International Journal of Advances in Mathematics
Volume 2018, Number 3, Pages 66-80, 2018
eISSN 2456-6098

Interpolation and Semi-group on Hilbert and Hardy Spaces of Dirichlet Series

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ABSTRACT. We investigate that a bounded sequence of points in the half plane $\sigma > \frac{1}{2}$ is an interpolating sequence for Hilbert space of ordinary Dirichlet series with square summable coefficiens. Also, in the half plane $\sigma > \frac{1}{2}$ we discuss the wold decomposition for the shift semi group on the Hardy space of square summable Dirichlet series. The local behavior of Hilbert space of Dirichlet series is considered, relative to the functions spaces on the infinite dimensional polydisk, and we generate some results for each case.

1 Introduction

We show that for the Hilbert space with respect to the half-plane $\sigma > \frac{1}{2}$ a bounded sequence of points is an interpolating sequence if and only if it is an inte-rpolating sequence for the Hardy space H^2 of the same half-plane. Similar local results are obtained for Hilbert spaces of ordinary Dirichlet series that relate to Bergman and Dirichlet spaces of the half-plane $\sigma > \frac{1}{2}$. Spaces be identified with functions spaces on the infinite-dimensional polydisk, this gives new results on the Dirichlet and Bergman spaces on the infinite dimensional polydisk, as well as the scale of BesovSobolev spaces containing the Drury Arveson space on the infinite-dimensional unit ball. We use both techniques from the theory of sampling in PaleyWiener spaces, and classical results from analytic number theory.

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Received November 19, 2017; accepted April 011, 2018.
2010 Mathematics Subject Classification:11C08,11C20, 11M06, 11N60, 32A05, 30B50, 42B30, 46B09, 46B30, 46G25, 47B35, 60G15, 60G70.
Key words and phrases: Interpolating sequence,Hilbert and Hardy spaces, Dirichlet series and infinite dimensional polydisk.

We develop a Wold decomposition for the shift semi-group on the Hardy space of square summable Dirichlet series convergent in the half-plane that greater than $\frac{1}{2}$. We show that the asymptotic behavior of the partial sums of a sequence of positive numbers determine the local behavior of the Hilbert space of Dirichlet series defined using these as weights. This extends results recently obtained desc-ribing the local behavior of Dirichlet series with square summable coefficients in terms of local integrability, boundary behavior, Carleson measures and Interpolat-ing sequences.

2 Local Interpolation Of Dirichlet Series

In this section we are going to shed more light on the function theory of the Hilbert space $\mathcal{H}$ that consists of all Dirichlet series (which are L-functions) of the form:

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

($s = \sigma + it$ a complex variable) with:
$\|f\|^2_{\mathcal{H}} = \sum_{n=1}^{\infty} |a_n|^2 < +\infty$. For the basics of $\mathcal{H}$ See [1] and [2].
The definition of $\mathcal{H}$ shows that it consists of functions analytic in the half-plane $\sigma > \frac{1}{2}$. The key to understand the local boundary behavior in this half-plane is the following embedding [1], [3]:

$$\int_{\theta+1}^{\theta} |f(\frac{1}{2}+it)|^2 dt \leq C\|f\|^2_{\mathcal{H}} \tag{2.1}$$

where C is an absolute constant independent of θ. Of course, we can see that, the embedding makes sense only when the Dirichlet series converges for $\sigma = \frac{1}{2}$. Moreover, when (1) has been established for, say, Dirichlet polynomials, it follows that $f(s)/s$ is in $\mathcal{H}^2$ of the half-plane $\sigma > \frac{1}{2}$ for every f in $\mathcal{H}$.That is, an implicit consequence of (1) is that f has nontangential limits almost everywhere on the line $\sigma = \frac{1}{2}$ and then, we have a well-defined boundary limit function which we can denote it by $f(\frac{1}{2}+it)$.
Also, we can see a link between $\mathcal{H}$ and $\mathcal{H}^2$ of the half-plane $\sigma > \frac{1}{2}$ by comparing reproducing kernels. For $\mathcal{H}$, it is immediate that its kernel $K^{\mathcal{H}}_w$ at the point w is a translation of the Riemann zeta-function:

$$K^{\mathcal{H}}_w(s) = \zeta(s+\overline{w}).$$

Using the fact that, zeta-function has a simple pole of residue 1 at $s = 1$, we have :

$$K^{\mathcal{H}}_w(s) = \frac{1}{s+\overline{w}-1} + h(s+\overline{w}) \tag{2.2}$$

Where h is an entire function. Here the first term is the kernel of H^2 of the half-plane $\sigma > \frac{1}{2}$, and so (2) says that, near the diagonals, the kernels for the two spaces coincide modulo a bounded term.

Considering these observations, one might suspect that locally functions in $\mathcal{H}$ look like functions in H^2. The main result of this section may be seen as a way of quantifying this similarity.Based on(1) and (2),we will prove that the interpolating sequences for the two spaces coincide, provided that we consider only bounded sequences of interpolation points.

Then, after that, one can briefly indicate that minor modifications of (1) and (2) yield similar interpolation results for a scale of Hilbert spaces studied by J. E. McCarthy in [4] Then we set some concluding remarks. In particular, we present there some simple observations on the problem of describing the unbounded interpolating sequences for $\mathcal{H}$, merely to hint at the complexity of the problem.

Let $\mathcal{H}$ be a Hilbert space of functions on some set Ω. We assume point evaluation $f \mapsto (w)$ is bounded for each w in Ω such that $\mathcal{H}$ has producing kernel. We denote this kernel by $K_w^{\mathcal{H}}(s)$ and say that a sequence $S = (s_j)_{j=1}^{\infty}$ of distinct points s_j from Ω is an interpolating sequence for $\mathcal{H}$ if $f(s_j) = a_j$ has a solution f in $\mathcal{H}$ whenever $(a_j/\|K_{s_j}^{\mathcal{H}}\|)_{j=i}^{\infty}$ is in ℓ^2.
We set $\mathbb{C}_{\frac{1}{2}} = \{s = \sigma + it : \sigma > \frac{1}{2}\}$, and let $H^2(\mathbb{C}_{\frac{1}{2}})$ denote the classical Hardy space of this half-plane.This is the space of all functions f analytic in $\mathbb{C}_{\frac{1}{2}}$ with

$$\|f\|_{H^2}^2 = sup_{\sigma > \frac{1}{2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(\sigma + it)|^2 dt < \infty.$$

One can get non-tangential boundary limits on the line $\sigma = \frac{1}{2}$ for almost every t and may express the square of the norm as

$$\|f\|_{H^2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(\frac{1}{2} + it)|^2 dt.$$

