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Two-Dimensional Conservative Solute Transport with Temporal and Scale-Dependent Dispersion: Analytical Solution

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ABSTRACT. This study develops a mathematical model for two-dimensional solute transport in a semi-infinite heterogeneous porous medium. The geological formation is initially not solute free. The nature of pollutants is considered conservative and gives out from space and time-dependent pulse type point source. Dispersion coefficient is considered as a linear multiple of spatially dependent function and seepage velocity where as seepage velocity is n^{th} power of spatially dependent function. Exponentially decreasing and sinusoidal form of seepage velocity are considered. The effects of retardation factor, spatial and temporal dependence on the concentration distribution are taken into account. New independent variables are introduced to covert advection dispersion equation into constant coefficients. Solutions of the proposed model are obtained using Laplace Transform Technique. Effects of parameters and value on the concentration behaviour are shown graphically.

1 Introduction

A wide variety of analytical models describing groundwater flow and solute transport in porous media have been developed in the last two-three decades. Analytical solutions for aquifers have been developed for several types of boundary conditions for a finite and semi-infinite domain. The majority of analytical solutions developed on transport issue from several stand points regarding different aspects or hypothesis in one, two and three-dimensional ground-water flow in aquifers with common assumptions, like constant porosity, steady and unsteady pore-water velocity with or without retardation factor. Retardation factor is one of the major processes

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affecting the contaminants dissolved in groundwater. In nature, most of the geological formation is heterogeneous nature which originates from the variability of the hydraulic conductivity. In other word heterogeneity of the porous medium means porosity or hydraulic conductivity are dependent upon position. Ground water velocities and hydrodynamic dispersion coefficients are key parameters for description of solute transport in porous media. Solute transport in soil, reservoir, and aquifer is generally governed by advection-dispersion equation which is a parabolic partial differential equation of second order. This equation infer from Darcys law and the law of conservation of mass (Freeze and Cherry (1979)). The effect of the velocity variations on solute transport generally has been incorporated in the transport equation using the concept of hydrodynamic dispersion. Crank (1956) obtained analytical solution in one-dimension for instantaneous point source. Al-Niami and Rushton (1997) and Marino (1978) obtained analytical solution in porous media with uniform flow. Yates (1990, 1992) obtained the analytical solutions for one-dimensional for linearly and exponentially increasing dispersion coefficient. Leij et al. (1991) and Park and Zhan (2001) derived analytical solutions for three-dimensional semi infinite porous media. Logan (1996) obtained analytical solutions considering space-dependent dispersivity of uniform and periodic input source respectively, along uniform flow. In subsurface transport spatially and temporal variation in hydraulic conductivity could be the factor in variation in pore-water velocity studied by Goode and Konikow (1990). Essa et al. (2016) explained analytical solution of advection diffusion equation by different methods. Das et al. (2017) resents mathematical modeling of groundwater contamination with varying velocity field. Moghaddam et al. (2017) developed a numerical model for one dimensional solute transport in rivers. Aral and Liao (1996) obtained analytical solutions with a time-dependent dispersion coefficient for two-dimensional advection-dispersion equation. Chen (2007) derived an analytical solution of two-dimensional advection-dispersion equation in cylindrical co-ordinates considering longitudinal and transverse dispersivity as linear function of distance. Massab et al. (2006) developed analytical solutions for the advection dispersion equation using a Bessel function expansion in cylindrical coordinates. Dreuzy et al. (2012) studied the effects of heterogeneity and temporal fluctuations of the flow conditions on the asymptotic dispersion coefficients in two-dimensional heterogeneous porous media. Yadav and Jaisawal (2011) obtained analytical solution of temporally dependent solute dispersion in two-dimensional shallow aquifer. Singh et al. (2013) obtained time-dependent analytical solutions with point-source contaminant in two-dimension finite homogeneous porous media. Yadav et al. (2012) derived analytical solutions with spatially dependent dispersion in two-dimensional porous medium. Bai et al. (2015) discussed solute transport in one-dimension using source function method. Han et al. (1985) observed that the solute dispersion may depend on the position. Singh and Chatterjee (2016) present a solution of three dimensional advection dispersion equation with non-point source of contamination in semi-infinite aquifer with specified concentration along an arbitrary plane using Laplace transform technique. Zoppou and Knight (1997) obtained analytical solution with especially variable coefficient in one, two and three-dimension. Jaiswal et al. (2009) considered dispersion is proportional to the square of seepage velocity and obtained analytical solution in a one-dimensional semi infinite inhomogeneous porous media. Djordjevich and Savovic (2013) obtained analytical solution from a pulse type point source with temporally and spatially dependent flow in two-dimensional domain. Later Savovic and Djordjevich (2013) derived numerical solution account for temporally and spatially dependent solute dispersion in semi-infinite media. Sanskrityayn et al. (2016) obtained analytical solution of advection dispersion equation with spatially dependent and temporally dependent dispersion using greens function. The objective of this study is to explore theoretical model with the effect of various parameters and pore-water velocity on the solute transport in heterogeneous porous media. The seepage flow velocity is considered exponentially decreasing and sinusoidal function of time. The source of the input concentration is considered pulse type. In the presence of the source, it is assumed to be a function of space and time, and just after the elimination of the source it is assumed to be zero. Retardation factor which is a dimension less quantity is also considered. In the present study the dispersion coefficient is considered as a linear multiple of spatially dependent function and seepage velocity where as seepage velocity is n^{th} power of spatially dependent function. It is assumed that the longitudinal and lateral dimension of the aquifer is semi-infinite. The initial concentration is supposed to be spatially dependent.

2 Mathematical Formulation and Solution of the Problem

The geological formation is assumed semi-infinite in the longitudinal and lateral directions taken as x and y axes, respectively. Due to heterogeneity both form of velocity are considered spatially and temporally dependent function. In developing the analytical solution it assumed that solute is conservative and transport through porous media in two-dimension which described by second order partial differential equation of parabolic type generally known as advection-dispersion equation.

$$R\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}(D_x\frac{\partial C}{\partial x} - uC) + \frac{\partial}{\partial y}(D_y\frac{\partial C}{\partial y} - vC) \tag{1}$$

In which $C[ML^{-3}]$ is the solute concentration of the pollutant, transporting along the flow field through the medium at a position (x, y) and time t . $D_x[L^2T^{-1}]$ and $D_y[L^2T^{-1}]$ are the longitudinal and transverse dispersion coefficients , $u[LT^{-1}]$ and $v[LT^{-1}]$ are the unsteady uniform pore seepage velocity along $x[L]$ and $y[L]$ directions respectively, $t[T]$ is the time. First term on the left hand side of the Eq.(1) is represents change in concentration with time in liquid phase and R is retardation factor which is a dimensionless quantity. First term on the right-hand side of the Eq.(1) describes the influence of the dispersion on the concentration distribution in longitudinal direction while second term is the change of the concentration due to advective transport in longitudinal direction. Third term on the right-hand side of the Eq.(1) describes the influence of the dispersion on the concentration distribution in transverse direction while fourth term is the change of the concentration due to advective transport in transverse direction. The medium is supposed to have a uniform solute concentration C_i before an injection of pollutant in the domain. The initial concentration is considered as a decreasing function of the space variable, tending to zero at the infinite extent. The input condition is considered of the pulse type. It means the source of the input concentration remains varying up to a certain time period and after its elimination the input becomes zero forever. The right boundaries along both directions are assumed that the rate of change of concentration is equal to zero. This type phenomenon may be defined mathematically as:

$$C(x, y, t) = \frac{C_i}{(1 + ax)^m(1 + by)^m}; \quad t = 0, x \geq 0, y \geq 0 \tag{2}$$

where $m \in I$, I is a set of integer.

