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Some Properties of Differential Sandwich Results of p-valent Functions Defined by Liu-Srivastava Operator

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ABSTRACT. In this paper, we obtain some results of differential subordination and superordination for p-valent function in the open unit disk involving operator. Also we derive some sandwich theorems.

1 Introduction

Let $\mathcal{H}(\Delta)$ denote the class of functions analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, 3..\}$, let $\mathcal{H}[a, n] = \{f \in \mathcal{H}(\Delta) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ... , z \in \Delta\}$, with $\mathcal{H}_0 = \mathcal{H}[0, 1]$ and $\mathcal{H}_1 = \mathcal{H}[1, 1]$.
Let $W\sum_p$ be the subclass of $\mathcal{H}(\Delta)$ consisting of functions of the form :

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (z \in \Delta, p \in \mathbb{N}), \tag{1}$$

and $W\sum = W\sum_1$. For functions $f(z) \in W\sum_p$, given by (1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \quad (z \in \Delta, p \in \mathbb{N}), \tag{2}$$

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the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z), \quad (z \in \Delta, p \in \mathbb{N}). \tag{3}$$

A function $f \in W\sum_p$ is said to be starlike function of order ρ, if

$$\Re\Big(\frac{zf'(z)}{f(z)}\Big) > \rho, \quad (z \in \Delta, 0 < \rho \leq 1).$$

Denote the class of starlike functions by $S^*(\rho)$. Inparticulary the class $S^*(0) = S^*$.

A functon $f \in W\sum_p$ is said to be convex function of order ρ, if

$$\Re\Big(1 + \frac{zf''(z)}{f(z)}\Big) > \rho, \quad (z \in \Delta, 0 < \rho \leq 1).$$

Denote the class of convex functions by $C(\rho)$. In particulary the class $C(0) = C$. Let f and F be members of $\mathcal{H}(\Delta)$, the function f is said to be subordinat to F , or F is said to be superordinate to f, if there exist a Schwarz function w analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$ for all $z \in \Delta$, and denote that by $f \prec F$ or $f(z) \prec F(z)$. If the function $F(z)$ is univalent in Δ, then, we have $f \prec F$ if and only if $f(0) = F(0)$ and $f(\Delta) \subset F(\Delta)$, see[19,20].

Suppose that h and k are two analytic functions in Δ, let

$$\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}.$$

If $h(z)$ be univalent in Δ and $k(z)$ is satisfies the second order subordination

$$\phi(k(z), zk^{'}(z), z^2k^{''}(z); z) \tag{4}$$

then $k(z)$ is called a solution of the differential subordination (4). A univalent function $q(z)$, is called a dominant of the solutions of the differential subordination (4), moreover simply a dominant if $k(z) \prec q(z)$ for all $k(z)$ satisfy (4).

A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q(z)$ of (4) is said to be the best dominant of (4). Related results on subordination can be found in [11,12,14,15,17,21,25].

Let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$, if k and $\phi(k(z), zk^{'}(z), z^2k^{''}(z); z) \prec h(z)$ are univalent in Δ and if k satisfies the second order superordination.

$$h(z) \prec \phi(k(z), zk^{'}(z), z^2k^{''}(z); z) \tag{5}$$

then $k(z)$ is called a solution of the differential superordination (5). Note that if f is subordinat to F, then F is superordinate to f. An analytic function $q(z)$ is called a subordinant if , $q(z) \prec k(z)$, for all $k(z)$ satisfy (5). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants $q(z)$ of (5) is said to be the best subordinant.

Miller and Mocanu [20] obtained sufficient conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(k(z), zk^{'}(z), z^2k^{''}(z); z) \Rightarrow q(z) \prec k(z) \tag{6}$$

Ali et al.[1], and Aouf et al.[5], obtained sufficient conditions for certain normalized analytic functions f to satisfy :

$$q_1(z) \prec \Big(\frac{zf'(z)}{f(z)}\Big) \prec q_2(z), \tag{7}$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = q_2(0) = 1$. So newly,Shanmugam et al.[26,27] and Goyal et al.[11] obtained it and called sandwich results for certain classes of analytic functions . Further superordination results can be found in [1-7].