The reproducing kernel of $H^2(\mathbb{C}_{\frac{1}{2}})$ at w is :

$$K_w^{H^2}(s) = \frac{1}{s + \overline{w} - 1}$$

Theorem(2.1)[22]: Suppose S is a bounded sequence of distinct points from $\mathbb{C}_{\frac{1}{2}}$. Then S is an interpolating sequence for $\mathcal{H}$ if and only if it is an interpolating sequence for $H^2\mathbb{C}_{\frac{1}{2}}$. Needless to say, now H. S. Shapiro and A. L. Shields's H^2 version [5] of L.Carleson's classical interpolation theorem [6] gives a geometric description of the bounded interpolating sequences for $\mathcal{H}$.
One implication is immediate from (2) and the fact that $\frac{f(s)}{s}$ is in $H^2\mathbb{C}_{\frac{1}{2}}$ when-ever f is in $\mathcal{H}$. Namely, when we solve the problem $f(s_j) = a_j$ with f in $\mathcal{H}$, we simultaneously solve the problem $F(s) = \frac{a_j}{s_j}$ with F in $H^2(\mathbb{C}_{\frac{1}{2}})$. Also, since S is bounded, $(a_j/\|K_{s_j}^{H^2}\|_{H^2})_{j=1}^{\infty}$ si in ℓ^2 if and only if $(s_j a_j/\|K_{s_j}^{\mathcal{H}}\|_{\mathcal{H}})_{j=1}^{\infty}$ si in ℓ^2: Let us now assume that the bounded sequence S is an interpolating sequence for $H^2\mathbb{C}_{\frac{1}{2}}$. We wish to prove that then S is also an interpolating sequence for $\mathcal{H}$. To begin with, we observe that it suffices to show that the subsequence:

$$S_\epsilon = \{s_j = \sigma_j + it \in S : \frac{1}{2} < \sigma_j \leq \frac{1}{2} + \epsilon\}$$

is an interpolating sequence for $\mathcal{H}$ for some small ϵ. Indeed, it is clear that S/S_ϵ is finite sequence, which we may write as

$$S/S_\epsilon = (s_j)_{j=1}^{N}$$

The finite interpolation problem $f_0(s_j) = a_j, j = 1, ..N$ can be solved explicitly as follows. Choose primes $P_1, ...P_N$ (not necessarily distinct) such that the product

$$B(s) = \prod_{j=1}^{N} (1 - P_j^{s_j - s})$$

has simple zeros at the points $s_1, ..., s_N$. If we set $B_j(s) = B(s)/(1 - P_j^{s_j - s})$ then the finite interpolation problem has solution

$$f_0(s) = \sum_{j=1}^{N} a_j \frac{B_j(s)}{B_j(s_j)}$$

To solve the full interpolation problem $f(s_j) = a_j$, we can now solve

$$f_\epsilon(s_j) = \frac{a_j - f_0(s_j)}{B(s_j)}, \ \ s_j \in S_\epsilon$$

so that we obtain the final solution $f = Bf_\epsilon + f_0$. Clearly,

$$\left(\frac{f_\epsilon(s_j)}{\|K_{s_j}^{\mathcal{H}}\|_{\mathcal{H}}}\right)_{s_j \in S_\epsilon} \in \ell^2 \Leftrightarrow \left(\frac{a_j}{\|K_{s_j}^{\mathcal{H}}\|_{\mathcal{H}}}\right)_{s_j \in S_\epsilon} \in \ell^2$$

So that we have reduced the problem to showing that S_ϵ is an interpolating sequence for $\mathcal{H}$. The reason for making the transition from S to S_ϵ is that it will allow us to make use of the fact that

$$lim_{\epsilon \to 0} \sum_{s_j \in S\epsilon} (\sigma_j - \frac{1}{2}) = 0 \tag{2.3}$$

We note that (3) is just a consequence of the trivial fact that an interpolating sequence for $H^2(\mathbb{C}_{\frac{1}{2}})$ is a Blaschke sequence in $\mathbb{C}_{\frac{1}{2}}$. Since S is a bounded sequence, this means that

$$\sum_{s_j \in S\epsilon} (\sigma_j - \frac{1}{2}) < +\infty$$

It seems difficult to obtain a direct solution of the interpolation problem since we do not have a reasonable substitute for Blaschke products.We will instead argue by duality, using the lemma of R. P. Boas[7], [8].

Theorem (2.2)[22]: Suppose S is a bounded sequence of distinct points from $(\mathbb{C}_{\frac{1}{2}})$ and assume $\alpha \leq 1$. Then S is an interpolating sequence for $\mathcal{H}_\alpha$ if and only if it is an interpolating sequence for $D_\alpha(\mathbb{C}_{\frac{1}{2}})$.

($\mathcal{H}_\alpha$ are the corresponding Dirichlet-type spaces, $0 < \alpha \leq 1$ and Bergman paces , $\alpha < 0$, and the spaces $D_\alpha(\mathbb{C}_{\frac{1}{2}})$ defined as follows, for $\alpha < 0$, is the weighted Bergman space of functions analytic in $\mathbb{C}_{\frac{1}{2}}$, and for $0 < \alpha < 1$, we let $D_\alpha(\mathbb{C}_{\frac{1}{2}})$ be the Dirichlet-type of space of functions analytic in $\mathbb{C}_{\frac{1}{2}}$ s.t $f(\sigma) \to 0$ as $\sigma \to \infty$). One may distinguish between interpolating sequences as defined above and so-called universal Interpolating sequences, i.e., sequences $(s_j)_{j=1}^{\infty}$ for which $f \mapsto (\frac{f(s_j)}{\|K_{s_j}^{\mathcal{H}}\|_{\mathcal{H}}})$ maps H both into and onto ℓ^2. In the latter case, one then has the Carleson embedding

$$\sum_{j=1}^{\infty} |f(s_j)|^2 \|K_{s_j}^{\mathcal{H}}\|_{\mathcal{H}}^{-2} \leq C\|f\|_{\mathcal{H}}^2 \tag{2.4}$$

with some positive constant C. In the case of bounded interpolating sequences for $\mathcal{H}$, there is no reason to make a distinction because every bounded interpolating sequence for $\mathcal{H}$ is also a universal interpolating sequence for $\mathcal{H}$. The same holds true for $\mathcal{H}_\alpha$ when $\alpha < 0$. However, for $\mathcal{H}_\alpha$ with $0 < \alpha \leq 1$ this is no longer the case [9], [10], and one should therefore make a distinction. Still it is plain that Theorem (2.2) remains valid if one replaces each occurrence of the string an interpolating sequence by a universal interpolating sequence.

There exist geometric descriptions of the (universal)interpolating sequences for all $\alpha \leq 1$. For $\alpha < 0$, Beurling-type density theorems were proved in [11]. Descri-ptions in terms of Carleson measures were found by W. Cohn in the case $0 < \alpha < 1$ [12] and independently by C. Bishop and by D. Marshall and C. Sundberg in the case $\alpha = 1$

[9], [10]. For further information, we refer to the monograph [13]. It may be noted that Theorem (2.1) gives the first general sufficient condition for zero sequences of functions in $\mathcal{H}$. The statement is that for each bounded interpolating sequence S for $H^2(\mathbb{C}_{\frac{1}{2}})$ there is a function f in $\mathcal{H}$ vanishing on S.

On the other hand, a simple argument shows that this sequence S cannot be the zero sequence of any function in $\mathcal{H}$ .Indeed, the almost periodicity of the function $t \mapsto f(\sigma + it)$ along with Rouch's theorem implies that for any zero $\sigma_0 + it_0$ of a function f in $\mathcal{H}$ and $0 < \epsilon < \sigma$ there is a positive number T such that every rectangle $|\sigma - \sigma_0| < \epsilon, \theta < t < \theta + T$ contains a zero of f. In particular, this means that no function in $\mathcal{H}$ has a nonempty finite zero sequence.