The boundary conditions

$$-D_x\frac{\partial C}{\partial x}+uC=\begin{cases} uC_0;\ 0<t\leq t_0 \\ 0,\ t>t_0 \end{cases} \quad x=0, y=0 \tag{3.a}$$

$$-D_y\frac{\partial C}{\partial y}+vC=\begin{cases} vC_0;\ 0<t\leq t_0 \\ 0,\ t>t_0 \end{cases} \quad x=0, y=0 \tag{3.b}$$

$$\frac{\partial C(x,y,t)}{\partial x}=0, \frac{\partial C(x,y,t)}{\partial y}=0; t\geq 0, x\to\infty, y\to\infty \tag{4}$$

By Darcys equation, specific discharge hence groundwater velocity is equal to the product of hydraulic conductivity and hydraulic gradient (with a negative sign). The temporal variation in velocity is the result of temporal variation in the hydraulic gradient and spatial variation is due to as the hydraulic conductivity is variable in space. Hydraulic gradient fluctuates due to some distant fluctuating boundary conditions such as a river or a recharge area. During natural gradient tracer experiments, researchers (Sykes et al. (1982) and Sudicky (1986)) have noted that the magnitude and direction of the spatial mean hydraulic gradient fluctuate over time. This comes as natural flow systems are rarely in a steady state. Hence dispersion coefficient and velocity both are considered spatially and temporally dependent in general form. Later particular expressions are chosen. In the present case retardation factor is also considered in degenerate forms as follows:

$$\left.\begin{aligned} u&=u_0(1+ax)^n(1+by)^{n-1}f(st), \\ v&=v_0(1+by)^n(1+ax)^{n-1}f(st), \\ D_x&=D_{x_0}(1+ax)^{n+1}(1+by)^{n-1}f(st), \\ D_y&=D_{y_0}(1+by)^{n+1}(1+ax)^{n-1}f(st), \\ R&=R_0(1+ax)^{n-1}(1+by)^{n-1} \end{aligned}\right\} \text{ where } \ n\in I \tag{5}$$

where D_{x_0} and D_{y_0} are initial dispersion coefficient in longitudinal and transverse directions respectively, u_0 and v_0 are initial unsteady uniform pore seepage velocity along x and y directions respectively, and R_0 is initial retardation factor. $a[L^{-1}]$ and $b[L^{-1}]$ are heterogeneity parameter along longitudinal and lateral directions respectively and is of dimension inverse of that of space variable (Kumar et al. (2010)). The various value of (a,b) represents different heterogeneity. Heterogeneity of the porous medium which means the transport properties like porosity or hydraulic conductivity is not uniform throughout the domain but depends upon the position. $s[T^{-1}]$ represents unsteady parameter whose dimension is inverse of that of time variable, $f(st)$ is a non-dimensional expression. It is assumed that $f(st)=1$ for $s=0$ or $t=0$. The first case represents the steady flow and second case represents the initial state. We consider two temporally dependent form of $f(st)$, first one $f(st)=exp(-st)$ and second one is f(st)=1-sin(st).

Substituting values from Eq.(5) in Eq.(1), we have

$$R_0 \frac{\partial C}{\partial t}\frac{1}{f(st)} = D_{x_0}(1+ax)^2\frac{\partial^2 C}{\partial x^2} + D_{y_0}(1+by)^2\frac{\partial^2 C}{\partial y^2} - \{u_0 - a(n+1)D_{x_0}\}(1+ax)\frac{\partial C}{\partial x} - nau_0 C - \{v_0 - b(n+1)D_{y_0}\}(1+by)\frac{\partial C}{\partial y} - nbv_0 C \tag{6}$$

The initial and boundary conditions in Eqs.(2-4) may be written as :

$$C(x,y,t) = \frac{C_i}{(1+ax)^m(1+by)^m}; \quad t=0, x\geq 0, y\geq 0 \tag{7}$$

$$-D_{x_0}\frac{\partial C}{\partial x} + u_0 C = \begin{cases} u_0 C_0;\ 0<t\leq t_0 \\ 0,\ t>t_0 \end{cases} \quad x=0, y=0 \tag{8.a}$$

$$-D_{y_0}\frac{\partial C}{\partial y} + v_0 C = \begin{cases} v_0 C_0;\ 0<t\leq t_0 \\ 0,\ t>t_0 \end{cases} \quad x=0, y=0 \tag{8.b}$$

$$\frac{\partial C(x,y,t)}{\partial x} = 0, \frac{\partial C(x,y,t)}{\partial y} = 0; t\geq 0, x\rightarrow\infty, y\rightarrow\infty \tag{9}$$

Let us introduce new independent space variables X and Y by transformations (Kumar et al., (2010));

$$\left.\begin{aligned} X = log(1+ax) \Rightarrow \frac{dX}{dx} &= \frac{a}{(1+ax)}, \\ Y = log(1+by) \Rightarrow \frac{dY}{dy} &= \frac{a}{(1+ay)} \end{aligned}\right\} \tag{10}$$

Applying transformation of Eq. (10) on Eq. (6), we have

$$R_0\frac{\partial C}{\partial t}\frac{1}{f(st)} = a^2 D_{x_0}\frac{\partial^2 C}{\partial X^2} + b^2 D_{y_0}\frac{\partial^2 C}{\partial Y^2} - (au_0 - na^2 Dx_0)\frac{\partial C}{\partial X} - (bv_0 - nb^2 Dy_0)\frac{\partial C}{\partial Y} - nau_0 C - nbv_0 C \tag{11}$$

Eqs.(7-9) may be written in terms of new independent space variables X and Y by using Eq. (10), as :

$$C(X,Y,t) = C_i exp\{-m(X+Y)\}; t=0, X\geq 0, Y\geq 0 \tag{12}$$

$$-a^2 D_{x_0}\frac{\partial C}{\partial X} + au_0 C = \begin{cases} au_0 C_0;\ 0<t\leq t_0 \\ 0,\ t>t_0 \end{cases} \quad X=0, Y=0 \tag{13.a}$$

$$-b^2 D_{y_0}\frac{\partial C}{\partial Y} + bv_0 C = \begin{cases} bv_0 C_0;\ 0<t\leq t_0 \\ 0,\ t>t_0 \end{cases} \quad X=0, Y=0 \tag{13.b}$$

$$\frac{\partial C(X,Y,t)}{\partial X} = 0, \frac{\partial C(X,Y,t)}{\partial Y} = 0, t\geq 0, X\rightarrow\infty Y\rightarrow\infty \tag{14}$$