For a complex parameters α_i where $(i = 0, 1, ...q)$ and β_i where $(i = 0, 1, ...s)$ such that $(\beta_i \neq 0, -1, -2, ...;\ i = 1, 2, ..s)$, the generalized hypergeometric function ${}_q^sF(\alpha_1, \alpha_2, ...\alpha_q;\ \beta_1, \beta_2, ...\beta_s; z)$ is given by,see [8,9,22]: as follows:

$$ {}_q^sF(\alpha_1, \alpha_2, ...\alpha_q;\ \beta_1, \beta_2, ...\beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_q)_n}{(\beta_1)_n ... (\beta_s)_n} \frac{z^n}{n!} $$
$$ (q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \Delta), $$

where $(x)_n$ is the Pochhammer symbol (or shifted factorial) defined in terms of the Gamma function by

$$ (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1;\ if\ \ n = 0, x \in \mathbb{C}^* = \mathbb{C}\backslash\{0\} \\ x(x+1)...(x+n-1);\ \ if\ \ n \in \mathbb{N}, x \in \mathbb{C} \end{cases} $$

Corresponding to a function $h_p(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s; z)$ defined by
$h_p(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s; z) = z^p\ {}_q^sF(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s; z)$.
Liu-Srivastava[16] consider a linear operator $H_{p,q,s}(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s; z) : W\sum_p \to W\sum_p$ defined by the following Hadamard product (or convolution):

$$ h_{p,q,s}(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s; z) = h_p(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s; z) * f(z) \tag{8} $$

This operator was encourage essentially by Dziok and Srivastava([8,9] ; see also [16]. The theory of differential subordination in $\mathbb{C}$ is a generalization of differential disparity in $\Re$, and this theory of differential subordination was initiated by the works of Miller and Mocanu[18],many important works on differential subordination were great by Miller and Mocanu,and their monograph[19] complied their huge efforts in introducing and developing the same. Newly Miller and Mocanuin[20] investigated the dual problem of differential superordination, while Bulboac[7] investigates both subordination and superordination.

For $v > -p$ and function $f \in W\sum_p$, in the form (1). The Ruscheweyh derivative of order $(v+p-1)th$ is denoted by D^{v+p-1} and consider as following: See[13,24],

$$ D^{v+p-1}f(z) = \frac{z^p(z^{v-1}f(z))^{v+p-1}}{(v+p-1)!} = \frac{z^p}{(1-z)^{v+p}} * f(z). $$

In [14] define the linear operator $F_{p,q,s}[\alpha_1, v]$ on $W\sum_p$ as follows:

$$ \begin{aligned} F_{p,q,s}[\alpha_1, v]f(z) &= H_{p,q,s}[\alpha_1] * D^{v+p-1}f(z) \\ &= z^p + \sum_{k=p+1}^{\infty} \Lambda\sigma_{k,p}(\alpha_1)\varphi(v+p-1, k)a_k z^k, \end{aligned} $$

where $\Lambda = \frac{\prod_{i=1}^{s}\Gamma(\beta_i)}{\prod_{i=1}^{q}\Gamma(\alpha_i)}$, $\sigma_{k,p}(\alpha_1) = \frac{\prod_{i=1}^{q}\Gamma(\alpha_i+k-p)}{\prod_{i=1}^{s}\Gamma(\beta_i+k-p)}$ and

$$ \varphi(v+p-1, k) = \begin{pmatrix} v+p-1+k-1 \\ v+p-1 \end{pmatrix} \tag{9} $$

Then, we have

$$ z(F_{p,q,s}[\alpha_1, v]f(z))' = \alpha_1 F_{p,q,s}[\alpha_1+1, v]f(z) - (\alpha_1 - p)F_{p,q,s}[\alpha_1, v]f(z), \tag{10} $$

that easily to verify it by applying (9).

2 Preliminary Results

Definition 2.1[20]: Let Q the set of all functions $f(z)$ that are analytic and injective on $\overline{\Delta}\backslash E(f)$, where $E(f) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\}$, and are such that $f^{'}(z) \neq 0$ for $\zeta \in \partial\Delta\backslash E(f)$.

In order to prove our sandwich results, the following lemmas are needed:

Lemma 2.2[19]: Let $q(z)$ be univalent in the open unit disk Δ, and let θ and ϕ be analytic in the domain D containing $q(\Delta)$, with $\phi(w) \neq 0$ when $w \in q(\Delta)$.

Set $\mathcal{Q}(z) = zq^{'}(z)\phi(q(z)),\ \ h(z) = \theta(q(z)) + \mathcal{Q}(z)$, and suppose that

1. $\mathcal{Q}(z)$ is a starlike function in Δ,
2. $\Re\left(\frac{zh^{'}(z)}{\mathcal{Q}(z)}\right) > 0$

for all $z \in \Delta$. If $r(z)$ is analytic in Δ, with $r(0) = q(0), r(\Delta) \subseteq D$ and

$$\theta(r(z)) + zr^{'}(z)\phi(r(z)) \prec \theta(q(z)) + zq^{'}(z)\phi(q(z)) \tag{11}$$

then $r(z) \prec q(z)$, and $q(z)$ is the best dominant of (11).

