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Generalized contractions and common fixed point theorems in ordered metric space for weakly compatible mappings

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ABSTRACT. In this paper, we prove common fixed point theorems for maps that satisfy a contraction principle involving a rational expression in complete metric spaces. Presented theorems extend and generalize some existence results in the literature.

1 Introduction

In spite of its simplicity, the Banach fixed point theorem still seems to be the most important result in metric fixed point theory. Fixed point theorems are very useful in the existence theory of differential equations, integral equations, functional equations and other related areas. Existence of a fixed point for contraction type mappings in partially metric spaces and its applications has been considered recently by many authors[1, 2, 4, 5, 6, 8, 9, 10, 11, 23, 24, 20, 21, 22].
The following theorem is an extension of Banach contraction principle for single valued mappings in partially ordered sets, which were obtained by Ran and Reurings[25] and then by Nieto and Lopez [19].

Theorem 1.1 *[19, 25] Let $(X, \leq)$ be a partially ordered set, and suppose that there is a metric d such that (X, d) be a complete metric space. Let $T : X \to X$ be a nondecreasing mapping satisfying the following inequality*

$$d(Tx, Ty) \leq k\, d(x, y),\ for\ all\ x, y \in X\ with\ x \leq y,$$

where $0 \leq k < 1$. Also, assume either

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(i) T is continuous or

(ii) X has the property:

if a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$ then $x_n \leq x$, $\forall\, n$.

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$ then T has a fixed point.

Besides, applications to ordinary differential equations and matrix equations were presented in [19, 25].

Alber and Guerre-Delabriere [3] defined weakly contractive mappings on a Hilbert spaces and established a fixed point theorem for such maps.

Definition 1.1 *[3] Let (X, d) be a metric space. A selfmapping f on X is said to be weakly contractive if*

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$, where $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.

A generalization of Theorem 1.1 was proved by Harjani and Sadarangani [13].

Theorem 1.2 *[13] Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d such that (X, d) be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping satisfying the following inequality*

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),\ for\ all\ x, y \in X\ with\ x \leq y,$$

where $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a continuous and non-decreasing function with $\phi(t) = 0$ if and only if $t = 0$. Also, assume either

(i) T is continuous or

(ii) X has the property:

if a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$ then $x_n \leq x$, $\forall\, n$.

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$ then T has a fixed point.

In [12, 13], Harjani and Sadarangani obtained some fixed point theorems in a complete ordered metric space using altering distance functions. They proved the following theorems.

Theorem 1.3 *[12] Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d such that (X, d) be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping satisfying the following inequality*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)),$$

for comparable $x, y \in X$, where ϕ and ϕ are altering distance functions. Also, assume either

(i) T is continuous or

(ii) X has the property:

if a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$ then $x_n \leq x$, $\forall\, n$.

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$ then T has a fixed point.

Recently an interesting result have been obtained by Harjani, Lopez and Sadarangani [14]. They extended Theorem [15] in partially ordered metric spaces where they replaced the condition (ii) in Theorem 1.2 by a stronger condition, that is

$$if\ a\ non-decreasing\ sequence\ \{x_n\}\ in\ Xconverges\ to\ x \in X\ then\ x = sup\{x_n\}. \tag{1.2}$$

Theorem 1.4 *[14] Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d such that (X, d) be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping satisfying the following inequality*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx).\, d(y, Ty)}{d(x, y)} + \beta d(x, y),\ for\ all\ x, y \in X\ with\ x \leq y, x \neq y, \tag{1.3}$$

where $0 \leq \alpha, \beta$ and $\alpha + \beta < 1$. Also, assume either

- (i) *T is continuous or*
- (ii) *X has the property (1.2).*

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$ then T has a fixed point.

Very recently the fixed point theorems 1.1,1.2 and 1.4 were extend by Luong et al.[18] in new direction.

Theorem 1.5 *[14] Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d such that (X, d) be a complete metric space. Let $T : X \to X$ be a non-decreasing mapping satisfying the following inequality*

$$d(Tx, Ty) \leq M(x, y) - \phi(M(x, y)),\ for\ all\ x, y \in X\ with\ x \leq y, x \neq y, \tag{1.4}$$

where $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$ and

$$M(x, y) = \max\{\frac{d(x, Tx).\, d(y, Ty)}{d(x, y)}, d(x, y)\}.$$

Also, assume either

- (i) *T is continuous or*
- (ii) *X has the property (1.2).*

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$ then T has a fixed point.

Definition 1.2 *[16] Two mappings $T, g : X \longrightarrow X$ are said to be weakly compatible if they commute at their coincidence points. i.e. if $fx = gx$ for some $x \in X$, then $fgx = gfx$.*

Definition 1.3 *[7] Let $(X, \leq)$ be a partially ordered set and $T, g : X \longrightarrow X$ are mappings of X into itself. One says T is g-non-decreasing if for $x, y \in X$, we have*

$$g(x) \leq g(y) \Longrightarrow T(x) \leq T(y).$$

2 Main results

Now, we shall prove a theorem which is a generalization of Theorems 1.1,1.2,1.3,1.4 and 1.5.
Set $\Psi = \{\psi : [0,\infty) \longrightarrow [0,\infty) : \psi$ is continuous and nondecreasing mapping with $\psi(t) = 0$ if and only if $t = 0\}$.
Now, we establish an existence of fixed point of mapping satisfying contractive condition involving $\psi - \phi$ functions in the setup of ordered partial metric spaces.
Our first main result is the following.

