT), & 0 < T \leq T_0 \\ x = 0 \\ 0, \ T > T_0 \end{cases} \quad (13)
\]
\[
\frac{\partial c(x, T)}{\partial x} = 0; \quad \text{as } x \to \infty, T \geq 0; \quad (14)
\]
where \(H(T) = b_0e^{mT} + b_1e^{2mT} + b_2e^{3mT} + ... + b_n e^{(n+1)mT} b/s\) are constants and \(T_0 = \frac{1}{m} \log(1 + mt_0)\). In order to reduce advection-dispersion equation in to constant coefficient, we introduce a new space variable \(X\) \([24]\).
\[
X = \frac{1}{a} \log(1 + ax) \quad (15)
\]
also,
\[
x = \frac{1}{a} \{\exp(ax) - 1\} \quad (16)
\]
With this new space variable in Eq. (15) and Eqs. (11-14) may be written as:

\[ R_0 \frac{dc}{dT} = D_0 \frac{d^2 c}{dX^2} - \left( u_0 - D_0 a \right) \frac{dc}{dX} - (\mu_0 + \mu_0 a) c + \gamma_0 \]  

or

\[ R_0 \frac{dc}{dT} = D_0 \frac{d^2 c}{dX^2} - w_0 \frac{dc}{dX} - \mu_1 c + \gamma_0 \]  

\[ c(X, T) = c_T; \ T = 0, \ X \geq 0 \]  

\[-D_0 \frac{dc(X, T)}{dX} + u_0 c = \begin{cases} 
\frac{u_0 C_0 H(T) exp(mT)}{2 D_0} & 0 < T \leq T_0 \\
X = 0 & T > T_0 
\end{cases} \]  

\[ \frac{dc(X, T)}{dX} = 0; \ as \ x \to \infty, \ T \geq 0; \]  

where \( w_0 = u_0 - D_0 a \) and \( \mu_1 = \mu_0 + u_0 a \).

\( H(T) = b_0 + b_1 e^{mT} + ... + b_{n-1} e^{(n-1)mT} + b_n e^{nT} \) Now using following transformation, convective term in advection-dispersion Eq. (17) is removed. Transformation is defined as:

\[ c(X, T) = k(X, T) \exp \left[ \frac{w_0}{2 D_0} X - \frac{1}{R} \left( \frac{w_0}{4 D_0} + \mu_1 \right) T \right] + \frac{\gamma_0}{\mu_1} \]  

Where \( k(X, T) \) is a new dependent variable. With this new transformation Eqs. (17-20) reduce into:

\[ R_0 \frac{dk}{dT} = D_0 \frac{d^2 k}{dX^2} \]  

\[ k(X, T) = \left( c_1 - \frac{\gamma_0}{\mu_1} \right) \exp(-\beta X); \ T = 0, \ X \geq 0 \]  

\[-D_0 \frac{dk(X, T)}{dX} + \left( u_0 - \frac{w_0}{2} \right) k = \begin{cases} 
\frac{u_0 C_0 H(T) \exp(\eta^2 T)}{2 D_0} & 0 < T \leq T_0 \\
X = 0 & T > T_0 
\end{cases} \]  

\[ \frac{dk(X, T)}{dX} + \frac{w_0}{2 D_0} k(X, T) = 0; \ as \ X \to \infty, \ T \geq 0 \]  

where \( \beta = \frac{w_0}{2 D_0} \) and \( \eta = \sqrt{\frac{1}{R} \left( \frac{w_0}{4 D_0} + \mu_1 \right)} \).

Applying Laplace transformations on Eqs. (22-25), mentioned equations reduce into following boundary problem of ordinary differential equation.