Lemma 2.3[27]: Let $\lambda, \gamma \in \mathbb{C}$ with $\gamma \neq 0$, and let the function $q(z)$ be convex univalent function in Δ with

$$\Re\left(1 + \frac{zq^{''}(z)}{q^{'}(z)}\right) > max\left(0, -\Re\begin{pmatrix}\lambda\\ \gamma\end{pmatrix}\right), z \in \Delta.$$

If $r(z)$ is analytic function in Δ and

$$\lambda r(z) + \gamma zr^{'}(z) \prec \lambda q(z) + \gamma zq^{'}(z) \tag{12}$$

then $r(z) \prec q(z)$, and $q(z)$ is the best dominant of (12).

Lemma 2.4[20]: Let $q(z)$ be a convex univalent in Δ with $q(0) = 1$. Let $\tau \in \mathbb{C}$, and $\Re(\tau) > 0$. If $r(z) \in \mathcal{H}[q(0), 1] \cap Q$ and $r(z) + \tau zr^{'}(z)$ is univalent in Δ, then

$$q(z) + \tau zq^{'}(z) \prec r(z) + \tau zr^{'}(z) \tag{13}$$

which implies that $q(z) \prec r(z)$, and $q(z)$ is the best subordinant of (13).

Lemma 2.5[1]: Let $q(z)$ be a convex univalent in Δ, and let θ and ϕ be analytic in a domain D containing $q(\Delta)$. Suppose that

1. $\mathcal{Q}(z) = zq^{'}(z)\theta(q(z))$ is a starlike function in Δ,
2. $\Re\left(\frac{\phi^{'}(q(z))}{\phi(q(z))}\right) > 0$,

for all $z \in \Delta$. If $r(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $r(\Delta) \subseteq D$ such that $\theta(r(z)) + zr^{'}(z)\phi(r(z))$ is univalent in Δ, and

$$\theta(q(z)) + zq^{'}(z)\phi(q(z)) \prec \theta(r(z)) + zr^{'}(z)\phi(r(z)) \tag{14}$$

then $q(z) \prec r(z)$, and $q(z)$ is the best subordinant of (14).

Lemma 2.6[23]: The function $q(z) = (1 - z)^{-2ab}$, where $a, b \in \mathbb{C}^*$ is univalent in Δ if and only if $|2ab \mp 1| \leq 1$.

Lemma 2.7[10]: Let f be analytic in D, with $f(0) = f^{'}(0) - 1 = 0$. Then $f \in S^*$ if and only if $zf^{'}(z)/f(z) \in$ P. (where P is the class of all function φ analytic and having positive real part in D, with $\varphi(0) = 1$)

3 Subordination Main Results

Theorem 3.1 Let $q(z)$ be a convex univalent in Δ with $q(0) = 1,\ 0 < \delta < 1.\ \beta \in \mathbb{C}^*$ and suppose that

$$\Re\Big(1 + \frac{zq''(z)}{q'(z)}\Big) > max\{0; -\Re(\frac{1}{\delta\beta})\}. \tag{15}$$

If $f \in W\Sigma_p$ satisfies the subordination

$$\Upsilon_1(z) \prec q(z) + \delta\beta zq'(z), \tag{16}$$

where

$$\Upsilon_1(z) = (1-\alpha_1\beta)\Big(\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\Big)^{\frac{1}{\delta}} + \alpha_1\beta\Big(\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\Big)^{\frac{1}{\delta}}\frac{F_{p,q,s}[\alpha_1+1, v]f(z)}{F_{p,q,s}[\alpha_1, v]f(z)} \tag{17}$$

then

$$\Big(\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\Big)^{\frac{1}{\delta}} \prec q(z), \tag{18}$$

and $q(z)$ is the best dominant of (16).

Proof. Define the analytic function

$$r(z) = \Big(\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\Big)^{\frac{1}{\delta}},\ z \in \Delta \tag{19}$$

differentiating (19) logarithmically with respect to z, we get

$$\frac{zr'(z)}{r(z)} = \frac{1}{\delta}\Big(\frac{z\Big(F_{p,q,s}[\alpha_1, v]f(z)\Big)'}{F_{p,q,s}[\alpha_1, v]f(z)} - p\Big)$$

and using the identity (10), we have

$$\frac{zr'(z)}{r(z)} = \frac{\alpha_1}{\delta}\Big(\frac{F_{p,q,s}[\alpha_1+1, v]f(z)}{F_{p,q,s}[\alpha_1, v]f(z)} - 1\Big).$$

Therefore

$$\delta\beta zr'(z) = \alpha_1\beta\Big(\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\Big)^{\frac{1}{\delta}}\Big(\frac{F_{p,q,s}[\alpha_1+1, v]f(z)}{F_{p,q,s}[\alpha_1, v]f(z)} - 1\Big),$$

hence the subordination (16) and from hypothesis, yield $r(z) + \delta\beta zr'(z) \prec q(z) + \delta\beta q'(z)$. By applying Lemma 2.3 for special case $\lambda = 1$, and $\gamma = \delta\beta$, leads to (18) consequently the proof of Theorem 3.1 is completed.