Let us introduce another non-dimensional space variable Z through a transformation, similar to the one used in an earlier work (Carnahan and Remer, (1984));

$$Z = X + Y \tag{15}$$

Applying the transformation of Eq. (15) on the Eq. (11), we have

$$R_0\frac{\partial C}{\partial t}\frac{1}{f(st)}=D_0\frac{\partial^2 C}{\partial Z^2}-U_0\frac{\partial C}{\partial Z}-\gamma_0 C \tag{16}$$

where $D_0=(a^2Dx_0+b^2Dy_0),\omega_0=(au_0+bv_0),U_0=(\omega_0-nD_0),\gamma_0=n\omega_0$.
Eqs.(12-14) may be written in terms of new independent space variable Z defined by Eq. (15) as:

$$C(Z,t)=C_i exp\{-mZ\};t=0,Z\geq 0 \tag{17}$$

$$-D_0\frac{\partial C}{\partial Z}+\omega_0 C=\begin{cases}\omega_0 C_0;\ 0<t\leq t_0\\ 0,\ t>t_0\end{cases}\quad Z=0 \tag{18}$$

$$\frac{\partial C(Z,t)}{\partial Z}=0;t\geq 0,Z\to\infty \tag{19}$$

Let us introduce a new independent time variable, T by a transformation defined as (Crank, (1975)):

$$T=\int_0^t f(st)dt \tag{20}$$

we have, $T=\frac{1}{s}\{1-exp(-st)\}$ when $f(st)=exp(-st)$ and $T=\frac{1}{s}[st-\{1-cos(st)\}]$ when $f(st)=1-sin(st)$. Eq. (16), reduces into

$$R_0\frac{\partial C}{\partial T}=D_0\frac{\partial^2 C}{\partial Z^2}-U_0\frac{\partial C}{\partial Z}-\gamma_0 C \tag{21}$$

Eqs.(17-19) may be written in terms of new independent variable T defined by
Eq. (20) as :

$$C(Z,T)=C_i exp(-mZ);T=0,Z\geq 0 \tag{22}$$

$$-D_0\frac{\partial C}{\partial Z}+\omega_0 C=\begin{cases}\omega_0 C_0;\ 0<T\leq T_0\\ 0,\ T>T_0\end{cases}\quad Z=0 \tag{23}$$

$$\frac{\partial C(Z,T)}{\partial Z}=0\quad T\geq 0,Z\to\infty \tag{24}$$

Using the following transformation (25) on Eqs.(21-24) , above equations may be written in terms of new dependent variable K(Z,T) as:

$$C(Z,T)=K(Z,T)exp\{\frac{U_0}{2D_0}Z-\frac{1}{R_0}(\frac{U_0^2}{4D_0}+\gamma_0)T\} \tag{25}$$

$$R_0\frac{\partial K}{\partial T}=D_0\frac{\partial^2 K}{\partial Z^2} \tag{26}$$

and corresponding initial and boundary conditions in terms of new dependent variable $K(Z,T)$ may be written as follows :

$$K(Z,T)=C_i exp\{-(m+\frac{U_0}{2D_0})Z\};T=0,Z\geq 0 \tag{27}$$

$$-D_0\frac{\partial K(Z,T)}{\partial Z}+(\frac{2\omega_0-U_0}{2})K=\begin{cases}\omega_0 C_0 exp(\alpha^2 T);\ 0<T\leq T_0\\ 0,\ T>T_0\end{cases}\quad Z=0 \tag{28}$$

$$\frac{\partial K(Z,T)}{\partial Z}+\frac{U_0}{2D_0}K=0;\;\; T\geq 0, Z\rightarrow\infty \tag{29}$$

where $\alpha^2=\frac{1}{R_0}(\frac{U_0^2}{4D_0}+\gamma_0)$.

Applying the Laplace transformation on above initial and boundary value problem, it reduces into an ordinary differential equation of second order, which comprises in following three equations:

$$\frac{d^2\overline{K}}{dZ^2}-\frac{pR_0}{D_0}\overline{K}=-\frac{R_0C_i}{D_0}exp\{-(m+\frac{U_0}{2D_0})Z\} \tag{30}$$

where $\overline{K}=\int\limits_0^\infty K(Z,T)e^{-pT}dT$

$$-D_0\frac{d\overline{K}}{dZ}+(\frac{2\omega_0-U_0}{2})\overline{K}=\frac{\omega_0C_0}{p-\alpha^2}(1-exp\{-(p-\alpha^2)T_0\})\;\;;Z=0 \tag{31}$$

$$\frac{d\overline{K}}{dZ}+\frac{U_0}{2D_0}\overline{K}=0\;\;;Z\rightarrow\infty \tag{32}$$

where p is a Laplace parameter.

Thus the general solution of ordinary differential equation (30) may be written as:

$$\overline{K}(Z,p)=c_1exp(-Z\sqrt{\frac{pR_0}{D_0}})+c_2exp(Z\sqrt{\frac{pR_0}{D_0}})+\frac{c_iexp(-(m+\frac{U_0}{2D_0})Z)}{(p-\beta^2)} \tag{33}$$

where $\beta^2=\frac{(U_0+2mD_0)^2}{4R_0D_0}$.

Now, using boundary conditions Eq.(31) and Eq.(32) in general solution Eq. (33) to eliminate an arbitrary constants c_1 and c_2 , we get the particular solution of the above boundary value problem as

$$\begin{aligned}\overline{K}(Z,p)=&\frac{\omega_0C_0}{\sqrt{R_0D_0}}\frac{(1-exp\{-(p-\alpha^2)T_0\})}{(p-\alpha^2)(\sqrt{p}+\alpha)}exp(-Z\sqrt{\frac{pR_0}{D_0}})-\frac{\omega_0C_i}{\sqrt{R_0D_0}}\frac{exp(-Z\sqrt{\frac{pR_0}{D_0}})}{(p-\beta^2)(\sqrt{p}+\alpha)}\\&+\frac{c_iexp(-(\frac{U_0+2mD_0}{2D_0})Z)}{(p-\beta^2)}\end{aligned} \tag{34}$$

Now, Applying inverse Laplace transformation on it, using the appropriate tables (van Genuchten and Alves, (1982)), using back transformations Eq.(25), Eq.(20), Eq.(15) and Eq.(10) the analytical solution of advection-diffusion equation for pulse type input point source may be written in terms of $C(Z,T)$ as:

$$C(Z,T)=C_0F(Z,T)+C_iG(Z,T);\;\; 0<T\leq T_0 \tag{35.a}$$

$$C(Z,T)=C_0F(Z,T)-C_0F(Z,T-T_0)+C_iG(Z,T);\;\; T>T_0 \tag{35.b}$$

where

$$\begin{aligned}F(Z,T)=&\frac{\omega_0T}{\sqrt{\pi D_0R_0T}}exp(\frac{U_0Z}{2D_0}-\frac{R_0Z^2}{4D_0T}-\frac{\gamma^2T}{4D_0R_0})+\frac{\omega_0}{2\gamma}exp(\frac{(U_0-\gamma)Z}{2D_0}).erfc(\frac{R_0Z-\gamma T}{\sqrt{4D_0R_0T}})\\&-\frac{\omega_0}{2\gamma}(1+\frac{\gamma Z}{D_0}+\frac{\gamma^2T}{R_0D_0})exp(\frac{(U_0+\gamma)Z}{2D_0})erfc(\frac{R_0Z+\gamma T}{\sqrt{4D_0R_0T}})\end{aligned}$$

$$\begin{aligned}G(Z,T)=&exp((\frac{m^2D_0+mU_0-\gamma_0}{R_0})T)[exp(-mZ)-\frac{\omega_0}{2(\omega_0+mD_0)}exp(-mZ)erfc(\frac{R_0Z-(U_0+2mD_0)T}{\sqrt{4D_0R_0T}})\\&-\frac{\omega_0}{(n-m)D_0}exp(\frac{(U_0+mD_0)Z}{D_0})erfc(\frac{R_0Z+(U_0+2mD_0)T}{\sqrt{4D_0R_0T}})]\\&-\frac{\omega_0(\omega_0+nD_0)}{2D_0(m^2D_0+mU_0-\gamma_0)}exp(\frac{(U_0+\gamma)Z}{2D_0})erfc(\frac{R_0Z+\gamma T}{\sqrt{4D_0R_0T}})\end{aligned}$$