\[ \frac{d^2 k}{dX^2} - \frac{p R}{D_0} \frac{dk}{dX} = \frac{R}{D_0} \left[ c_1 - \frac{\gamma_0}{\mu_1} \right] \exp(-\beta X) \]  

\[-D_0 \frac{dk}{dX} + \left( u_0 - \frac{w_0}{2} \right) k = -\frac{u_0 C_0}{\mu_1} \frac{[1 - \exp(-p - \eta^2)] T_0}{(p - \eta^2)} \]  

\[ + u_0 C_0 b_0 \frac{[1 - \exp(-(p - (\eta^2 + m)) T_0)]}{(p - (\eta^2 + m))} \]  

\[ + u_0 C_0 b_1 \frac{[1 - \exp(-(p - (\eta^2 + 2m)) T_0)]}{(p - (\eta^2 + 2m))} \]  

\[ + ... + u_0 C_0 b_n \frac{[1 - \exp(-(p - (\eta^2 + (n+1)m)) T_0)]}{(p - (\eta^2 + nm))} - \frac{u_0 C_0}{\mu_1} \frac{[\exp(-(p - \eta^2)] T_0)}{(p - \eta^2)} , \ X = 0 \]
\[
\frac{dK}{dX} + \frac{u_0}{2D_0} \frac{dK}{dx} = 0, \quad \text{as} \quad X \to \infty
\]  
(28)

where, \( f(x, p) = L\{f(x, p)\} = \int_0^\infty e^{-pt} f(x, t) dt, \ p > 0 \) where \( P \) is a Laplace parameter.

Here, solution of Eq. (26), by using Eq. (27) and Eq. (28) may be written as:

\[
k(X, p) = -\frac{u_0\mu_0}{\sqrt{R\sigma}D_0\gamma_0} \left[ 1 - \exp\left(-\frac{p}{\sqrt{\sigma}}\right) \right] \times \exp\left(-\frac{1}{\sqrt{D_0}} X\right) + T \right] + \frac{\gamma_0}{\mu_0} \frac{1}{\sqrt{D_0}} X \right] \times \exp\left(-\frac{1}{\sqrt{D_0}} X\right) + \left( c_i - \frac{\gamma_0}{\mu_0} \right) \left( \frac{p - \rho^2}{\rho^2} \right) \times \exp\left(-\frac{1}{\sqrt{D_0}} X\right) + \left( c_i - \frac{\gamma_0}{\mu_0} \right) \left( \frac{p - \rho^2}{\rho^2} \right) \times \exp\left(-\frac{1}{\sqrt{D_0}} X\right)
\]

where \( \rho = \sqrt{\frac{1}{R} \beta^2} \) and \( \sigma = \frac{1}{\sqrt{\gamma_0}} (u_0 - \frac{\mu_0}{\gamma_0}) \).

Solution of present problem is generated by taking inverse Laplace Transform of Eq. (29) and then applying the transformation Eq. (21). Hence solution can be written as:

\[
c(X, T) = \left[ -\frac{u_0\gamma_0}{\sqrt{R\sigma}D_0\mu_1} H_{\mu_0}(X, T) + \frac{u_0\gamma_0}{\sqrt{R\sigma}D_0\mu_1} H_{\sqrt{\eta^2 + \mu_0}}(X, T) + \ldots \right.
\]

\[
+ \frac{u_0\gamma_0}{\sqrt{R\sigma}D_0\mu_1} H_{\sqrt{\eta^2 + \mu_0}}(X, T) - \frac{u_0\gamma_0}{\sqrt{R\sigma}D_0\mu_1} (c_i - \frac{\gamma_0}{\mu_0}) H_{\rho^2}(X, T) + \left( c_i - \frac{\gamma_0}{\mu_0} \right) \exp(\rho^2 T - \beta X))
\]

(30a)

\[
c(X, T) = \left[ -\frac{u_0\gamma_0}{\sqrt{R\sigma}D_0\mu_1} H_{\mu_0}(X, T) + \frac{u_0\gamma_0}{\sqrt{R\sigma}D_0\mu_1} H_{\sqrt{\eta^2 + \mu_0}}(X, T) - H_{\sqrt{\eta^2 + \mu_0}}(X, T - T_0) \exp(\eta^2 + m)T_0 \right.
\]

\[
+ \frac{u_0\gamma_0}{\sqrt{R\sigma}D_0\mu_1} H_{\sqrt{\eta^2 + \mu_0}}(X, T) - H_{\sqrt{\eta^2 + \mu_0}}(X, T - T_0) \exp(\eta^2 + 2m)T_0 \right.
\]

\[
+ \ldots
\]

\[
\left. + \frac{u_0\gamma_0}{\sqrt{R\sigma}D_0\mu_1} H_{\sqrt{\eta^2 + \mu_0}}(X, T) - H_{\sqrt{\eta^2 + \mu_0}}(X, T - T_0) \exp(\eta^2 + m)T_0 \right.
\]

\[
- \frac{u_0}{\sqrt{R\sigma}D_0}(c_i - \frac{\gamma_0}{\mu_1}) H_{\rho^2}(X, T) + \left( c_i - \frac{\gamma_0}{\mu_1} \right) \exp(\rho^2 T - \beta X))
\]

