Dynamics of Superior Anti-fractals in a New Orbit

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ABSTRACT. Anti-fractals have interesting features in the complex graphics of dynamical system. The aim of this paper is to visualize Superior tricorns and multicorns using a new iteration introduced by M. Abbas et al.[8] and study the pattern among the anti-fractals in the complex dynamics of anti-polynomial $z \rightarrow z^m + c$, for $m \geq 2$.

1 Introduction

Fractals are defined as objects that appear to be broken into number of pieces and each piece is a copy of the entire shape. Fractal is the word taken from the Latin word fractus which means broken. The term fractal was first used by a young mathematician, Julia [5] in 1918. Julia introduced the concept of iterative function system (IFS) and derived the Julia set in 1919. After that, in 1982, Mandelbrot [2] extended the work of Gaston Julia and introduced the Mandelbrot set, a set of all connected Julia sets. Many researchers have studied Julia and Mandelbrot sets from different aspects.

The polynomials $z \rightarrow z^m + c$, for $m \geq 2$, have been studied mathematically using one step feedback process. Crowe et. al.[15] considered it as a formal analogy with Mandelbrot sets and named it as Mandelbar set. They also brought their bifurcation features along arcs rather than at points. Multicorns have been found in a real slice of the cubic connectedness locus [15]. Winter[13] showed that the boundary of the tricorn contains arc. The symmetries of tricorn and multicorns have been analyzed by Lau and Schleicher[4]. In 2003, Nakane and Schleicher[14] presented beautiful figures and quoted that multicorns are the generalized tricorns or the tricorns of higher order.

* Corresponding Author.
Received September 02, 2017; revised October 12, 2017; accepted October 18, 2017.
2010 Mathematics Subject Classification: 37F45, 37F50.
Key words and phrases: Complex polynomials, Superior anti-fractals, tricorn, multicorn, new orbit.
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The dynamics of anti-holomorphic complex polynomials \( z \rightarrow z^m + c \), for \( m \geq 2 \), was studied and explored to visualize interesting tri-corns and multi-corns anti-fractals with respect to one-step feedback process\[12\], two step-feedback process \[10, 11\], three-step feedback process\[16\] and four step feedback process\[1, 3\]. In this paper we generate a new class of tricorns and multicorns using a new four-step feedback process\[8\] and analyze them.

2 Preliminaries and notations

Definition 2.1[12]. The multicorns \( A_c \) for the quadratic function \( A_c(z) = z^m + c \) is defined as the collection of all \( c \in \mathbb{C} \) for which the orbit of the point 0 is bounded, that is

\[
A_c = \{ c \in \mathbb{C} : A_c^n(0) \text{ do not tend to } \infty \}
\]

(2.1)

where \( \mathbb{C} \) is a complex space. \( A_c^n \) is the \( n \)th iterate of the function \( A_c(z) \). An equivalent formulation is that the connectedness of loci for higher degree anti-holomorphic polynomials \( A_c(z) = z^m + c \) are called multicorns.

Note that at \( m = 2 \), multicorns reduce to tricorn. Naturally, the tricorns lives in the real slice \( d = c \) in the two dimensional parameter space of maps \( z \rightarrow (z^2 + d)^2 + c \). They have \((m+1)\)-fold rotational symmetries. Also, by dividing these symmetries, the resulting multicorns are called unicorns \[14\].

Definition 2.2[6]. The filled in Julia set of the function \( g \) is defined as

\[
K(g) = \{ z \in \mathbb{C} : g^k(z) \text{ does not tend to } \infty \},
\]

where \( \mathbb{C} \) is the complex space, \( g^k(z) \) is \( k \)th iterate of function \( g \) and \( K(g) \) denotes the filled Julia set. The Julia set of the function \( g \) is defined to be the boundary of \( K(g) \), i.e., \( J(g) = \partial K(g) \), where \( J(g) \) denotes the Julia set.

Definition 2.3[12]. The Mandelbrot set \( M \) consists of all parameters \( c \) for which the filled Julia set of \( Q_c \) is connected, that is

\[
M = \{ c \in \mathbb{C} : K(Q_c) \text{ is connected} \}.
\]

In fact, \( M \) contains an enormous amount of information about the structure of Julia sets. The Mandelbrot set \( M \) for the Quadratic \( Q_c(z) = z^2 + c \) is defined as the collection of all \( c \in \mathbb{C} \) for which the orbit of the point 0 is bounded, that is

\[
M = \{ c \in \mathbb{C} : \{ Q_c^n \} ; n = 0, 1, 2, .. \text{is bounded} \}.
\]

We choose the initial point 0 as 0 is the only critical point of \( Q_c \).

Now, we give definition of the new orbit, which will be used in the paper to implement four-step feedback process in the dynamics of polynomial \( z \rightarrow z^m + c \).