$Z = log\{(1+ax)(1+by)\}, T = \int_0^t f(st)dt,$
$D_0 = a^2 D_{x_0} + b^2 D_{y_0}, \omega_0 = (au_0 + bv_0),$
$U_0 = \omega_0 - nD_0, \gamma_0 = n\omega_0, \gamma^2 = U_0^2 + 4n\omega_0 D_0,$
m and n both are integers but $m \neq n$.

3 Result and Discussions

In this study the input parameters, values and the ranges of these parameters within which they are varied taken either from published literature or empirical relationship. For example the range of seepage velocity, keeping in view the different types of soils, aquifers is lies between 2m/ day to 2 m/year (Todd, (1980)). The concentration values C/C_0 are evaluated assuming reference concentration as $C_0 = 1.0, C_i = 0.1$ in a finite domain $0 \leq x(m) \leq 4$ and $0 \leq y(m) \leq 2$ along longitudinal and transverse directions respectively. The transverse components of velocity and dispersion coefficient are approximately one-tenth of the respective longitudinal components because transverse mixing, although smaller than longitudinal mixing, cannot be ignored in shallow aquifers and surface water bodies. The medium is supposed heterogeneous along both directions. The source of pollution is supposed to be eliminated at a time t_0=6(day). The units of distance and time are considered in meter and day, respectively. The common input values are taken $m = 2, n = 5, u_0 = 0.50(m/day), v_0 = 0.050(m/day)$ for all cases dicussed below. Since m and n belongs to any integer and results holds for all m, n. The solution obtained in Eq. (35.a) and (35.b) are represents the concentration scenario in the presence and absence of source respectively.

3.1 Case-I: Figs. (1-5) demonstrate the concentration behaviour in presence of source $(0 < t \leq t_0)$. The common input data are taken as $a = 0.01(m^{-1})$ and $b = 0.01(m^{-1})$ for Figs(1-4).

Figs.1 (a, b) illustrated the spatial distribution of the dimensionless concentration predicted by the present solution at time $t = 2$ and 5(days) computed for the common parameters $R_0 = 1.15, D_{x_0} = 0.85(m^2/day), D_{y_0} = 0.085(m^2/day), s = 0.1(day^{-1})$ the ground water velocity are taken exponentially decreasing and sinusoidal function of time. The input concentration, C/C_0 at the origin, $x = 0, y = 0$ for seepage velocity exponentially decreasing and sinusoidal form at time $t = 2$ and 5(days)are respectively (0.6104,0.7379) and (0.6093,0.7313). It attenuates with position and time in both forms of the velocity but at particular position along longitudinal and transverse directions the concentration level for sinusoidal velocity is lower in comparison to exponential. At particular position the concentration level is lower for lower time and higher for larger time in both form of velocity. The concentration pattern increases with respect to time, whereas it decreases with respect to the space. It is observed that the solute transport and the rehabilitation are slightly faster in case of sinusoidal velocity than those of exponential velocity in both longitudinal and transverse direction. The decreasing tendency of contaminant concentration with time and distance travelled may help to predict the harmless concentration level in the domain.

Figs. 2(a, b) illustrates the effect of retardation factor on concentration distribution behaviour with various retardation factors $R_0 = 1.15$ and 1.55 along the longitudinal and lateral directions of the medium, described by the solution in Eq. (35.a), i.e., in the presence of the source of pollutant in the time domain $0 < t \leq t_0$, at common parameters $t = 5(day), D_{x_0} = 0.85(m^2/day), D_{y_0} = 0.085(m^2/day), s = 0.1(day^{-1})$ The input concentration, C/C_0 at the origin, $x = 0$ and $y = 0$ for seepage velocity exponentially decreasing and sinusoidal form at retardation factor $R_0 = 1.15$ and 1.55 are respectively $(0.7378, 0.6896)$ and $(0.7313, 0.6828)$. Higher values of retardation factor reduce the solute concentration. At particular position the concentration level is lower for higher retardation factor while higher for lower retardation factor in both form of velocity but the concentration for sinusoidal velocity is decreases slightly faster in comparison to exponential velocity in both longitudinal and transverse directions. It may be observed that the trends of concentration profiles are similar for both forms of the velocities at fixed time. Figs. 3(a, b) illustrates the effect of dispersion coefficient on concentration distribution behaviour with different form of pore-water velocity with common parameters $t = 5(day)$ and $R_0 = 1.15$ and $s = 0.1(day^{-1})$. Variation in longitudinal and lateral dispersion coefficients are taken as $D_{x_0} = 0.85(m^2/day), D_{y_0} = 0.085(m^2/day)$ and $D_{x_0} = 1.25(m^2/day), D_{y_0} = 0.125(m^2/day)$. It may be observed that the trends of concentration profiles at particular position along longitudinal and transverse direction are same but at particular position the concentration level is lower for the lower dispersion coefficient and higher for higher dispersion in both forms of velocity. It may also observed that concentration profile for sinusoidal velocity decreases slightly faster in comparison to exponential velocity in both longitudinal and transverse directions.

Figs. 4(a, b) illustrates the effect of unsteady parameter on concentration profiles computed for $s = 0.1(day^{-1})$ and $s = 0.3(day^{-1})$ with parameters $R_0 = 1.15, D_{x_0} = 0.85(m^2/day), D_{y_0} = 0.085(m^2/day)$ and $t = 5(day)$. It reveals that the trends of contaminant concentration profiles at particular position is higher for smaller unsteady parameter and lower for larger in both form of velocity but the concentration profiles for sinusoidal velocity is decreases slightly faster in comparison to exponential velocity in both longitudinal and transverse directions.