(30b)

\[
\exp\left(\frac{u_0}{\sqrt{D_0}} X - \frac{1}{\sqrt{D_0}} \left( \frac{u_0}{\mu_1} \right) + \frac{\gamma_0}{\mu_1} \right), \quad T_0 < T
\]

where value of \( H_{\phi}(X, T) \) is given as [26]:

\[
H_{\phi}(X, T) = \frac{1}{2\sqrt{\pi \rho^2}} \exp(\phi^2 T - \phi \sqrt{T}) \text{erfc} \left( \frac{X}{2\sqrt{\rho^2 t}} \right) - \frac{1}{2\sqrt{\pi \rho^2}} \exp(\phi^2 T + \phi \sqrt{T}) \text{erfc} \left( \frac{X}{2\sqrt{\rho^2 t}} \right) + \frac{\phi}{(\rho^2 - \rho^2)^2} \exp\left(\frac{X^2}{2\rho^2 t} \right) \text{erfc} \left( \frac{X}{2\sqrt{\rho^2 t}} + 2 \sqrt{T} \right) + \frac{\phi}{(\rho^2 - \rho^2)^2} \exp\left(\frac{X^2}{2\rho^2 t} \right) \text{erfc} \left( \frac{X}{2\sqrt{\rho^2 t}} + 2 \sqrt{T} \right)
\]
where \( \rho = \sqrt{\frac{D_0}{\pi}} \).

3 Results and discussions

The obtained solution Eqs. (30a, b) are elaborated with two forms of input function, namely \( F(m't) = c_0(l + \sin(m't)) \) (sinusoidal) and \( F(m't) = c_0 \exp(m't) \) (exponential) for a chosen set of data taken from the experimental and theoretical published literature. In former \( F(m't) \), \( l \) is a constant and can be chosen such that input concentration never takes negative value. In order to evaluate the concentration values \( c/c_0 \) reference concentrations \( c_0, c_i \), are assumed 1.0, 0.01 respectively. The figures are plotted for finite region \( 0 < x(km) \leq 5 \) of semi-infinite domain. Presence of Source is assumed up to time \( t_0 \) (year) = 9. For figures 1, 2, 3, 4, 9, 10, 11, 12 the values of initial parameter and dispersion coefficient \( D_0 \) taken 0.085 (km\(^{-1}\)) and 1.2 (km\(^2\)year\(^{-1}\)) respectively. Groundwater velocity depending upon geometrical conditions of porous medium is taken \( u_0 = 0.03 \) (km year\(^{-1}\)) lying between 2m/day to 2m/year [25]. Values of other common parameters described above are taken as: \( R_0 = 1.25, \mu \) (year\(^{-1}\)) = 0.05, and \( \gamma_0 = 0.0007 \). Hermite interpolation method is used to get the polynomial \( G_n(t) \) as mentioned in Eq. (9) in following two cases.

3.1 Case I - When input is in the form \( F(m't) = (l + \sin(m't)) \)

For the present case, the value of unsteady parameter \( m', m \) and parameter \( l \) are taken 0.8, 0.8 and 2 respectively. Concentration value are computed at times \( t \) (year) = 3, 6 and 9 in the time domain \( 0 \leq t \) (year) \leq 9. While for \( t \) (year) > 9 (\( t_0 \)) the same is computed at times \( t \) (year) = 9.5, 12 and 14.5. Fig. 1 depicts the concentration pattern in presence of the source. At \( x(km) = 0 \) concentration level increases with different rates. It is because of the periodic nature of \( F(m't) \). At all the three levels of time concentration attenuates up to the distance \( x(km) = 5 \).