Putting $q(z) = \frac{1+AZ}{1+BZ}$, where $-1 \le B < A \le 1$ in the Theorem 3.1,the condition (15) reduces to:(see[6,17]).

$$\Re\Big(\frac{1-BZ}{1+BZ}\Big) > max\{0; -\Re(\frac{1}{\delta\beta})\},\ z \in \Delta. \tag{20}$$

It is easy to verify that the function $\phi(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < |B|$, is convex in Δ, and since $\phi(\bar{\zeta}) = \overline{\phi(\zeta)}$ for all $|\zeta| < |B|$ it follows that $\phi(\Delta)$ is a convex domain symmetric with respect to the real axis , hence

$$inf\Big(\Re\Big(\frac{1-BZ}{1+BZ}\Big) : z \in \Delta\Big) = \frac{1-|B|}{1+|B|} \tag{21}$$

Then, the inequality (20) is equivalent to

$$\Re\Big(\frac{1}{\delta\beta}\Big) \ge \frac{|B|-1}{|B|+1},$$

hence, we have the following result.

Corollary 3.2. Let $-1 \le B < A \le 1$, and $0 < \delta < 1, \beta \in \mathbb{C}^*$ with

$$max\Big(0; -\Re\Big(\frac{1}{\delta\beta}\Big)\Big) \le \frac{1-|B|}{1+|B|}$$

with $f \in W\Sigma_p$ and $\Upsilon_1(z)$ is given by (17), satisfies the subordination

$$\Upsilon_1(z) \prec \frac{1+Az}{1+Bz} + \frac{\delta\beta(A-B)z}{(1+Bz)^2}, \tag{22}$$

then,

$$\Big(\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\Big)^{\frac{1}{\delta}} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant of (22).

For $A = 1$ and $B = -1$, the last corollary becomes.

Corollary 3.3. Let $0 < \delta < 1$ and $\beta \in \mathbb{C}^*$ with $\Re\Big(\frac{1}{\delta\beta}\Big) \geq 0$. If $f \in W\Sigma_p$ and $\Upsilon_1(z)$ is given by (17), satisfies the subordination

$$\Upsilon_1(z) \prec \frac{1+z}{1-z} + \frac{2\delta\beta z}{(1-z)^2}, \tag{23}$$

then

$$\Big(\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\Big)^{\frac{1}{\delta}} \prec \frac{1+z}{1-z},$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant of (23).

Theorem 3.4. Let $q(z)$ be univalent in Δ with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in \Delta$. Let $\delta, \gamma \in \mathbb{C}^*$ and $\alpha, x, y \in \mathbb{C}$ with $x + y \neq 0$. Let $f \in W\sum_p$ and suppose that f and q satisfy the following condition:

$$(x+y)^{-1}z^p\{xF_{p,q,s}[\alpha_1+1, v]f(z) + yF_{p,q,s}[\alpha_1, v]f(z)\} \neq 0,\ z \in \Delta \tag{24}$$

and

$$\Re\Big(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\Big) > 0,\ z \in \Delta. \tag{25}$$

If

$$\alpha + \frac{\gamma}{\delta}\Big(\frac{xz\Big(F_{p,q,s}[\alpha_1+1, v]f(z)\Big)' + yz\Big(F_{p,q,s}[\alpha_1, v]f(z)\Big)'}{xF_{p,q,s}[\alpha_1+1, v]f(z) + yF_{p,q,s}[\alpha_1, v]f(z)} - p\Big) \prec \alpha + \gamma\frac{zq'(z)}{q(z)}, \tag{26}$$

then

$$\Big((x+y)^{-1}z^p\{xF_{p,q,s}[\alpha_1+1, v]f(z) + yF_{p,q,s}[\alpha_1, v]f(z)\}\Big)^{\frac{1}{\delta}} \prec q(z),$$

and $q(z)$ is the best dominant of (26).