Definition 2.4[8]. Let us consider a sequence \( \{ x_n \} \) of iterates for initial point \( x_0 \in X \) such that

\[
\begin{align*}
x_{n+1} : x_{n+1} &= (1 - \alpha_n) Ty_n + \alpha_n Tz_n; \\
y_n &= (1 - \beta_n) Tx_n + \beta_n Tz_n; \\
z_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n; & n = 0, 1, 2, ..
\end{align*}
\]

(2.2)
where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences of positive numbers. Then the sequence (1) is a function (New Orbit) of five tuples $(T, x_0, \alpha_n, \beta_n, \gamma_n)$.

For visualizing new Superior anti-fractals, the required escape criterion with respect to the new orbit for $z \rightarrow z^m + c$ is $\max\{ |c|, (2/\alpha)^{m-1}, (2/\beta)^{m-1}, (2/\gamma)^{m-1} \}$[7].

3 Multicorns in New Orbit

In this section, we programmed the polynomial $z \rightarrow z^m + c$ in the software Mathematica 9.0 and generate Superior tricorns and multicorns in a new orbit (see Figs. 1-14).

We find the following observations from generated Superior multicorns:

- The number of branches in the Superior tricorns and multicorns is $m + 1$, where $m$ is the power of $z$. Also, few branches have $m$ subbranches (see Figs. 6, 7).
- Superior Multicorns exhibit $(m + 1)$-fold rotational symmetries.
- There exist many Superior multicorns for any $m$.
- We also find that higher degree Superior multicorns become circular saw (Figs. 13-14).

Some authors [1,3,11] had also found the similar conclusion while generating multicorns using two-step, three-step, four-step feedback processes. The name circular saw was, first, given by Rani and Kumar to Mandelbrot sets [9].

3.1 Superior Tricorns for $m = 2$:

Figure 1: $\alpha = \beta = 0.3, \gamma = 0.1$

Figure 2: $\alpha = \beta = 0.3, \gamma = 0.1$

Figure 3: $\alpha = \beta = \gamma = 0.3$
3.2 Superior Multicorns for $m = 3$: 

Figure 4: $\alpha = 0.1, \beta = 0.9, \gamma = 0.1$

Figure 5: $\alpha = \beta = 0.9, \gamma = 0.1$

Figure 6: $\alpha = \beta = 0.6, \gamma = 0.1$

Figure 7: $\alpha = 0.1, \beta = \gamma = 0.6$

Figure 8: $\alpha = 0.6, \beta = 0.1, \gamma = 0.6$

Figure 9: $\alpha = 0.1, \beta = \gamma = 0.9$
3.3 Superior Multicorns for higher degrees:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10.png}
\caption{Figure 10: $m = 4, \alpha = 0.1, \beta = 0.9, \gamma = 0.1$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11.png}
\caption{Figure 11: $m = 6, \alpha = 0.6, \beta = 0.1, \gamma = 0.6$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{Figure 12: $m = 10, \alpha = 0.6, \beta = 0.1, \gamma = 0.6$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure13.png}
\caption{Figure 13: Circular saw multicorn for $m = 50, \alpha = \beta = 0.6, \gamma = 0.1$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure14.png}
\caption{Figure 14: Circular saw multicorn for $m = 100, \alpha = \beta = 0.6, \gamma = 0.1$}
\end{figure}

4 New Superior Anti-Julia Sets

Superior Anti Julia sets have been generated for $z \rightarrow z^m + c$ in a new orbit. In Figures 15-17, we can see that the Superior Anti Julia sets look like Superior tricorns or multicorns for $m = 2$. Also, we observed that the higher degree Superior anti Julia sets took different shapes for different values of $m, \alpha, \beta, \gamma$ and $c$. 
Figure 15: Superior AntiJulia set for $m = 2, \alpha = 0.4, \beta = 1.0, \gamma = 1.0, c = 0.3 + 0.5i$

Figure 16: Superior AntiJulia set for $m = 2, \alpha = \beta = \gamma = 0.5, c = 0.3 + 0.5i$

Figure 17: Superior AntiJulia set for $m = 2, \alpha = \beta = \gamma = 0.5, c = 0.1 + 0.1i$

Figure 18: Superior AntiJulia set for $m = 3, \alpha = \beta = \gamma = 0.4, c = 0.7 + 0.7i$

Figure 19: Superior AntiJulia set for $m = 3, \alpha = \beta = 0.1, \gamma = 0.05, c = 0.6 + 0.5i$
5 Conclusion

In the dynamics of anti-polynomials $z \rightarrow z^m + c$, where $m \geq 2$, there exist many Superior multicom for the same value of $m$ in the new orbit. We also generate some Superior Anti- Julia sets in the new orbit. In our results, we
found that for higher degrees of the polynomial, all the Superior anti-fractals become circular saw.

References