Figs. 5(a, b) illustrated the effect of heterogeneity parameters (a, b) on concentration profiles computed at various heterogeneity parameters $\{a = 0.01(m^{-1}), b = 0.01(m^{-1})\}, \{a = 0.03(m^{-1}), b = 0.03(m^{-1})\}$ and $\{a = 0.05(m^{-1}), b = 0.05(m^{-1})\}$ and other common values are taken $t = 5(day), R_0 = 1.15, D_{x_0} = 0.85(m^2/day), D_{y_0} = 0.085(m^2/day)$ and $s = 0.1(day^{-1})$ with different form the ground water velocity. The input concentration, C/C_0 at the origin, $x = 0, y = 0$ for seepage velocity exponentially decreasing and sinusoidal form at heterogeneity parameters $\{a = 0.01(m^{-1}), b = 0.01(m^{-1})\}, \{a = 0.03(m^{-1}), b = 0.03(m^{-1})\}$ and $\{a = 0.05(m^{-1}), b = 0.05(m^{-1})\}$ are respectively (0.7378,0.6840,0.6341) and (0.7313,0.6789,0.6303). It depict that contaminant attenuates with position and time in both forms of the velocity but at particular position along longitudinal and transverse directions the concentration level for sinusoidal velocity is lower in comparison to exponential. It may also observe that at particular position the concentration level is lower for larger heterogeneity parameters and higher for smaller heterogeneity parameters in both form of velocity. The concentration pattern increases with increasing in heterogeneity parameters, whereas it decreases with respect to the space. It is observed that the solute transport and the rehabilitation are slightly faster in case of sinusoidal velocity than those of exponential velocity in both longitudinal and transverse direction. The decreasing tendency of contaminant concentration with time and distance travelled may help to predict the harmless concentration level in the domain.

3.2 Case-II: Figs. (6-10) illustrates the concentration pattern in absence of source$(t > t_0)$ for the solution obtained as in Eq.(35.b). The time of elimination of the source of pollution is considered as $t_0 = 6(day)$. The common input data taken as $a = 0.01m^{-1}$ and $b = 0.01m^{-1}$ for Figs.(6-9).

Figs. 6(a, b) shows the effect of various time on concentration distribution pattern computed with common parameters $R_0 = 1.15, D_{x_0} = 0.85(m^2/day), D_{y_0} = 0.085(m^2/day)$,$s = 0.1(day^{-1})$ at time $t = 8$ and 11(day). It reveals that concentration values near the origin for the exponentially decreasing nature of velocity are less than the sinusoidal form of velocity. At particular position along horizontal and transverse direction the concentration level for exponential velocity is lower in comparison to sinusoidal. The concentration level for exponential velocity is decreases slightly faster than sinusoidal form of velocity in longitudinal and transverse directions as time or space or both increases.

Figs. 7(a, b) demonstrates the effect of retardation factors $R_0 = 1.15$ and $R_0 = 1.55$ on concentration behavior commutated with common parameters $t = 11(day)$,$D_{x_0} = 0.85(m^2/day), D_{y_0} = 0.085(m^2/day)$ and $s = 0.1(day^{-1})$. It depict that the trends of contaminant concentration profiles are nearly similar in both forms of velocity. Concentration profile is increases initially near the boundary and starts rehabilitate toward harmless level. It may observed that at particular position along longitudinal and transverse direction the concentration level for exponential form of velocity is lower as compared to sinusoidal. The concentration level near the boundary is lower for lower retardation factor and higher for higher retardation factor in both forms of velocity but away from source boundary these trends reverse. Rehabilitation rate is slightly faster when pore water velocity is in exponential form in both longitudinal and transverse as retardation increases.

Figs. 8(a, b) are drawn for various dispersion parameters $D_{x_0} = 0.85(m^2/day), D_{y_0} = 0.085(m^2/day)$ and $D_{x_0} = 1.25(m^2/day), D_{y_0} = 0.125(m^2/day)$ for the common parameters $t = 11(day), R_0 = 1.15, s = 0.1(day^{-1})$. The ground water velocity is taken exponentially decreasing and sinusoidal function of time. It reveals that the trends of contaminant concentration profiles are nearly similar in both forms of velocity. Concentrations are increases initially near to boundary up to certain distance after then decreases with distance travelled in domain from the boundary but at a particular position along longitudinal and transverse direction the concentration level for exponential velocity is lower in comparison to sinusoidal. At particular position the concentration level near to boundary is lower for higher dispersion parameter and higher for lower dispersion parameter in both forms of velocity but after certain distance travelled from boundary the concentration level is lower for lower dispersion parameter and higher for higher dispersion parameter in both forms of velocity. The concentration level for sinusoidal velocity is decreases slightly faster in comparison to exponential velocity in both longitudinal and transverse directions as dispersion parameter increases.

Figs. 9(a, b) demonstrates the effect of unsteady parameters $s = 0.1(day^{-1})$ and $s = 0.2(day^{-1})$ on concentration profile computed at common values $t = 11(day), R_0 = 1.15, D_{x_0} = 0.85(m^2/day), D_{y_0} = 0.085(m^2/day)$. It may observed that at particular position the concentration level near the boundary is lower for lower unsteady parameters and higher for higher in both forms of velocity but along both axis concentration level is lower for higher unsteady parameters and higher for lower unsteady parameters. The concentration level for sinusoidal velocity is decreases little faster in comparison to exponential velocity in both longitudinal and transverse directions as unsteady parameters or distance increases.

Figs. 10(a, b) are drawn for various heterogeneity parameters $\{a = 0.01(m^{-1}), b = 0.01(m^{-1})\}, \{a = 0.03(m^{-1}), b = 0.03(m^{-1})\}$ and $\{a = 0.05(m^{-1}), b = 0.05(m^{-1})\}$ with the common parameters and value $t = 11(day)$, $R_0 = 1.15, D_{x_0} = 0.85(m^2/day), D_{y_0} = 0.085(m^2/day)$, $s = 0.1(day^{-1})$. The input concentration, C/C_0 at the origin,

$x = 0, y = 0$ for seepage velocity exponentially decreasing and sinusoidal form at heterogeneity parameters $\{a = 0.01(m^{-1}), b = 0.01(m^{-1})\}, \{a = 0.03(m^{-1}), b = 0.03(m^{-1})\}$ and $\{a = 0.05(m^{-1}), b = 0.05(m^{-1})\}$ are respectively $(0.2131, 0.1734, 0.1394)$ and $(0.2833, 0.2393, 0.2004)$. It reveals that the concentration values are decreases as heterogeneity parameters increases. The concentration level for sinusoidal velocity is decreases slightly faster in comparison to exponentially decreasing velocity in both longitudinal and transverse directions as heterogeneity parameters increases. At particular position the concentration level is lower for higher heterogeneity parameters and higher for lower heterogeneity parameters in both forms of velocity. It may also observe that heterogeneity causes larger variations in velocity and dispersion, ultimately concentration attenuation at a position is faster than that in a medium of lesser heterogeneity. Thus by varying the value of parameter (a, b) one may easily access the effect of heterogeneity on concentration distribution.