We finally make some remarks about unbounded interpolating sequences for $\mathcal{H}$. Take first an arbitrary interpolating sequence located on the real line. We may assume with no essential loss of generality that the sequence is $\sigma_j = \frac{1}{2} + 2^{-j}, j = 1, 2, 3...$ Then any sequence $s_j = \sigma_j + it_j$ will give us a Carleson embedding of the form (4). To see this, we associate with

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

The Dirichlet series

$$f^+(s) = \sum_{n=1}^{\infty} |a_n| n^{-s}$$

This means that $|f(s_j)| \leq f^+(\sigma_j)$, and since $\|f\|_{\mathcal{H}} = \|f^+\|_{\mathcal{H}}$, we obtain the result from the fact that $(\sigma_j)_{j=1}^{\infty}$ yields a Carleson embedding.

We may next observe that the sequence $(s_j)_{j=1}^{\infty}$ can be split into a finite number of interpolating sequences for $\mathcal{H}$. (The number of sequences depends only on $(\sigma_j)_{j=1}^{\infty}$). This follows from Gerschgorin's circle theorem. Indeed, it is enough to check that the normalized Grammian

$$\left(\frac{K_{s_j}^{\mathcal{H}}(s_l)}{\|K_{s_j}^{\mathcal{H}}\|_{\mathcal{H}} \|K_{s_l}^{\mathcal{H}}\|_{\mathcal{H}}}\right)_{j,l=1}^{\infty}$$

is invertible as a map on ℓ^2. We observe that in our case the entries of the matrix decay exponentially and monotonically away from the main diagonal. Thus by splitting $(\sigma_j)_{j=1}^{\infty}$ into sufficiently sparse subsequences, we obtain the invertibility from Gerschgorin's criterion.

To illustrate a different point, we construct the following sequence. For each positive integer j pick points equidistributed on the line segment $\sigma = \frac{1}{2} + 2^{-j}$, $0 \leq t \leq 1$ i.e., choose

$$s_{j,l} = \frac{1}{2} + 2^{-j} + i\frac{l}{j}, \ \ l = 1, 2, ..j$$

Then Carleson's theorem along with our Theorem (2.1) shows that (s_j, l) is an interpolating sequence for $\mathcal{H}$. In particular, the Carleson embedding (4) holds. if we move the points vertically and far apart, the Carleson embedding may fail. This is a consequence of the almost periodicity of $t \mapsto \zeta(\sigma + it)$. If we measure the distance between two points in terms of the angle between the corresponding reproducing kernels, this almost periodicity implies that points that are far apart in the hyperbolic sense of the half-plane may be arbitrarily close in the geometry induced by $\mathcal{H}$. The reproducing kernels for $D_0(\mathbb{C}_{\frac{1}{2}})$ at w is

$$K_w^{D_0}(s) = c_\theta(\overline{w} + s - 1)^{\theta - 1}$$

when $\theta < 1$, with $c_\theta = (1-\theta)2^{-\theta-1}$ for $\theta < 0$ and $c_\theta = 2^{\theta-1}(1-\theta)^{-1}$ for $0 < \theta < 1$. In the limiting case $\theta = 1$, we have

$$K_w^{D_1}(s) = \frac{(3-2\overline{w})(3+2s)}{(1-2\overline{w})(1+2s)}(log\frac{(1+2\overline{w})(1+2s)}{2^3} + log\frac{1}{\overline{w}+s-1}).$$

What is essential here is that for and in a bounded set we have $K_w^{D_1}(s) = log\frac{1}{\overline{w}+s-1}$ a bounded term. Now as for the classical zeta-function, we have a singularity at $s = 1$ in agreement with the behavior of the kernel of $D_0(\mathbb{C}_{\frac{1}{2}})$, we generate the following result: **Corollary 2.3:** For $\epsilon > 0$, we have $\Sigma_{n=1}^{\infty}\frac{n^{-(\frac{1}{2}+\epsilon+i\theta)}}{log^{1-\epsilon}(n+1)} = \Gamma(\epsilon)(-\frac{1}{2}+\epsilon+i\theta)^{-\epsilon} + o(1)$. For $\epsilon \to \frac{1}{2} - i\theta$ we get:

$$\Sigma_{n=1}^{\infty}\frac{n^{-(\frac{1}{2}+\epsilon+i\theta)}}{log^{1-\epsilon}(n+1)} = log\frac{1}{\epsilon+i\theta-\frac{1}{2}} + o(1)$$

Proof: From [14] we have,
$\Sigma_{n=1}^{\infty}\frac{n^{-(\frac{1}{2}+\epsilon+i\theta)}}{log^{1-\epsilon}(n+1)} = \int_1^\infty \frac{x^{-(\frac{1}{2}+\epsilon+i\theta)}}{log^{1-\epsilon}(n+1)}d[x]\int_1^\infty \frac{x^{-(\frac{1}{2}+\epsilon+i\theta)}[x]}{log^{1-\epsilon}(n+1)}((\frac{1}{2}+\epsilon+i\theta) + \frac{1-\epsilon}{log(1+x)}\frac{x}{x+1})dx$
The integral:
$\int_1^\infty \frac{x^{-(\frac{1}{2}+\epsilon+i\theta)}[x]}{log^{1-\epsilon}(n+1)}((\frac{1}{2}+\epsilon+i\theta) + \frac{1-\epsilon}{log(1+x)}\frac{x}{x+1})dx - \int_1^\infty \frac{x^{-(\frac{1}{2}+\epsilon+i\theta)}}{log^{1-\epsilon}(n+1)}((\frac{1}{2}+\epsilon+i\theta) + \frac{1-\epsilon}{log(1+x)})dx$ Converges absolutely and defines an analytic function in the right half plane, then $[x] = x$, in the integral now: $\frac{x}{x+1} \approx 1$ and $log(x+1) \approx logx$, when $\epsilon > 0$, we can find:
$\int_1^\infty \frac{x^{-(\frac{1}{2}+\epsilon+i\theta)}}{log^{1-\epsilon}(x)}dx = \Gamma(\epsilon)(-\frac{1}{2}+\epsilon+i\theta)^{-\epsilon}$ which gives the result. When $\epsilon > 1$, we find by using the functional equation for the gamma function, that:
$\int_1^\infty \frac{x^{-(\frac{1}{2}+\epsilon+i\theta)}}{log^{1-\epsilon}(x)}((\frac{1}{2}+\epsilon+i\theta)\frac{1-\epsilon}{logx})dx = \Gamma(\epsilon)(-\frac{1}{2}+\epsilon+i\theta)^{-\epsilon}$ as well when $\epsilon = 1$, we find that:
$\int_1^\infty \frac{x^{-(\frac{1}{2}+i\theta)}}{logx}dx = log\frac{1}{i\theta-\frac{1}{2}} + o(1)$ as $i\theta \to \frac{1}{2}$.