Fig. 2 illustrates the concentration profiles obtained at \( t \) (year) = 9.5, 12 and 14.5 from the solution in Eq. (30b), i.e., in the time domain \( t > 9 \). Unlike uniform nature input, presence of concentration at \( x(km) = 0 \) even after elimination of the source ascertains the behaviour of varying nature input. A steep decrement in level of concentration recorded in duration 9.5 (years) to 12 (years) in comparison to duration 12 (years) to 14.5 (years) at \( x(km) = 0 \).

Surface plots are presented to gauge the concentration pattern in every time and space combination of the proposed time and space domain in presence and absence of the source for fixed dispersion coefficient \( D_0 = 1.2 \) (km\(^2\)year\(^{-1}\)) and parameter \( a = 0.085 \) (km\(^{-1}\)). In Fig. 3 i.e. in presence of the source a smooth wave like phenomena at \( x = 0 \) with varying time may be observed. It is due to sinusoidal nature of input. Also throughout the space domain concentration increases with increase of time. In Fig. 4 at \( x(km) = 0 \) concentration drops fast in very first quarter of year just after elimination of source then after it decreases slowly in coming years.

Fig. 5 and Fig. 6 are drawn to study the effect of dispersion coefficient in presence and absence of the source respectively for fixed \( a = 0.085 \) (km\(^{-1}\)) and Concentration attenuates slight fast as the dispersion coefficient increases in both presence and absence of source. Trend of attenuation with space remains same in both cases for different values of dispersion.

The effect of parameter \( a \) is demonstrated from Fig. 7 and Fig. 8 in presence and absence of the source re-
respectively for fixed dispersion coefficient $D_0 = 1.2(km^2/year^{-1})$. As the value of the parameter $a$ increases the dispersion at each point of the space domain increases, the net concentration injected in to the domain in the varying input for a particular time decreases in presence of the source. And in absence of the source, the concentration level decreases almost in first half of the space domain with increases of the value parameter $a$. Concentration pattern remains same for all the three values of $a$ in both presence and absence of the source.

### 3.2 Case II- When value of input is $F(m't) = exp(m't)$

The unsteady parameters $m'$ and $m$ are 0.2 and 0.1 respectively for the present case. Concentration value in the time domain $0 \leq t(year) \leq 9$ are computed at times $t(year) = 5.7$ and $9$. While for $t(year) > 9 (= t_0)$ the same is computed at times $t(year) = 9.5, 12$ and $14.5$. Fig.9 displays the solute concentration in the presence of the source at times $t(year) = 5.7$ and $9$ in the time domain $(0 \leq t \leq t_0)$. The input concentration, $c/c_0$ at the origin, $x(km) = 0$ are $0.14, 0.24$ and $0.38$ respectively. Almost equal increment in injected input at $x(km) = 0$ may be recorded. But concentration attenuates almost up to same level for all three values of times. Concentration pattern in absence of the source in depicted from Fig. 10. at three values time $t(year) = 9.5, 12$ and $14.5$. It is recorded that concentration at near end decreases sharply as time elapses. And it spreads fast away from the source end.

Surface plots of concentration are obtained in presence and absence of the source in Fig. 11 and Fig. 12 respectively for fixed dispersion coefficient $D_0 = 1.2(km^2/year^{-1})$ and parameter $a = 0.085(km^{-1})$. Unlike Fig 3., a continuous increase in concentration with time may be observed from Fig. 11 in presence of the source at each point of the space. As the source eliminated concentration at near end drops fast but rate of diminution slows down with time. In contrast at the far end of our study domain concentration level increase with time. Effect of dispersion for current case of $F(m't)$ has shown from Fig. 13 and Fig. 14 in presence and absence of the source respectively for fixed $a = 0.085(km^{-1})$. Pattern of variations in concentration level with dispersion are alike the same of previous case of $F(m't)$ in both presence and absence of the source.

Fig.15 and Fig.16 are drawn to study the effect parameter $a$ on the concentration pattern in presence and absence of the source respectively for fixed dispersion coefficient $D_0 = 1.2(km^2/year^{-1})$. The dispersion increases as the value of the parameter $a$ increases at each point of the space domain. It is noticed that net concentration injected in presence of the source decreases with increase of parameter $a$. Moreover a drop in concentration is recorded for higher value of parameter $a$ at the near end in absence of the source.