3.3 Particular cases:

3.3.1. For steady flow:

If we extend in Eq.(35a,b) $n = 1, m = 0, s = 0$ and $R = 1$, the Eq.(35a,b) agrees with solution derived by Jaiswal et al. (2011) , given as:

$$C(Z,t) = C_0 F(Z,t) + C_i G(Z,t);\ \ 0 < t \le t_0 \tag{36.a}$$

$$C(Z,t) = C_0 F(Z,t) - C_0 F(Z, t - t_0) + C_i G(Z,t);\ \ t > t_0 \tag{36.b}$$

where,

$F(Z,t) = \frac{\omega_0 t}{\sqrt{\pi D_0 t}}.exp(\frac{U_0 Z}{2D_0} - \frac{Z^2}{4D_0 t} - \frac{\gamma^2 t}{4D_0}) + \frac{\omega_0}{2\gamma}.exp(\frac{(U_0-\gamma)Z}{2D_0})erfc(\frac{Z-\gamma t}{2\sqrt{D_0 t}}) - \frac{\omega_0}{2\gamma}(1 + \frac{\gamma Z}{D_0} + \frac{\gamma^2 t}{D_0}).exp(\frac{(U_0+\gamma)Z}{2D_0})erfc(\frac{Z+\gamma t}{2\sqrt{D_0 t}})$

$G(Z,t) = exp(\gamma_0 t)[1 - \frac{1}{2}erfc(\frac{Z-U_0 t}{2\sqrt{D_0 t}}) - \frac{\omega_0}{D_0}exp(\frac{U_0 Z}{D_0})erfc(\frac{Z+U_0 t}{2\sqrt{D_0 t}})] - \frac{(\omega_0+D_0)}{2D_0}exp(\frac{(U_0+\gamma)Z}{2D_0})erfc(\frac{Z+\gamma t}{2\sqrt{D_0 t}})$.

$Z = log\{(1+ax)(1+by)\}, D_0 = a^2 D_{x_0} + b^2 D_{y_0}, \omega_0 = (au_0 + bv_0)$,

$U_0 = (\omega_0 - D_0), \gamma_0 = \omega_0, \gamma^2 = (U_0^2 + 4\omega_0 D_0)$.

3.3.2 For One-dimensional solution along steady flow:

If we extend in Eq.(35a,b) $n = -1, s = 0, y = 0, D_{y_0} = 0, v_0 = 0$ and $C_i = 0$ the Eq.(35a,b) agrees with solution derived by Kumar and Yadav, (2015), given as

$$C(X,t) = C_0 F(X,t));\ \ 0 < t \le t_0 \tag{37.a}$$

$$C(X,t) = C_0 F(X,t) - C_0 F(X, t - t_0);\ \ t > t_0 \tag{37.b}$$

$F(X,t) = \frac{\omega_0 t}{\sqrt{\pi D_0 R_0 t}}.exp(\frac{U_0 X}{2D_0} - \frac{R_0 X^2}{4D_0 t} - \frac{\gamma^2 t}{4D_0 R_0}) + \frac{\omega_0}{2\gamma}.exp(\frac{(U_0-\gamma)X}{2D_0})erfc(\frac{R_0 X - \gamma t}{2\sqrt{D_0 R_0 T}})$

$- \frac{\omega_0}{2\gamma}(1 + \frac{\gamma X}{D_0} + \frac{\gamma^2 t}{D_0 R_0}).exp(\frac{(U_0+\gamma)X}{2D_0})erfc(\frac{R_0 X + \gamma T}{2\sqrt{D_0 R_0 t}})$

$X = log(1+ax), D_0 = a^2 D_{x_0}, \omega_0 = au_0$,

$U_0 = (\omega_0 + D_0), \gamma_0 = -\omega_0, \gamma^2 = U_0^2 - 4\omega_0 D_0$.

3.3.3 For One-dimensional solution along steady flow:

If we extend in Eq.(35a,b) $n = 1, s = 0, y = 0, D_{y_0} = 0, v_0 = 0$ and $C_i = 0$ the Eq.(35a,b) agrees with solution derived by Kumar et al. (2010), given as

$$C(X,t) = C_0 F(X,t));\ \ 0 < t \le t_0 \tag{38.a}$$

$$C(X,t) = C_0F(X,t) - C_0F(X,t-t_0);\ \ t > t_0 \tag{38.b}$$

$F(X,t) = \frac{\omega_0 t}{\sqrt{\pi D_0 R_0 t}}.exp(\frac{U_0 X}{2D_0} - \frac{R_0 Z^2}{4D_0 t} - \frac{\gamma^2 t}{4D_0 R_0}) + \frac{\omega_0}{2\gamma}.exp(\frac{(U_0-\gamma)X}{2D_0})erfc(\frac{R_0 X - \gamma T}{2\sqrt{D_0 R_0 T}})$
$- \frac{\omega_0}{2\gamma}(1 + \frac{\gamma X}{D_0} + \frac{\gamma^2 t}{D_0 R_0}).exp(\frac{(U_0+\gamma)X}{2D_0})erfc(\frac{R_0 X + \gamma T}{2\sqrt{D_0 R_0 t}})$
$X = log(1+ax), D_0 = a^2 D_{x_0}, \omega_0 = au_0,$
$U_0 = (\omega_0 - D_0), \gamma_0 = \omega_0, \gamma^2 = U_0^2 + 4\omega_0 D_0.$

It may be noted that the derived result can be extended in three-dimensions which may be given by Eqs.(39a, b) as

$$C(Z,T) = C_0F(Z,T) + C_iG(Z,T);\ \ 0 < T \leq T_0 \tag{39.a}$$

$$C(Z,T) = C_0F(Z,T) - C_0F(Z,T-T_0) + C_iG(Z,T);\ \ T > T_0 \tag{39.b}$$

where
$F(Z,T) = \frac{\omega_0 T}{\sqrt{\pi D_0 R_0 T}}.exp(\frac{U_0 Z}{2D_0} - \frac{R_0 Z^2}{4D_0 T} - \frac{\gamma^2 T}{4D_0 R_0}) + \frac{\omega_0}{2\gamma}.exp(\frac{(U_0-\gamma)Z}{2D_0})erfc(\frac{R_0 Z - \gamma T}{2\sqrt{D_0 R_0 T}})$
$- \frac{\omega_0}{2\gamma}(1 + \frac{\gamma Z}{D_0} + \frac{\gamma^2 T}{D_0 R_0}).exp(\frac{(U_0+\gamma)Z}{2D_0})erfc(\frac{R_0 Z + \gamma T}{2\sqrt{D_0 R_0 T}})$
$G(Z,T) = exp\{(\frac{m^2 D_0 + mU_0 - \gamma_0}{R_0})T\}[exp(-mZ) - \frac{\omega_0}{2(\omega_0 + mD_0)}exp(-mZ)erfc(\frac{R_0 Z - (U_0 + 2mD_0)T}{2\sqrt{D_0 R_0 T}})$
$- \frac{\omega_0}{(n-m)D_0}exp\frac{(U_0 + mD_0)Z}{D_0}erfc(\frac{R_0 Z + (U_0 + 2mD_0)T}{2\sqrt{D_0 R_0 T}})] - \frac{\omega_0(\omega_0 + nD_0)}{2D_0(m^2 D_0 + mU_0 - \gamma_0)}.exp(\frac{(U_0+\gamma)Z}{2D_0})erfc(\frac{R_0 Z + \gamma T}{2\sqrt{D_0 R_0 T}})$
$Z = log\{(1+ax)(1+by)(1+cz)\}, T = \int_0^t f(st)dt$, $D_0 = a^2 D_{x_0} + b^2 D_{y_0} + c^2 D_{z_0}, \omega_0 = (au_0 + bv_0 + cw_0),$
$U_0 = \omega_0 - nD_0, \gamma_0 = n\omega_0, \gamma^2 = U_0^2 + 4n\omega_0 D_0.$