3 Shift Semi-group of Dirichlet Series

Let ϵ be an auxiliary Hilbert space and denote by $\mathcal{H}^2(\epsilon)$ the space of all ϵ-valued Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{3.1}$$

Where $a_n \in \epsilon$, for $n \geq 1$, with finite norm

$$\|f\|_{\mathcal{H}^2}^2 = \sum_{n=1}^{\infty}\|a_n\|^2 < +\infty.$$

The Dirichlet series (5) of a function $f \in \mathcal{H}^2(\epsilon)$ converges in the half-plane $C_{\frac{1}{2}}$, where $C_\alpha = \{s\alpha = \sigma + it \in C : \mathcal{R}(s) = \sigma > \alpha\}$.

For α real, as follows by an application of the Cauchy-Schwarz inequality. It is easy to see that the space $\mathcal{H}^2(\epsilon)$ becomes in $C_{\frac{1}{2}}$ a Hilbert space of ϵ-valued analytic functions.

In the scalar context, the study of the space $\mathcal{H}^2$ was initiated by Hedenmalm, Lindqvist and Seip [1] as a Dirichlet series analogue of the classical Hardy space $H^2(\mathbb{D})$ of square summable power series in the unit disc $\mathbb{D}$. A characteristic feature of the space $\mathcal{H}^2$ is that of its reproducing kernel function being essentially the Riemann zeta function from number theory:

$K_{H^2}(s,s') = \zeta(s+\bar{s})$ for $s,s' \in C_{\frac{1}{2}}$ where $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$.
For every positive integer $n \in Z^+$ we have a natural operator $S(n)$ acting on H^2 defined by

$$S(s)f(s) = n^{-s}f(s), \;\; s \in C_{\frac{1}{2}} \tag{3.2}$$

for functions $f \in \mathcal{H}^2$. This provides us with a function $S : Z^+ \ni n \mapsto S(n)$ which is easily seen to be a multiplicative semigroup of operators in the sense that

$$S(n,m) = S(n), S(m), \;\; n,m \in Z^+$$

and $S(1) = I$, where I is the identity operator. We call this semi-group S the Shift semi-group on $\mathcal{H}^2$. One can think of the shift semi-group as a counterpart for the space $\mathcal{H}^2$ of the classical shift operator on the Hardy space $H^2(\mathbb{D})$ given by mult-iplication by the complex coordinate.

Some properties of the shift semi-group S are more or less evident. For every integer $n \geq 2$ the operator $S(n)$ is a pure isometry. For relatively prime integers $n, m \in Z^+$ the operators $S(n)$ and $S(m)$ commute. It is also fairly straightforward to see that

$$\cap_{n=1}^{\infty}(\vee_{k=n}^{\infty} S(p_k)(\mathcal{H}^2)) = \{0\} \tag{3.3}$$

where $p_1 = 2, p_2 = 3, p_3 = 5, ...$ is the sequence of prime numbers. Recall that the symbol $\vee$ is used to denote a closed linear span. The main purpose of the present section is to prove that the three properties quoted in the previous paragraph completely determine the shift semi-group S up to unitary equivalence allowing for a general multiplicity ϵ as indicated above.The unitary equivalence can be chosen of a certain canonical form similar to what has previously been done for pure isometries and some related classes of operators using a so-called Wold decomposition. Let us proceed to describe our results. Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$. Let $T : Z^+ \to \mathcal{L}(\mathcal{H})$ be a multiplicative semi-group of operators satisfying the analogous properties to those quoted above for the shift semi-group, the exact properties are listed as conditions $(a, b \; and \; c)$ as will be described bellow. We call the subspace

$$\epsilon = \mathcal{H}\Theta(\vee_{k=1}^{\infty} T(p_k)(\mathcal{H})) \tag{3.4}$$

of $\mathcal{H}$ for the wandering subspace for T, and we denote by P the orthogonal proje-ction of $\mathcal{H}$ onto ϵ. We associate to every element $x \in \mathcal{H}$ the ϵ valued function given by the Dirichlet series

$$(\vee x)(s) = \sum_{n=1}^{\infty} (PT(n)^* x) n^{-s}$$

which is initially seen to converge in the smaller half-plane C_1. The task at hand is then to prove that the map $\vee : x \mapsto \vee_x$ gives an isometry of $\mathcal{H}$ onto $\mathcal{H}^2(\epsilon)$ which intertwines T with the shift semigroup S in the sense that

$$\vee T(n) = S(n) \vee, \;\; n \in Z^+$$

The map $\vee$ then provides a unitary equivalence of semi-groups T and S. It should be mentioned here that Wold decompositions for pairs of doubly commuting iso-metries have been studied by Slocinski [15]. We recall that two operators $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ are said to doubly commute if $T_1, T_2 = T_2 T_1$ and $T_1, T_2^* = T_2^* T_1$. Popovici [16] has studied Wold decompositions for pairs of commuting isometr-ies dropping the assumption of double commutativity.

Following Bohr [17] we can for the k^{th}-prime p_k identify p_k^{-s} with a complex variable $z_k(k = 1, 2...)$ to make the Dirichlet series (5) correspond to a power series

$$f(z) = \sum_{n=1}^{\infty} a_n z_1^{v_1}, z_2^{v_2}, ...z_m^{v_m}$$

in infinitely many variables $z_1, z_2, ...$ where in the sum $n = p_1^{v_1}, p_2^{v_2}, ...p_m^{v_m}$ is the factorization of n into a product of prime powers.The power series obtained from functions $f \in \mathcal{H}^2$ in this way extend by means of bounded point evaluations to the set

$$\triangle = \{\{z_k\}_{k=1}^{\infty} : |z_k| < 1 \;\; for \;\; k \geq 1 \;\; \sum_{k=1}^{\infty} |z_k|^2 < +\infty\}$$

It is natural to think of $\mathcal{H}^2$ as an equivalent of the Hardy space $H^2(\triangle)$ on the set $\triangle = \mathbb{D}^w \cap \ell^2$. The operator on $\mathcal{H}^2$ corresponds to the shift S_k on $H^2(\triangle)$ which is the operator given by multiplication by the variable $z_k(k = 1, 2, ..)$. An infinite sequence $\{T_k\}_{k=1}^{\infty}$ of operators on a Hilbert space $\mathcal{H}$ is unitarily equivalent to the shift sequence $\{S_k\}_{k=1}^{\infty}$ on a vector-valued Hardy space $H^2(\triangle, \epsilon)$ if and only if every operator T_k is a pure isometry, the operators T_k and T_j doubly commute for $j \neq k$, and the condition

$$\cap_{n=1}^{\infty} (\vee_{k=n}^{\infty} T_k(\mathcal{H})) = \{0\} \tag{3.5}$$

is satisfied. This result is in some contrast to the case of a finite operator tuple $(T_1, ...T_n) \in \mathcal{L}(\mathcal{H})^n$ on a Hilbert space $\mathcal{H}$ which is unitarily equivalent to the shift tuple $(S_1, ...S_n)$ on a vector-valued polydisc Hardy space $H^2(\mathbb{D}^n, \epsilon)$ if and only if every operator T_k is a pure isometry and the operators T_j and T_k doubly commute for all $j \neq k$.