Proof. According to (24), we consider the analytic function

$$r(z) = \Big((x+y)^{-1}z^p\{xF_{p,q,s}[\alpha_1+1, v]f(z) + yF_{p,q,s}[\alpha_1, v]f(z)\}\Big)^{\frac{1}{\delta}},\ z \in \Delta \tag{27}$$

with $r(0) = 1$. By logarithmically differentiating of (27) yields

$$\frac{zr'(z)}{r(z)} = \frac{1}{\delta}\Big(\frac{xz\Big(F_{p,q,s}[\alpha_1+1, v]f(z)\Big)' + yz\Big(F_{p,q,s}[\alpha_1, v]f(z)\Big)'}{xF_{p,q,s}[\alpha_1+1, v]f(z) + yF_{p,q,s}[\alpha_1, v]f(z)} - p\Big),$$

let us consider the function

$$\theta(w) = \alpha \text{ and } \phi(w) = \frac{\gamma}{w},$$

then θ is analytic in $\mathbb{C}$ and $\phi(w) \neq 0$ is analytic in $\mathbb{C}^*$. If we suppose

$$\Omega(z) = zq'(z)\phi(q(z)) = \gamma\frac{zq'(z)}{q(z)},\ z \in \Delta, \text{and}$$

$$h(z) = \theta(q(z)) + \Omega(z) = \alpha + \gamma \frac{zq'(z)}{q(z)},\ z \in \Delta.$$

From the assumption (25),we see that $\Omega(z)$ is starlike function in Δ, and also have

$$\Re\Big(\frac{zh'(z)}{\Omega(z)}\Big) = \Re\Big(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\Big) > 0,\ z \in \Delta.$$

Now,using Lemma 2.2,we derive the subordination (26) implies $r(z) \prec q(z)$ and the function $q(z)$ is the best dominant of (26).

Letting $x = 0, y = \alpha = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ in the last Theorem3.4, it is easy to view that the assumption (25) holds whenever $-1 \le B < A \le 1$ which leads to the following result.

Corollary 3.5. Let $-1 \le B < A \le 1$ and let $\delta, \gamma \in \mathbb{C}^*$. Let $f \in W\sum_p$ and suppose that $z^{-p}F_{p,q,s}[\alpha_1, v]f(z) \ne 0, z \in \Delta$.

$$1 + \frac{\gamma}{\delta}\Big(\frac{z\Big(F_{p,q,s}[\alpha_1, v]f(z)\Big)'}{F_{p,q,s}[\alpha_1, v]f(z)} - p\Big) \prec 1 + \gamma \frac{(A - B)z}{(1 + Az)(1 + Bz)} \tag{28}$$

then $\Big(z^{-p}F_{p,q,s}[\alpha_1, v]f(z)\Big)^{\frac{1}{\delta}} \prec \frac{(1+Az)}{(1+Bz)}$, and $q(z) = \frac{(1+Az)}{(1+Bz)}$ is the best dominant of (28). Taking $x = 0, p = y = \alpha = 1, \alpha_i = \beta_i (i = 1, 2, ..s), \gamma = \frac{1}{ab}, \delta = \frac{1}{b}$ where $a, b \in \mathbb{C}^*$ and $q(z) = (1 - z)^{-2ab}$ in Theorem 3.4, then merge this together with Lemma2.6,we obtain the next result.

Corollary 3.6. Let $a, b \in \mathbb{C}^*$ such that $|2ab \mp 1| \le 1$. Let $f \in W\sum_p$ and suppose that $z^{-1}f(z) \ne 0$ for all $z \in \Delta$. If

$$1 + \frac{1}{a}\Big(1 + \frac{zf'(z)}{f(z)}\Big) \prec \frac{1 + z}{1 - z}, \tag{29}$$

then

$$[zf(z)]^b \prec (1 - z)^{-2ab},$$

and $q(z) = (1 - z)^{-2ab}$ is the best dominant of (29).

Again by setting $x = 0, p = y = \alpha = 1, \alpha_i = \beta_i(i = 1, 2, ..s), \gamma = \frac{e^{im}}{ab\ cosm}$, where $a, b \in \mathbb{C}^*, |m| < \frac{\pi}{2}, \delta = \frac{1}{b}$ and $q(z) = (1 - z)^{-2ab\ cosme^{-im}}$ in Theorem3.4, we obtain the next result due to Aouf et al[6].

Corollary 3.7. Let $a, b \in \mathbb{C}^*$ and assume that $|2abcosme^{-im} \mp 1| \le 1$ such that $|m| < \frac{\pi}{2}$. Let $f \in W\Sigma_p$ and $z^{-1}f(z) \ne 0$ for all $z \in \Delta$. If

$$1 + \frac{e^{im}}{a\ cosm}\Big(\frac{zf'(z)}{f(z)} - 1\Big) \prec \frac{1 + z}{1 - z}, \tag{30}$$

then

$$[z^{-1}f(z)]^b \prec (1 - z)^{-2ab\ cosme^{-im}}$$

and $q(z) = (1 - z)^{-2ab\ cosme^{-im}}$ is the best dominant of (30).