### 3.3 Verification of solution

In order to verify the solution we may consider the input function $F(m't) = \frac{1}{1+m't}$ in Eq. (4). This particular input may be solved for solution of advection-dispersion equation by method used in the present paper discussed above as well as the way given below:

The input boundary conditions with $F(m't) = \frac{1}{1+m't}$ may be described as

$$c(x,t) = c_0; \quad t = 0, \quad x \geq 0$$

(A1)
\[-D(x, t) \frac{\partial c(x, t)}{\partial x} + u(x, t) c = \begin{cases} \frac{u_0 c_0}{\gamma + \rho}, & 0 < t \leq t_0 \\ x = 0 \\ 0; & t > t_0 \end{cases} \] (A2a)

\[\frac{\partial c(x, t)}{\partial x} = 0; \text{ as } x \to \infty, \ t \geq 0. \] (A3)

Here, \( m = m' \)

With transformation Eq. (10), Eqs. (A1-A3) may written as:

\[ c(x, T) = c_i; \ T = 0, \ x \geq 0 \] (A4)

\[-D_0(1 + ax)^2 \frac{\partial c(x, T)}{\partial x} + u_0(1 + ax)c = \begin{cases} u_0 c_0, & 0 < T \leq T_0 \\ x = 0 \\ 0; & T > T_0 \end{cases} \] (A5a)

\[ \frac{\partial c(x, T)}{\partial x} = 0; \text{ as } x \to \infty, \ T \geq 0. \] (A6)

With transformation Eq. (15), Eq.(A4-A6) may written as:

\[ c(X, T) = c_i; \ T = 0, \ X \geq 0 \] (A7)

\[-D_0 \frac{\partial c(X, T)}{\partial X} + u_0 c = \begin{cases} u_0 c_0, & 0 < T \leq T_0 \\ X = 0 \\ 0; & T > T_0 \end{cases} \] (A8a)

\[ \frac{\partial c(X, T)}{\partial X} = 0; \text{ as } X \to \infty, \ T \geq 0. \] (A9)

Now, After using transformation Eq.(21) and then applying Laplace Transformation Technique, solution may be written as:

\[ c(x, T) = [(c_i - \frac{70}{\mu_1})H_{\rho e}(X, T) - (c_i - \frac{70}{\mu_1})H_{\rho e}(X, T) + (c_i - \frac{70}{\mu_1})\exp(\rho^2 T - \beta X)] \times \exp[\frac{u_0}{2D_0} X - \frac{1}{R} \frac{u_0^2}{4D_0} + \frac{\mu_1}{\gamma}] + \frac{70}{\mu_1}; \quad 0 < T \leq T_0 \] (A10a)

\[ c(X, T) = [(c_i[H_{\rho e}(X, T) - H_{\rho e}(X, T - T_0)\exp(\gamma T_0))] - \frac{70}{\mu_1}H_{\rho e} - (c_i - \frac{70}{\mu_1})H_{\rho e}(X, T) + (c_i - \frac{70}{\mu_1})\exp(\rho^2 T - \beta X)] \times \exp[\frac{u_0}{2D_0} X - \frac{1}{R} \frac{u_0^2}{4D_0} + \frac{\mu_1}{\gamma}] + \frac{70}{\mu_1}; \quad 0 < T \leq T_0 \] (A10b)

In order to compare the results obtained from solutions Eqs. (30a&30b) and Eqs. (A10a&A10b) with help of figures and tables, the values of constants and parameters are taken as: \( c_0 = 1.0, \mu_i = 0.01, \ u_0 = 0.03 \text{ (km year}^{-1} \text{)} \) and \( D_0 = 1.2 \text{ (km}^2 \text{ year}^{-1} \text{)} \), \( R = 1.25 \text{ (year}^{-1} \text{)} \), \( \gamma = 0.05, \gamma_0 = 0.0007, a(km - 1) = 0.085 \) and \( m = m' = 0.1 \) and time of elimination of source \( t_0 \text{ (year)} = 9 \).