It should be pointed out that condition (9) does not follow from the mere fact that the operator sequence $\{T_k\}_{k=1}^{\infty}$ is such that every operator T_k is a pure isometry and the operators T_j and T_k doubly commute for $j \neq k$. A multiplicative semi-group of operators $T : Z^+ \to \mathcal{L}(\mathcal{H})$ satisfying conditions (a-c) which will come later can have null wandering subspace in the sense of (8). By a shift invariant subspace of $\mathcal{H}^2$ we mean a closed subspace I of $\mathcal{H}^2$ such that $S(n)(I) \subset I$ for every positive integer $n \in Z^+$. As an application of the Wold decomposition. By an $\mathcal{H}^2$-inner function we mean a function φ in $\mathcal{H}^2$ of unit norm such that $S(n)\varphi \perp S(m)\varphi$ in $\mathcal{H}^2$ for $n, m \in Z^+$ with $n \neq m$. It is to a bounded analytic function in the right half-plane C_0 with operator norm given by $\|m\| = sup_{s \in C_0} |m(s)|$. The space of such Dirichlet series m is denoted $\mathcal{H}^\infty$. An $\mathcal{H}^2$ inner function φ is a function in $\mathcal{H}^\infty$ such that $|\varphi(x)| = 1$ for a.e. $x \in \Xi$ with respect to Haar measure (Ξ character group). Let us recall first the Wold decomposition of an isometry. A Hilbert space operator $T \in \mathcal{L}(\mathcal{H})$ is called an isometry if $T * T = I$ in $\mathcal{L}(\mathcal{H})$, where I is the identity operator. An isometry $T \in \mathcal{L}(\mathcal{H})$ is called pure if

$$\cap_{k=1}^{\infty} T^k(\mathcal{H}) = \{0\}.$$

The subspace $\epsilon = \mathcal{H} \Theta T(\mathcal{H})$ is called a wandering subspace for $\in \mathcal{L}(\mathcal{H})$. The next proposition gives some basic properties of the shift S.

Proposition(3.1)[23] For every integer $n \geq 2$ the operator $S(n) \in \mathcal{L}(\mathcal{H}^2)$ is a pure isometry. For relatively prime integers $n, m \in Z^+$ the operators $S(n)$ and $S(m)^*$ commute: $S(n)S(m)^* = S(m)^* S(n)$. Furthermore

$$\cap_{n=1}^{\infty} (\vee_{k=n}^{\infty} S(p_k)(\mathcal{H})^2) = \{0\},$$

where $\{p_k\}_{k=1}^{\infty}$ is the sequence of prime numbers.

Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$. We shall first consider a multiplicative semi-group $T : Z^+ \to \mathcal{L}(\mathcal{H})$ of operators satisfying the conditions:

(a) The operator $T(p) \in \mathcal{L}(\mathcal{H})$ is a pure isometry for every prime number p.

(b) For different primes p and q,the operators $T(p)$ and $T(q)$ doubly commute.

Let $T : Z^+ \to \mathcal{L}(\mathcal{H})$ be a multiplicative semi-group of operators satisfying (a)-(b). We shall make use of the subspace

$$\epsilon = \mathcal{H}\Theta(\vee_{k=1}^{\infty} T(p_k)(\mathcal{H})) = \vee_{k=1}^{\infty} ker T(p_k)^* \tag{3.6}$$

of $\mathcal{H}$ where $\{p_k\}_{k=1}^{\infty}$ is the sequence of prime numbers. This subspace ϵ has the property that:

$$T(n)\epsilon \perp T(m)\epsilon \;\; in \;\; \mathcal{H} \tag{3.7}$$

For all $n, m \in Z^+$ with $n \neq m$. We call a subspace ϵ of $\mathcal{H}$ with the property that $T(n)\epsilon \perp T(m)\epsilon$ in $\mathcal{H}$ for all relatively prime integers $n, m \in Z^+$ with $n \neq m$ for a wandering subspace for the semi-group T.The particular wandering subspace ϵ in (10) is referred to as the wandering subspace for $T : Z^+ \to \mathcal{L}(\mathcal{H})$. (c) The semi-group $T : Z^+ \to \mathcal{L}(\mathcal{H})$ is such that

$$\cap_{n=1}^{\infty}(\vee_{k=n}^{\infty} T(p_k)(\mathcal{H})) = \{0\}$$

Where $\{p_k\}_{k=1}^{\infty}$ is a sequence of prime numbers.

Remark that condition (c) inherits to invariant subspaces. The condition also has some resemblance to the notion of pureness of an isometry.

Theorem(3.2)[23]: Let $T : Z^+ \to \mathcal{L}(\mathcal{H})$ be a multiplicative semi-group satisfying conditions (a)-(b), and denote by ϵ the wandering subspace for T given by (10). Then

$$\vee_{n=1}^{\infty} T(n)\epsilon = \mathcal{H}\Theta(\cap_{n=1}^{\infty}(\vee_{k=n}^{\infty} T(p_k)(\mathcal{H}))$$

Proof: Consider the subspaces

$$\mathcal{H}_n = \mathcal{H}\Theta(\vee_{k\geq n} T(p_k)(\mathcal{H})), \;\; n = 1, 2, 3..$$

of $\mathcal{H}$. Notice that $\mathcal{H}_1 = \epsilon$ is the wandering subspace given by (10). Observe also that the subspace $\mathcal{H}_{n+1}$ is reducing for the operator $T(p_n)$ for $n = 1, 2, ...$ We shall next prove that

$$\mathcal{H}_{n+1}\Theta T(p_n)(\mathcal{H}_{n+1}) = \mathcal{H}_n, \;\; n = 1, 2, .. \tag{3.8}$$

It is obvious that $\mathcal{H}_n$ is contained in the space on the left hand side in(12).To prove the reverse inclusion it suffices to show that if $x \in \mathcal{H}_{n+1}$ is such that $x \perp T(p_n)(\mathcal{H}_{n+1})$ then $x \perp T(p_n)(\mathcal{H})$. For this purpose let $y in \mathcal{H}$ and write $y = y_1 + y_2$, where $y_1 \in \mathcal{H}_{n+1}$ and $y_2 \in \mathcal{H}\Theta\mathcal{H}_{n+1}$. Since the subspace $\mathcal{H}_{n+1}$ reduces $T(p_n)$, we have that $T(p_n)y_2 \in \mathcal{H}\Theta\mathcal{H}_{n+1}$. Now

$$\langle x, T(p_n)y\rangle = \langle x, T(p_n)y_1\rangle + \langle x, T(p_n)y_2\rangle = 0$$

where $\langle x, T(p_n)y_1\rangle = 0$ because $y_1 \in \mathcal{H}_{n+1}$ and $\langle x, T(p_n)y_2\rangle = 0$ because $T(p_n)y_2 \in \mathcal{H}\Theta\mathcal{H}_{n+1}$. Notice that the operator Tp_n restricted to $\mathcal{H}_{n+1}$ is a pure isometry and that the wandering subspace for $T(p_n)|_{\mathcal{H}_{n+1}}$ is $\mathcal{H}_n$ by (12). Since we can include that:

$$\mathcal{H}_{n+1} = \vee_{k\geq 0} T(p_n)^k(\mathcal{H}_n), \;\;\; n = 1, 2, ..$$