Theorem 3.8. Let $q(z)$ be univalent in Δ with $q(0) = 1$, let $\delta, \gamma \in \mathbb{C}^*$ and $\alpha, x, y \in \mathbb{C}$ such that $x + y \ne 0$. Let $f \in W\Sigma_p$ and suppose that f and q satisfy the next conditions:

$$(x + y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1 + 1, v]f(z) + yF_{p,q,s}[\alpha_1, v]f(z)\} \ne 0,\ z \in \Delta, \tag{31}$$

and

$$\Re\Big(1 + \frac{zq''(z)}{q'(z)}\Big) > max\{0; -\Re(\frac{\alpha}{\gamma})\}. \tag{32}$$

If

$$\begin{aligned} \Upsilon_2(z) = &\Big((x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\}\Big)^{\frac{1}{\delta}} \\ &\times\Big(\alpha+\frac{\gamma}{\delta}\Big(\frac{xz\Big(F_{p,q,s}[\alpha_1+1,v]f(z)\Big)'+yz\Big(F_{p,q,s}[\alpha_1,v]f(z)\Big)'}{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)}-p\Big)\Big) \end{aligned} \tag{33}$$

and

$$\Upsilon_2(z) \prec \alpha q(z)+\gamma z q'(z), \tag{34}$$

then

$$\Big((x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\}\Big)^{\frac{1}{\delta}} \prec q(z),$$

and $q(z)$ is the best dominant of (34).

Proof. We begin by define the function

$$r(z) = \Big((x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\}\Big)^{\frac{1}{\delta}} \tag{35}$$

From (31) the function $r(z)$ is analytic in Δ, with $r(0)=1$, and differentiating (35) logarithmically with respect to z, we have

$$\frac{zr'(z)}{r(z)} = \frac{1}{\delta}\Big(\frac{xz\Big(F_{p,q,s}[\alpha_1+1,v]f(z)\Big)'+yz\Big(F_{p,q,s}[\alpha_1,v]f(z)\Big)'}{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)}-p\Big),$$

and hence

$$zr'(z) = \frac{r(z)}{\delta}\Big(\frac{xz\Big(F_{p,q,s}[\alpha_1+1,v]f(z)\Big)'+yz\Big(F_{p,q,s}[\alpha_1,v]f(z)\Big)'}{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)}-p\Big). \tag{36}$$

By setting $\theta(w)=\alpha w, \phi(w)=\gamma,\ w\in\mathbb{C}$. Then we get

$$\mathcal{Q}(z) = zq'(z)\phi(q(z)) = \gamma z q'(z),\ z\in\Delta, \text{and}$$

$$h(z) = \theta(q(z))+\mathcal{Q}(z) = \alpha q(z)+\gamma z q'(z),\ z\in\Delta.$$

From the assumption (32), we see that $\mathcal{Q}(z)$ is starlike function in Δ and we also have

$$\Re\Big(\frac{zh'(z)}{\mathcal{Q}(z)}\Big) = \Re\Big(\frac{\alpha}{\gamma}+1+\frac{zq''(z)}{q'(z)}\Big) > 0;\ z\in\Delta. \tag{37}$$

Now, application of Lemma 2.2 the proof of Theorem 3.8 is complete.

Letting $x=\alpha=1, y=0, q(z)=\frac{1+Az}{1+Bz}$ in Theorem 3.8, where $-1\le B<A\le 1$ and according to (21) the condition (32) becomes $max\Big(0;-\Re\frac{\alpha}{\gamma}\Big)\le\frac{1-|B|}{1+|B|}$. we obtain the next result.

Corollary 3.9. Let $-1\le B<A\le 1$ and let $\delta,\gamma\in\mathbb{C}^*$ such that $max\Big(0;-\Re\frac{\alpha}{\gamma}\Big)\le\frac{1-|B|}{1+|B|}$. Let $f\in W\sum_p$ and suppose that $z^{-p}F_{p,q,s}[\alpha_1+1,v]f(z)\neq 0,\ z\in\Delta$. If

$$\begin{aligned} \Big(z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)\Big)^{\frac{1}{\delta}}\Big(1+\frac{\gamma}{\delta}\Big(\frac{z\Big(F_{p,q,s}[\alpha_1,v]f(z)\Big)'}{F_{p,q,s}[\alpha_1,v]f(z)}-p\Big)\Big) \\ \prec \frac{1+Az}{1+Bz}+\frac{\gamma(A-B)z}{(1+Bz)^2} \end{aligned} \tag{38}$$

then $\left(z^{-p}F_{p,q,s}[\alpha_1+1,v]f(z)\right)^{\frac{1}{\delta}} \prec \frac{(1+Az)}{(1+Bz)}$, and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant of (38). Taking $x=\gamma=p=1, y=0, \alpha_i=\beta_i(i=1,2,..s)$ and $q(z)=\frac{1+z}{1-z}$ in Theorem 3.8, we obtain the next result.