Concentration patterns obtained from solution Eq.(A10a&A10b) are in good agreement with same obtained from the Eq.(30a&30b) as shown with help of figures 4(17&18). It verifies the authenticity of the obtained solution Eq.(30a&30b). Table 1 represents the error occurring in measurement of the concentration from solution Eqs.
In the present study, solute transport phenomena in one-dimensional semi-infinite porous media has been studied for a varying pulse type input source. Solution maintains the trends of such varying nature inputs in which concentration of the solute does not reduce to zero at origin as the source eliminated rather the same decreases gradually. Laplace transformation technique is employed to get the solution of the present problem. Two different time dependent functions taken as input concentration in numerical examples are feasible.

- The obtained result predicts the concentration profile for non-reactive contaminants and an appropriate interpolation method provides optimum result.
- Authenticity of the solution has been provided with help of a particular input.
- The solution may help to determine minimum/maximum or harmless concentration at any position and time. Generalization of the input source as any function smooth function that acts up to certain duration has enhanced utility of the solution.
- Model may be helpful to validate the some other solute transport problem.
- The concentration pattern depicting the solute transport with varying input phenomena provides the different trend in absence and presence of input comparison to uniform input source. Concentration presence in domain may be noticed less in comparison to uniform input.

4 Conclusion

Table 1. Representing Error in Measurement in solute concentration from Solution Eq. (30a) Comparison to Solution Eq. (A10a) in Presence of Source.

Table 2. Representing Error in Measurement in solute concentration from Solution Eq. (30b) Comparison to Solution Eq. (A10b)

Errors measured in table 1 (presence of source) and table 2 (absence of source) are nominal in all the combination of time and space. Tables 1 & 2 validates Eq. (30a) and Eq. (30b) at large extent.

References


[24] A. Kumar, D.K. Jaiswal and N. Kumar, One-dimensional solute dispersion along unsteady flow through a heterogeneous medium, dispersion being proportional to the square of velocity, Hydrological Sciences Journal Journal des Sciences Hydrologiques. 57(6) (2012), 1223-1230,


Fig. 1. Distribution of dimensionless concentration for various time \(0 \leq t \leq t_0\).

Fig. 2. Distribution of dimensionless concentration for various time \(t_0 < t\).

Fig. 3. Surface Plot in presence of source i.e. \(0 \leq t \leq t_0\).

Fig. 4. Surface plot in absence of source i.e. \(t_0 < t\).

Fig. 5. Distribution of dimensionless concentration for various dispersion coefficient at time \(7 \leq t \leq t_0\).

Fig. 6. Distribution of dimensionless concentration for various dispersion coefficient at time \(t \leq t_0\).
Fig. 7. Distribution of dimensionless concentration for various values of parameter $a$ at time $t (\text{year}) = 7 \ (0 \leq t \leq t_0)$.

Fig. 8. Distribution of dimensionless concentration for various values of parameter $a$ at time $t (\text{year}) = 12 \ (t_0 < t)$.

Fig. 9. Distribution of dimensionless concentration for various times $0 \leq t \leq t_0$.

Fig. 10. Distribution of dimensionless concentration for various times $(t_0 < t)$.

Fig. 11. Surface Plot in presence of source i.e. $(0 \leq t \leq t_0)$.

Fig. 12. Surface plot in absence of source i.e. $(t_0 < t)$. 
Fig. 13. Distribution of dimensionless concentration for various dispersion coefficient at time $t(\text{year}) = 7$ ($0 \leq t \leq t_0$).

Fig. 14. Distribution of dimensionless concentration for various dispersion coefficient at time $t(\text{year}) = 12$ ($t_0 < t$).

Fig. 15. Distribution of dimensionless concentration for various value of parameter $\alpha$ at time $t(\text{year}) = 7$ ($0 \leq t \leq t_0$).

Fig. 16. Distribution of dimensionless concentration for various value of parameter $\alpha$ at time $t(\text{year}) = 12$ ($t_0 < t$).

Fig. 17. Distribution of dimensionless concentration for various times in presence of source.
Fig. 18. Distribution of dimensionless concentration for various times in absence of source.

### Table 1. Representing Error in Measurement in solute concentration from Solution Eq. (39a) Comparison to Solution Eq. (A10.1) in Presence of Source

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Table 2. Representing Error in Measurement in solute concentration from Solution Eq. (30b) Comparison to Solution Eq. (A10b)