Proceeding inductively we find that

$$\vee_{k_1,k_2,..k_n\geq 0}T(p_1^{k_1},p_2^{k_2},..p_n^{k_n})(\epsilon)=\mathcal{H}\Theta(\vee_{k=n+1}^{\infty}T(p_k\mathcal{H})$$

for positive integers n.By this last formula the conclusion of the theorem follows. We remark that

$$\vee_{n=1}^{\infty}T(n)(\epsilon)=\bigoplus_{n=1}^{\infty}T(n)(\epsilon).$$

In Theorem (3.2) by the orthogonality property (10). In the following develop-pments we shall rephrase this last equality in a somewhat more appealing form. Let $T: Z^{+}\to\mathcal{L}(\mathcal{H})$ be as in Theorem (3.2). It is straightforward to see that the subspace

$$\mathcal{H}_0=\bigcap_{n=1}^{\infty}(\vee_{k=n}^{\infty}T(p_k)(\mathcal{H}))$$

of $\mathcal{H}$ is reducing for T, that is, the inclusions $T(n)(\mathcal{H}_0)\subset\mathcal{H}_0$ and $T(n)^{*}(\mathcal{H}_0)\subset\mathcal{H}_0$ hold for every $n\in Z^{+}$. It should be pointed out that the subspace $\mathcal{H}_0$ in(13)can be nonzero assuming only conditions (a)-(b). Motivated by Theorem (3.2) we are led to consider semi-groups $T: Z^{+}\to\mathcal{L}(\mathcal{H})$ that in addition to (a),(b) and (c). Now we establish the following results:

Results:

Corollary(3.3) For any $\epsilon>0$, the operator $S(2+\epsilon)\in\mathcal{L}(\mathcal{H}^2)$ is pure isometry. For relatively prime integers $2+\epsilon, 3+\epsilon\in Z^{+}$, the operators $S(2+\epsilon)$ and $S(3+\epsilon)^{*}$ commute: $S(2+\epsilon)S(3+\epsilon)^{*}=S(3+\epsilon)^{*}S(2+\epsilon)$. Hence: $\cap_{\epsilon\geq 1}(\vee_{k=2+\epsilon}^{\infty}S(p_k)(\mathcal{H}^2))=\{0\}$, so that $\{p_k\}_{k=1}^{\infty}$ is the sequence of prime numbers Proof: Since we have, $S(3+\epsilon)^{*}S(2+\epsilon)f=f$, for $f\in\mathcal{H}^2$, Implies that $S(2+\epsilon)\in\mathcal{L}\mathcal{H}^2$, is an isometry, we can verify that: $S(2+\epsilon)S(3+\epsilon)^{*}f=S(3+\epsilon)^{*}S(2+\epsilon)f$, for any $f\in\mathcal{H}^2$ if $2+\epsilon, 3+\epsilon\in Z^{+}$ are relatively prime.

As a consequence of (15)the range $S(2+\epsilon)\mathcal{H}^2$ consists of all functions $f\in\mathcal{H}^2$ such that $\hat{f}(k)=0$ for all positive integers $k\in Z^{+}$ not divisible by $2+\epsilon$, or be put in another way:

$$S(2+\epsilon)(\mathcal{H}^2)=\{f\in\mathcal{H}^2: Sup(\hat{f})\subset(2+\epsilon)Z^{+}\}$$

(14) shows that the isometry $S(2+\epsilon)$ is pure for $\epsilon\geq 0$. An iteration of (14) gives that $\vee_{k=2+\epsilon}^{\infty}S(p_k)(\mathcal{H}^2)=\{f\in\mathcal{H}^2: Sup(\hat{f}\subset\cup_{k=2+\epsilon}^{\infty}(p_kZ^{+})\}$, if we take an Intersection, we find that: ($Sup(\hat{f})$ denotes the support of $\hat{f}$ as a function on Z^{+}).

$$\cap_{\epsilon\geq 1}(\vee_{k=2+\epsilon}^{\infty}S(p_k)((\mathcal{H}^2))=\{f\in\mathcal{H}^2: Sup(\hat{f}\subset\cap\epsilon\geq 1(\cup_{k=2+\epsilon}^{\infty}(p_kZ^{+}))\}=\{0\}$$

Where the last equality is obvious since: $\cap\epsilon\geq 1(\cup_{k=2+\epsilon}^{\infty}(p_kZ^{+}))=\emptyset$.

Corollary(3.4): Let $T^{*}: Z^{+}\to\mathcal{L}(\mathcal{H})$ be a multiplicative self adjoint semi-group satisfying conditions (a)-(b). Then the infinite product: $P=\prod_{k=1}^{\infty}(1-T^{*2}_{k}(p_k))$ in $\mathcal{L}(\mathcal{H})$ Converges in the strong operator topology in $\mathcal{L}(\mathcal{H})$. The operator P is the orthog-onal projection of $\mathcal{H}$ onto the wandering subspace ϵ for T^{*} given by (10).

Proof: Since $p_k=1-T_k^{*2}(p_k)$ is the orthogonal projection of $\mathcal{H}$ onto the wandering subspace $\mathcal{H}\Theta T^{*2}(p_k)(\mathcal{H})$ for $T^{*}(p_k)$. Since $P_jP_k=P_kP_j$. Since the operator $T^{*}(p_j)$ and $T^{*}(p_k)$ doubly commute for $j\neq k$. Hence $\Pi_{k=1}^{N}$ is orthogonal projection of $\mathcal{H}$ onto the subspace

$$\mathcal{H}\Theta(\vee_{k=1}^{N}T^{*}(p_k)(\mathcal{H})=\Pi_{k=1}^{N}(\mathcal{H}\Theta T^{*}(p_k)(\mathcal{H}))\ \ on\ \ \mathcal{H}.$$

Therefore, $P = lim_{N\to\infty}\Sigma_{k=1}^{N} P_k$ in $\mathcal{L}(\mathcal{H})$, exists in the operator topology in $\mathcal{L}(\mathcal{H})$ and if $P_1, P_2, ... = P_k$, then $P = lim_N P_k^N$.