Corollary 3.10. Let $f\in W\sum_p$ such that $z^{-1}f(z)\neq 0$ for all $z\in\Delta$, and let $\delta\in\mathbb{C}^*$. If

$$[z^{-1}f(z)]^{\frac{1}{\delta}}\left(\alpha+\frac{1}{\delta}\Big(\frac{zf'(z)}{f(z)}\Big)-1\right)\prec\alpha\frac{1+z}{1-z}+\frac{2z}{(1-z)^2} \tag{39}$$

then $[z^{-1}f(z)]^{\frac{1}{\delta}}\prec\frac{1+z}{1-z}$ and $q(z)=\frac{1+z}{1-z}$ is the best dominant of (39).

4 Superordination Main Results

Theorem 4.1. Let $q(z)$ be a convex univalent function in Δ with $q(0)=1$, let $0<\delta<1, \beta\in\mathbb{C}^*$ with $\Re(\beta)>0$. Let $f\in W\sum_p$ such that $\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p}\neq 0,\ z\in\Delta$, and suppose that f satisfies the condition:

$$\Big(\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p}\Big)^{\frac{1}{\delta}}\in\mathcal{H}[q(0),1]\cap Q.$$

If the function $\Upsilon_1(z)$ given by (17) is univalent in Δ and satisfies

$$q(z)+\delta\beta zq'(z)\prec\Upsilon_1(z), \tag{40}$$

then $q(z)\prec\Big(\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p}\Big)^{\frac{1}{\delta}},\ z\in\Delta$, and $q(z)$ is the best subordinat of (40).

Proof. We begin by setting

$$r(z)=\Big(\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p}\Big)^{\frac{1}{\delta}},\ z\in\Delta, \tag{41}$$

then $r(z)$ is analytic function in Δ with $r(0)=1$. By differentiating (41) logarithmically with respect to z, we have

$$\frac{zr'(z)}{r(z)}=\frac{1}{\delta}\Big(\Big(\frac{zF_{p,q,s}[\alpha_1,v]f(z)\Big)'}{F_{p,q,s}[\alpha_1,v]f(z)}-p\Big),$$

A simple computation and using the identity (10), shows that

$$\begin{aligned}r(z)+\delta\beta zr'(z)&=(1-\alpha_1\beta)\Big(\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p}\Big)^{\frac{1}{\delta}}\\&+\alpha_1\beta\Big(\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p}\Big)^{\frac{1}{\delta}}\frac{F_{p,q,s}[\alpha_1+1,v]f(z)\Big)}{F_{p,q,s}[\alpha_1,v]f(z)}\end{aligned}$$

now by applying Lemma 2.4, we obtain the required result.

By taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 4.1, where $-1\le B<A\le 1$, we get the next result.

Corollary 4.2. Let $q(z)$ be a convex in Δ with $q(0)=1$, let $0<\delta<1, \beta\in\mathbb{C}^*$ with $\Re(\beta)>0$. If $f\in W\Sigma_p$ such that

$$\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p}\neq 0,\ z\in\Delta,$$

and suppose that f satisfies the condition

$$\Big(\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p}\Big)^{\frac{1}{\delta}}\in\mathcal{H}[q(0),1]\cap Q.$$

If $\Upsilon_1(z)$ given by (17) is univalent in Δ and satisfies the superordination

$$\frac{1+Az}{1+Bz}+\frac{\delta\beta(A-B)z}{(1+Bz)^2}\prec\Upsilon_1(z), \tag{42}$$

then $\frac{1+Az}{1+Bz} \prec \left(\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p}\right)^{\frac{1}{\delta}}$ and $q(z) = \frac{1+Az}{1+Bz}$ is the best subordinat of (42).

Theorem 4.3. Let $q(z)$ be a convex univalent in Δ with $q(0) = 1$, let $\delta, \gamma \in \mathbb{C}^*$, and $\alpha, x, y \in \mathbb{C}$ such that $x + y \neq 0$ and $\Re\{\alpha q'(z)/\gamma\} > 0$. Let $f \in W\Sigma_p$ and f satisfies the following condition.

$$(x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z) + yF_{p,q,s}[\alpha_1,v]f(z)\} \neq 0,\ z \in \Delta, \tag{43}$$

and

$$\Big((x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z) + yF_{p,q,s}[\alpha_1,v]f(z)\}\Big)^{\frac{1}{\delta}} \in \mathcal{H}[q(0),1] \cap Q.$$

If the function $\Upsilon_2(z)$ given by (33) is univalent in Δ, and

$$\alpha q(z) + \gamma z q'(z) \prec \Upsilon_2(z), \tag{44}$$

then $q(z) \prec \Big((x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z) + yF_{p,q,s}[\alpha_1,v]f(z)\}\Big)^{\frac{1}{\delta}}$, and $q(z)$ is the best subordinat of (44).