4 Conclusion

Analytical solutions have been obtained for spatially dependent solute concentration for a source of the varying pulse type input along a temporally exponential decreasing and sinusoidal form of seepage flow in two -dimensional semi-infinite heterogeneous porous medium. The spatial dependence is due to heterogeneity of the medium. Laplace transform technique is used to obtain the analytical solutions which is simpler, more viable and commonly used. The derived result may helpful to predict the concentration levels at space and time which may help to reduce/eliminate the concentration level in domain. It may also be observed that in a medium for which the values of a and b are less, the concentration values at a particular position and at a particular time are less than those in a medium of higher heterogeneity. It is observed that the solute transport along the medium for sinusoidal form velocity is faster than exponentially decreasing nature. The decreasing tendency of contaminant concentration with time and position may help to understand rehabilitation tendency of the contaminated aquifer. It may observe that if source of pollution is fully eliminated the concentration peak value lowers down with time and tramp away from the origin and hence obtained solutions may help to predict the time period of rehabilitation.

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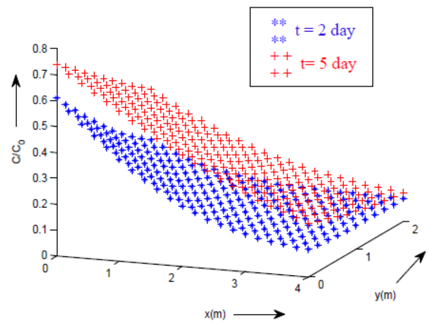

Fig. 1(a). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.a) for the exponentially decreasing form of velocity at various time.

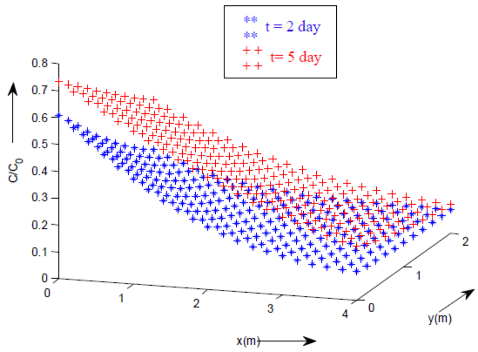

Fig. 1(b). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.a) for the sinusoidal form of velocity at various time.

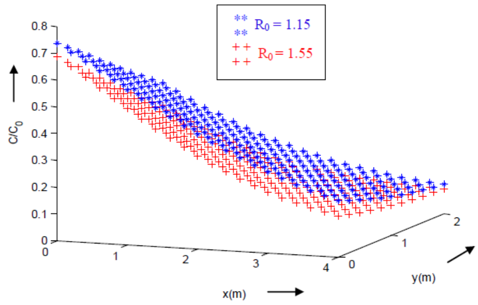

Fig. 2(a). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.a) for the exponential decreasing form of velocity at various retardation factor.

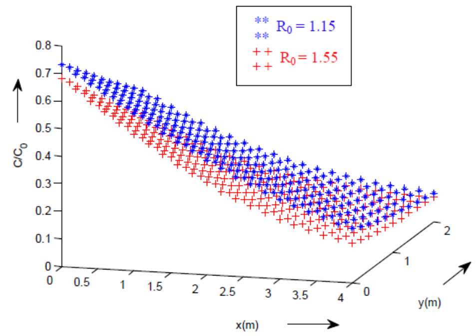

Fig. 2(b). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.a) for the sinusoidal form of velocity at various retardation factor.

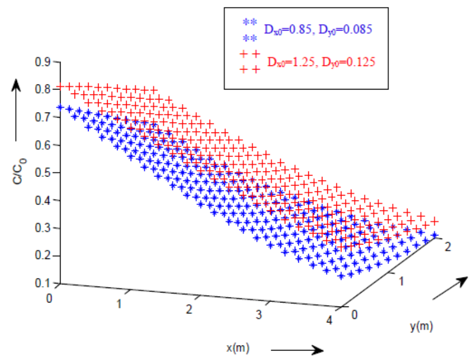

Fig. 3(a). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.a) for the exponentially decreasing form of velocity at various dispersion coefficient.

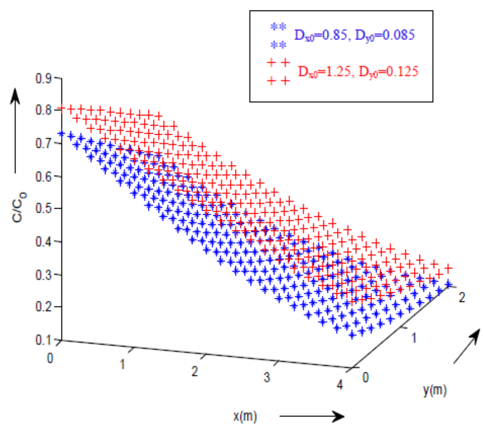

Fig. 3(b) Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.a) for the sinusoidal form of velocity at various dispersion coefficient.

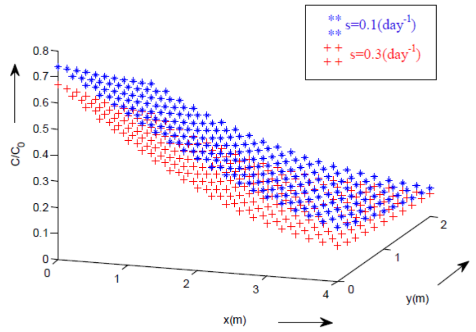

Fig. 4(a). Dimensionless concentration distribution evaluated by analytical solution presented in (35.a) for the exponentially decreasing form of velocity at various unsteady parameter.

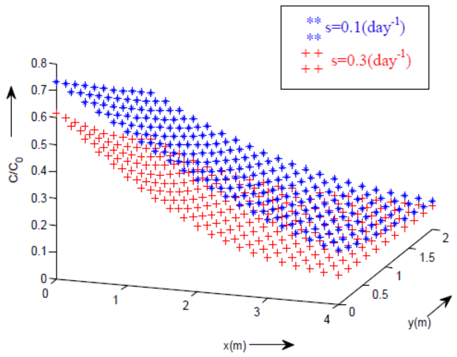

Fig. 4(b). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.a) for the sinusoidal form of velocity at various unsteady parameter.

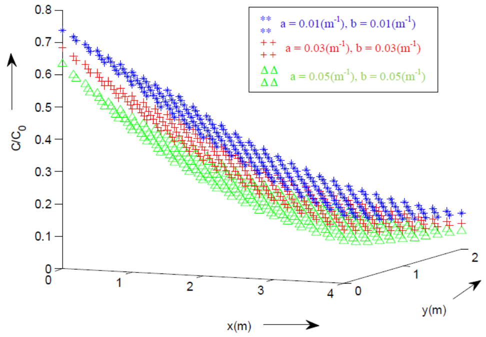

Fig. 5(a). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.a) for the exponentially decreasing form of velocity at various heterogeneity parameters.