4 Local properties of Dirichlet series

The theory of Dirichlet series, i.e. functions of the form $f(s) = \sum_{n\in N} a_n n^{-s}$ with $s = \sigma + it$ as the complex variable, offers a bridge between number theory and analysis. Perhaps the most appealing example of the power of this connection is given by the tauberian approach to the classical prime number theorem. One way to state the prime number theorem is to say that the Chebyshev-type inequalities

$$A\frac{x}{(logx)^{\alpha}} \le \sum_{n\le x} w_n \le B\frac{x}{(logx)^{\alpha}}, \tag{4.1}$$

with coefficients

$$w_n = \begin{cases} 1, & \text{n is prime;} \\ 0, & \text{otherwise.} \end{cases}$$

and $\alpha = 1$, holds for any $A, B > 1$ as long as $x > 0$ is taken to be sufficiently large. Originally due to Ikehara, the general idea of the tauberian approach is to connect the function theoretic properties of the Riemann zeta function $\zeta(s) = \Sigma_{n\in N} n^{-s}$ to the growth of these partial sums [18]. As is well known, the properties of the Riemann zeta function is closely related to the behavior of the prime numbers through the Euler product formula

$$\zeta(s) = \Pi_p \frac{1}{1-p^{-s}},$$

We study the connection between the asymptotic behavior in terms of the inequa-lities (15) for general sequences $(w_n)_{n\in N}$ of non-negative numbers, and local func-tion theoretic properties of the Hilbert spaces

$$\mathcal{H}_w = \{\Sigma_{n\in N} a_n n^{-s} : \sum \frac{|a_n|^2}{w_n} < \infty\}.$$

(By convention, if $w_n = 0$, we exclude the basis vector n^{-s} from this definition.) where in particular the local behavior of functions in the DirichletHardy space H^2, which corresponds to the choice $w_n \equiv 1$, is studied. By the CauchySchwarz inequality, the space H^2 is seen to consist of functions analytic on the half-plane $\mathbb{C}\frac{1}{2} = \{Res > \frac{1}{2}\}$. The results of this and later contributions [2, 16, 19, 20] can be summarized as saying that locally H^2 looks much like the classical Hardy space,

$$H^2(\mathbb{C}\frac{1}{2}) = \{f \ \ analytic \ \ on \ \ \mathbb{C}\frac{1}{2} : sup_{\sigma>\frac{1}{2}} \frac{1}{2\pi}\int_R |f(\sigma+it)|^2 < \infty\}$$

One of the starting points of the function theory for the DirichletHardy space is a simple, but striking, local connection indicated by comparing reproducing kernels, i.e. functions k_w such that $\langle f|k_w = f(w)\rangle$ for all f in the space, and points w in the domain of definition. For the space H^2, the reproducing kernel at $w \in \mathbb{C}\frac{1}{2}$ is the translate $k_w(s) = \zeta(s+\overline{w})$ of the Riemann zeta function. The Riemann zeta function is known to be a meromorphic function with a single pole of residue one at $s = 1$. This yields the formula

$$k_w(s) = \frac{1}{s+\overline{w}-1} + h(s+\overline{w}),$$

where h is an entire function. This reveals that k_w is an analytic perturbation of the reproducing kernel for $H^2(\mathbb{C}\frac{1}{2})$, namely the Szego'-kernel: $k^s_w(s) = (s + \overline{w} - 1)^{-1}$. The following results strengthens this local connection.

Lemma(4.1)[24]: Let (w_n) be sequence of non-negative numbers and suppose that $\alpha \in R$. Then there exist $\eta \in (0,1)$ such that $\sum_{n\in(\eta x,x)} w_n \lesssim x(logx)^\alpha \Leftrightarrow \sum_{n\in(0,x)} w_n \lesssim x(logx)^\alpha$.

Moreover, suppose that the upper Chebyshev-type inequality (15) holds, then there exist $\eta \in (0,1)$ such that $\sum_{n\in(\eta x,x)} w_n \gtrsim x(logx)^\alpha \Leftrightarrow \sum_{n\in(0,x)} w_n \gtrsim x(logx)^\alpha$.

Proof. It is clear that for each statement, one implication is trivial. As for the $\Rightarrow$ part of the first statement, note that

$$\sum_{n\leq e^\xi} w_n = \sum_{k\leq\xi}\sum_{n\in(e^{k-1}e^k)} w_n \leq C\sum_{k\leq\xi} e^k k^{-\alpha} = Ce^\xi\xi^{-\alpha}\sum_{k\leq\xi} e^{k-\xi}(\frac{k}{\xi})^{-alpha}$$

This gives the desired conclusion since, by a simple calculation, the sum on the right-hand side is bounded by a constant. Finally, $\Leftarrow$ part of the second statement follows from an argument by contradiction. Indeed, assume it holds for no $\eta > 0$, and set $\psi(x) = x(logx)^{-alpha}$. Then there exist sequences $x_k \to \infty$ and $\eta_k \to 0$ for which $\sum_{n\in(x_k\eta_k,x_k)} w_n \leq \frac{\psi(x_k)}{k}$. Then, this, and the upper Chebyshev-type inequality, imply

$$\sum_{n\in(1,x_k)} w_n \leq \sum_{n\in(1,x_k\eta_k)} w_n + \sum_{n\in(x_k\eta_k,x_k)} w_n \lesssim \psi(x_k\eta_k) + \frac{\psi(x_k)}{k}.$$

Applying the lower Chebyshev inequality to the left-hand side now yields a contradiction, since the quotient $\frac{\psi(x_k\eta_k)}{\psi(x_k)}$ goes to zero as $k \to \infty$.

Theorem(4.2)[24]. Let $(w_n)_{n\in N}$ be a sequence of non-negative numbers, and $\alpha \in (-\infty, 1]$. The following statements are equivalent.

(a). There exists a constant $C > 0$ such that for all $x \geq 2$, $\sum_{n\leq x} w_n \leq C\frac{x}{(logx)^\alpha}$

(b). $\mathcal{H}_w$ is embedded locally into the space $D_\alpha(\mathbb{C}\frac{1}{2})$

By analogy to the prime number theorem, the inequality in (a) can be considered as an upper Chebyshev-type inequality. Proof: $(a) \Rightarrow (b)$ when $\alpha = 0$. For $F \in \mathcal{H}_w$ and $\sigma > \frac{1}{2}$, we calculate by duality

$$\begin{aligned}(\int_I |F(\sigma+it)|^2 dt)^{\frac{1}{2}} &= Sup_{g\in L^2}\int_I F(\sigma+it)g(it)dt,\ \ \|g\| = 1,\\ &= Sup_{g\in L^2}\sum_{n=1}^N a_n n^{-\sigma}\int_I g(it)n^{-it}dt\\ &= \sqrt{2\pi}Sup_{g\in L^2}\sum_{n=1}^N a_n\frac{\hat{g}(logn)}{n^\sigma}\end{aligned} \tag{4.2}$$

If we multiply and divide by $\sqrt{w_n}$, apply the CauchySchwarz inequality, and take the appropriate limits, this yields

$$(\int_I |F(\sigma+it)|^2 dt)^{\frac{1}{2}} \lesssim \|F\|_{\mathcal{H}_w} sup_{g\in L^{2(I)}}\underbrace{\sum_{n\geq 1}\frac{|\hat{g}(logn)|^2}{n}w_n}_{(i)}$$

The functions $\hat{g}$ are Fourier transforms of functions with compact support in a fixed interval in R, which implies that they are very regular in the sense that for $\xi \in (k, k+1)$ we get the easy estimate $|\hat{g}(\xi)| \leq |\hat{g}(k)| + \|\hat{g}'\|_{L^2(k,k+1)}$. This is sufficient to conclude, since by this estimate, the upper Chebyshev inequality for (w_n), and basic properties of the Fourier transform, we obtain

$$(i)=\sum_{k=1}^{\infty}\sum_{n\in(e^k,e^{k+1})}\frac{|\hat{g}(logn)|^2}{n}w_n \quad \leq \sum_{k=1}^{\infty}\frac{|\hat{g}(k)|^2+\|\hat{g}'\|_{L^2(k,k+1)}}{e^k}\sum_{n\leq e^{k+1}}w_n \lesssim \|g\|^2_{L^2(I)} \tag{4.3}$$

The following result generalizes the theorem on boundary functions mentioned before, and demonstrates the function theoretic significance of lower Chebyshev-type inequalities.