Proof. Consider the analytic function

$$r(z) = \Big((x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z) + yF_{p,q,s}[\alpha_1,v]f(z)\}\Big)^{\frac{1}{\delta}} \tag{45}$$

with $r(0) = 1$. By differentiating (45) logarithmically with respect to z, yields

$$\frac{zr'(z)}{r(z)} = \frac{1}{\delta}\Big(\frac{xz(F_{p,q,s}[\alpha_1+1,v]f(z))' + yz(F_{p,q,s}[\alpha_1,v]f(z))'}{xF_{p,q,s}[\alpha_1+1,v]f(z) + yF_{p,q,s}[\alpha_1,v]f(z)} - p\Big),$$

then

$$zr'(z) = \frac{r(z)}{\delta}\Big(\frac{xz(F_{p,q,s}[\alpha_1+1,v]f(z))' + yz(F_{p,q,s}[\alpha_1,v]f(z))'}{xF_{p,q,s}[\alpha_1+1,v]f(z) + yF_{p,q,s}[\alpha_1,v]f(z)} - p\Big).$$

Setting the function $\theta(w) = \alpha w, \phi(w) = \gamma, w \in \mathbb{C}$, then θ and ϕ is analytic in $\mathbb{C}$, with $\phi(w) \neq 0$ for all $w \in \mathbb{C}$. Also, we have $\Omega(z) = zq'(z)\phi(q(z)) = \gamma z q'(z)$, is starlike univalent function in Δ, and

$$\Re\Big(\frac{\theta'(q(z))}{\phi(q(z))}\Big) = \Re\Big(\frac{\alpha q'(z)}{\gamma}\Big) > 0,\ z \in \Delta,$$

by simple computation , shows that

$$\Upsilon_2(z) = \alpha r(z) + \gamma z r'(z). \tag{46}$$

From (44) and (46), with applying of Lemma 2.5, we obtain $q(z) \prec r(z)$ and using (45), we have required result.

5 Sandwich Main Results

Combining results of differential subordinations and superordinations, to get at the following sandwich results.

Theorem 5.1. Let $q_1(z)$ and $q_2(z)$ be a convex univalent functions in Δ, with $q_1(0) = q_2(0) = 1$, let $0 < \delta < 1, \beta \in \mathbb{C}^*$ with $\Re(\beta) > 0$. Let $f \in W\Sigma_p$ such that $\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p} \neq 0$ and suppose that f satisfies the condition:

$$\Big(\frac{F_{p,q,s}[\alpha_1,v]f(z)}{z^p}\Big)^{\frac{1}{\delta}} \in \mathcal{H}[q(0),1] \cap Q.$$

If the function $\Upsilon_1(z)$ given by (17) is univalent in Δ and satisfies

$$q_1(z) + \delta\beta z q_1'(z) \prec \Upsilon_1(z) \prec q_2(z) + \delta\beta q_2'(z), \tag{47}$$

then

$$q_1(z) \prec \Big(\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\Big)^{\frac{1}{\delta}} \prec q_2(z),$$

and q_1, q_2 are respectively, the best subordinant and the best dominant of (47).

Theorem 5.2. Let $q_1(z)$ and $q_2(z)$ be a convex univalent functions in Δ, with $q_1(0) = q_2(0) = 1$, let $\delta, \gamma \in \mathbb{C}^*$ and $\alpha, x, y \in \mathbb{C}$ such that $x + y \neq 0$, suppose q_1 satisfies $\Re\{\alpha q_1'(z)/\gamma\} > 0$ and q_2 satisfies (32). Let $f \in W\Sigma_p$ satisfy the next conditions:

$$\Big((x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1, v]f(z) + yF_{p,q,s}[\alpha_1, v]f(z)\}\Big) \neq 0,\ z \in \Delta,$$

and

$$\Big((x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1, v]f(z) + yF_{p,q,s}[\alpha_1, v]f(z)\}\Big)^{\frac{1}{\delta}} \in \mathcal{H}[q(0), 1] \cap Q.$$

If the function $\Upsilon_2(z)$ given by equation (33) is univalent in Δ, and

$$\alpha q_1(z) + \gamma q_1'(z) \prec \Upsilon_2(z) \prec \alpha q_2(z) + \gamma q_2'(z),$$

then

$$q_1(z) \prec \Big((x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1, v]f(z) + yF_{p,q,s}[\alpha_1, v]f(z)\}\Big)^{\frac{1}{\delta}} \prec q_2(z),$$

and q_1, q_2 are respectively, the best subordinant and the best dominant of (48).

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