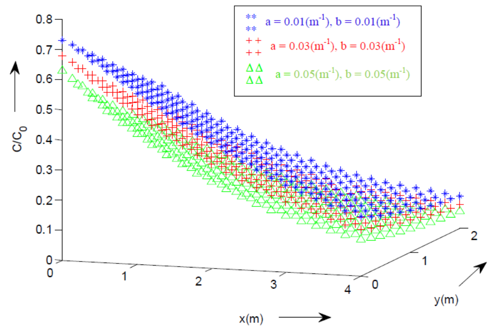

Fig. 5(b). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.a) for the sinusoidal form of velocity at various heterogeneity parameters.

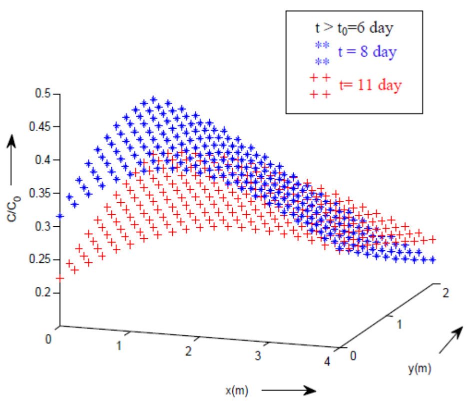

Fig. 6(a). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.b) for the exponentially decreasing form of velocity at various time.

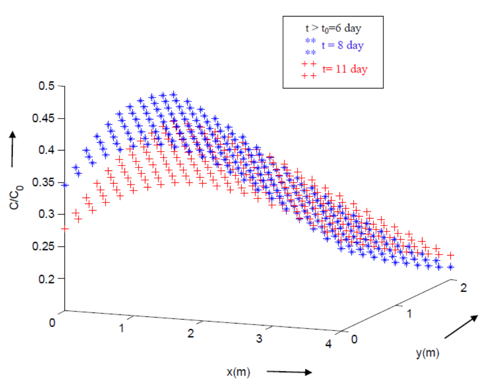

Fig. 6(b). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.b) for the sinusoidal form of velocity at various time.

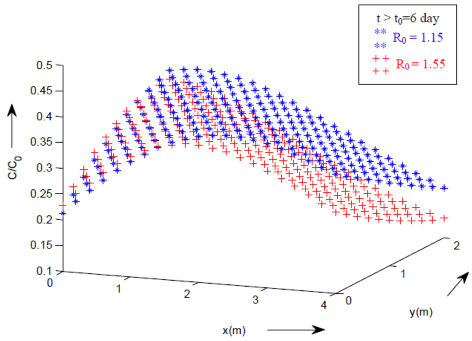

Fig. 7(a). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.b) for the exponentially decreasing form of velocity at various retardation coefficient.

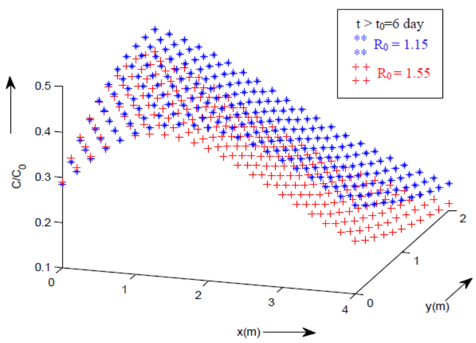

Fig. 7(b). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.b) for the sinusoidal form of velocity at various retardation coefficient.

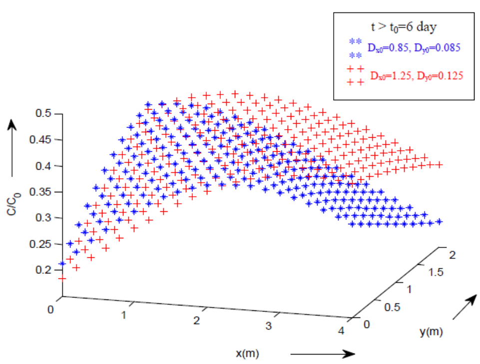

Fig. 8(a). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.b) for the exponentially decreasing form of velocity at various dispersion coefficient.

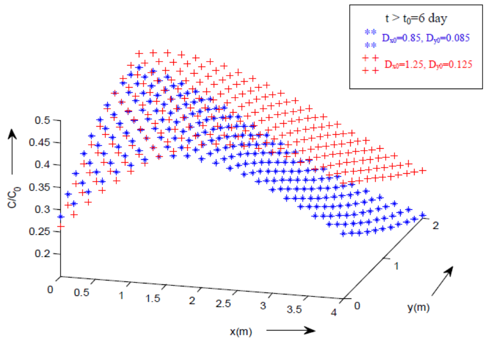

Fig. 8(b). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.b) for the sinusoidal form of velocity at various dispersion coefficient.

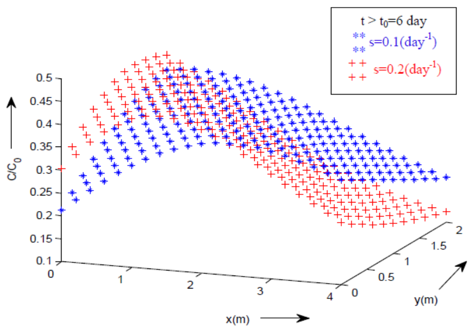

Fig. 9(a). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.b) for the exponentially decreasing form of velocity at various unsteady parameter.

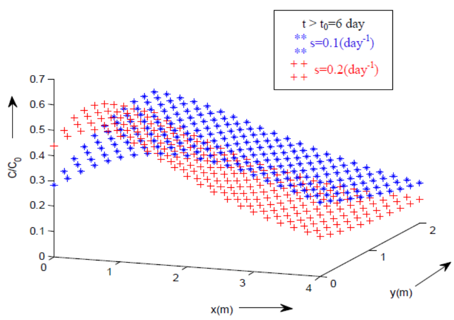

Fig. 9(b). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.b) for the sinusoidal form of velocity at various unsteady parameter.

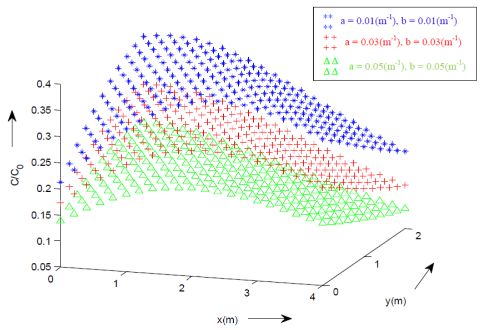

Fig. 10(a). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.b) for the exponentially decreasing form of velocity at various heterogeneity parameters.

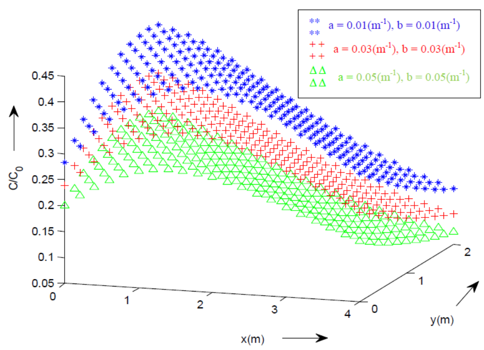

Fig. 10(b). Dimensionless concentration distribution evaluated by analytical solution presented in Eq.(35.b) for the sinusoidal form of velocity at various heterogeneity parameters.