$(a)\Rightarrow(b)$: We have already proved this for the case $\alpha=0$. For $\alpha\neq 0$,the argument holds with minor modifications. Namely, if we multiply and divide $\sqrt{log^{\alpha}nw_n}$ on the right-hand side of (16), then for $\sigma>\frac{1}{2}$ we obtain

$$(\int_I|F(\sigma+it)|^2dt)\lesssim \underbrace{\sum_{n=1}^{N}(\frac{|a_n|^2}{w_n}\frac{1}{(logn)^{\alpha}n^{2\sigma-1}})}_{(1)}\underbrace{sup_{g\in L^{2(I)}\|g=1\|}\sum_{n=1}^{N}|\hat{g}(logn)|^2\frac{(logn)^{\alpha}w_n}{n}}_{(ii)}$$

The factor (ii) can be dealt with exactly as before, using the compact support of the functions g, to yield $(ii)\leq C$. For $\alpha<0$, we use this to evaluate

$$\int_{\frac{1}{2}}^{1}\int_I|F(\sigma+it)|^2(\sigma-\frac{1}{2})^{-\alpha-1}dtd\sigma\leq C\sum_{n=1}^{N}(\frac{|a_n|^2}{w_n}\frac{1}{(logn)^{\alpha}})\int_{\frac{1}{2}}^{1}n^{-(2\sigma-1)}(\sigma-\frac{1}{2})^{(-\sigma-1)}d\alpha\simeq\|F\|^2_{\mathcal{H}}$$

Here, for $n\geq 1$, we used the change of variables $\xi=(2\sigma-1)logn$ to bound the integrals in the sum by $\Gamma(-\alpha)(2logn)^{\alpha}$, where Γ is the gamma-function. A similar argument holds for $\alpha\in(0,1]$.

$(b)\Rightarrow(a)$: Define the function $g_k(s)=\sum_{n\in(e_k,e_{k+1})}w_nn^{-s}$. Suppose that $\alpha<0$. Then the local embedding of $\mathcal{H}_w$ into $D_{\alpha}(\mathbb{C}\frac{1}{2})$ implies that for any $\delta>0$ there exists a constant $C>0$ such that

$$\int_{\frac{1}{2}}^{1}\int_{-\delta}^{\delta}|g_k(s)|^2(\sigma-\frac{1}{2})^{-\alpha-1}dtd\sigma\leq C\sum_{n\in(e_k,e_{k+1})}w_n.$$

By expanding $|g_k(s)|^2$, we find that the left-hand side of the above expression is equal to

$$2\delta\sum_{n,m\in(e_k,e_{k+1})}w_nw_m\frac{sin\delta ln(\frac{n}{m})}{\delta ln(\frac{n}{m})}\int_{\frac{1}{2}}^{1}(nm)^{-\sigma}(\sigma-\frac{1}{2})^{1-\alpha}d\sigma.$$

We fix $\delta>0$ small enough so that $\delta ln(n/m)\leq\frac{\pi}{2}$. By evaluating the integral with respect to σ, then up to a constant the previous expression is seen to be greater than or equal to

$$\sum_{n,m\in(e_k,e_{k+1})}w_nw_m\frac{(lognm)^{\alpha}}{\sqrt{nm}}\geq\frac{(2k)^{\alpha}}{e^{k+1}}(\sum_{n\in(e_k,e_{k+1})}w_n)^2.$$

By combining the above estimates, we obtain

$$\sum_{n,m\in(e_k,e_{k+1})}w_n\lesssim\frac{e^k}{k^{\alpha}}.$$

By Lemma(4.1), this implies the desired conclusion. The cases $\alpha=0$ and $0<\alpha<2$ are treated in the same way. This made it possible to study the span of the boundary values of functions in the DirichletHardy space H^2, as well as the more general spaces introduced by McCarthy.

Now we can generate the following results: **Corollary 4.3:** Let ω_n be a sequence of non-negative numbers and $\epsilon>0$.Then the conditions of Theorem (3.2) are equivalent to either of the statements:

(i) For any interval I of fixed length, there exists a constant such that for all $f_j \in W_0^{\frac{\epsilon-1}{2}}(I)$ then $\sum_{n\in N}\sum_{j\geq 1}\frac{|\hat{f}_j(logn)|^2}{n}w_n \lesssim \|f_j\|^2_{W_0^{\frac{\epsilon-1}{2}}(I)}$

(ii) The operator E_I is bounded for $\mathcal{H}_w$ to $W^{\frac{\epsilon-1}{2}}(I)$.

Proof:$(i) \Rightarrow (ii)$ is given exactly in [21]. The adjoint operator of E_I with respect to the natural non-weighted pairing is shown by:

$$E_I^* : g_i \in W_0^{\frac{\epsilon-1}{2}}(I) \mapsto \sum_{n\in N}\sum_{j\geq 1}\frac{\overline{g}_j(logn)}{\sqrt{n}}n^{-s} \in \mathcal{H}_{\frac{1}{w}}$$

For $(i) \Leftarrow (ii)$ upon using the measure $v = \sum_{n\in N}\delta_{logn}\frac{w_n}{n}$ Therefore (i) is equivalent to the following:

$$\int\sum_{j\geq 1}|\hat{f}_j(1+\epsilon_1)|^2 dv \leq \int\sum_{j\geq 1}|\hat{f}_j(1+\epsilon_1)|^2(2+2\epsilon_1+\epsilon_1^2)^{\frac{\epsilon-1}{2}}d(1+\epsilon_1)$$

which is equivalent to :

$$v(1+\epsilon_1, 2+\epsilon_1) \leq (2+2\epsilon_1+\epsilon_1^2)^{\frac{\epsilon-1}{2}}$$

And which is equivalent to :

$$\sum_{n\in(e^{\epsilon_1}, e^{1+\epsilon_1})} w_n \leq e^{1+\epsilon_1}e^{\epsilon-1} = e^{\epsilon_2}, \ \ where \ \ \epsilon_2 = \epsilon_1 + \epsilon.$$

Corollary 4.4: For suitable $\epsilon > 0$ how that $\frac{x}{(logx)^\alpha} \sim e^A + \epsilon, \ -\infty < \alpha \leq 1$.
Proof: Since $\sum_{n\leq x} w_n \sim C\frac{x}{(logx)^\alpha}$, as $x \to \infty$, given that: $\sum_{n\in(1,e^\xi)} w_n \geq e^{kr}, r \geq 0, k \in N$, upon taking $(1, e^\xi) = x \geq N$ and $n \geq N$. Hence $e^{kr} + \epsilon \sim C\frac{x}{(logx)^\alpha}$, for $x \geq 2$, we can find that $\frac{x}{(logx)^\alpha} \sim e^A + \epsilon$, where $A = kr$.

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