Theorem 2.1 *Let $(X, d, \leq)$ be a partially ordered complete metric space. Suppose $T, g : X \longrightarrow X$ are such that T is a $g-$non-decreasing and for every two comparable gx and gy, we have*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{2.1}$$

where

$$M(x, y) = max\{d(gx, gy), d(gx, Tx)\varphi(d(gx, gy), d(gy, Ty))\}, \tag{2.2}$$

$\psi \in \Psi$, and $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$, and $\varphi : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ is a continuous function such that $\varphi(t, t) = 1$ for all $t \in [0,\infty)$. Suppose $TX \subseteq gX$, g is continuous and also suppose either

(i) T is continuous and the pair (T, g) is compatible or

(ii) gX is closed and X has the property (1.2).

If there exists an $x_0 \in X$ with $g(x_0) \leq T(x_0)$ then T and g have a coincidence point.

Proof: Let x_0 be an arbitrary point of X such that $g(x_0) \leq T(x_0)$. Since $TX \subseteq gX$ we can choose $x_1 \in X$ so that $g(x_1) = T(x_0)$. Again from $TX \subseteq gX$ we can choose $x_2 \in X$ so that $g(x_2) = T(x_1)$. Since $g(x_0) \leq T(x_0) = g(x_1)$ and T is $g-$non-decreasing, we have $T(x_0) \leq T(x_1)$.
Continuing this process we can choose a sequence $\{x_n\}$ in X such that $g(x_{n+1}) = T(x_n)$ with

$$T(x_0) \leq T(x_1) \leq T(x_2) \leq ... \leq T(x_n) \leq T(x_{n+1}) \leq ....$$

Therefore,

$$g(x_1) \leq g(x_2) \leq g(x_3) \leq ... \leq g(x_n) \leq g(x_{n+1}) \leq .... \tag{2.3}$$

If there exists $n_0 \in \mathbb{N}$ such that $d(gx_{n_0}, gx_{n_0+1}) = 0$, then we have $gx_{n_0} = gx_{n_0+1} = Tx_{n_0}$. Hence x_{n_0} is a coincidence of T and g. So we assume that $d(gx_n, gx_{n+1}) > 0$, for all $n \in \mathbb{N}$.
We will show that

$$d(gx_{n+1}, gx_n) \leq d(gx_n, gx_{n-1}) \quad \forall\, n \geq 1. \tag{2.4}$$

From (2.1) and (2.3) with $x = x_n$ and $y = x_{n+1}$, we have

$$\psi(d(gx_{n+1}, gx_n)) = \psi(d(Tx_n, Tx_{n-1})) \leq \psi(M(x_n, x_{n-1})) - \phi(M(x_n, x_{n-1})), \tag{2.5}$$

where

$$\begin{aligned} M(x_n, x_{n-1}) &= max\{d(gx_n, gx_{n-1}), d(gx_n, Tx_n)\varphi(d(gx_n, gx_{n-1}), d(gx_{n-1}, Tx_{n-1}))\} \\ &= max\{d(gx_n, gx_{n-1}), d(gx_n, gx_{n+1})\varphi(d(gx_n, gx_{n-1}), d(gx_{n-1}, gx_n))\} \\ &= max\{d(gx_n, gx_{n-1}), d(gx_n, gx_{n+1})\}. \end{aligned}$$

Suppose that there exists $k_0 \in \mathbb{N}$ such that $d(gx_{k_0}, gx_{k_0+1}) > d(gx_{k_0-1}, gx_{k_0})$, from (2.5), we have

$$\begin{aligned} \psi(d(gx_{k_0+1}, gx_{k_0})) &\leq \psi(max\{p(gx_{k_0}, gx_{k_0-1}), p(gx_{k_0}, gx_{k_0+1})\}) \\ &\quad - \phi(max\{p(gx_{k_0}, gx_{k_0-1}), p(gx_{k_0}, gx_{k_0+1})\}) \\ &= \psi(p(gx_{k_0+1}, gx_{k_0})) - \phi(p(gx_{k_0+1}, gx_{k_0})) \end{aligned}$$

$\phi(d(gx_{k_0+1}, gx_{k_0+2})) < 0$, a contradiction. Therefore, we proved that (2.4) holds. Then, the sequence $\{d(gx_n, gx_{n+1})\}$ of real numbers is monotone decreasing. Hence there exists a real number $\delta \geq 0$ such that,

$$\lim_{n \longrightarrow \infty} d(gx_n, gx_{n+1}) = \delta. \tag{2.6}$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Taking the upper limit as $n \longrightarrow \infty$ in (2.5) and using the properties of the function ϕ, we get

$$\psi(\delta) \leq \psi(\delta) - \liminf_{n \longrightarrow \infty} \phi(d(gx_n, gx_{n+1})) \leq \psi(\delta) - \phi(\delta) < \psi(\delta),$$

which is a contradiction. Therefore, $\delta = 0$, that is,

$$\lim_{n \longrightarrow \infty} d(gx_n, gx_{n+1}) = 0. \tag{2.7}$$

Similarly [23, 24], we can show that $\{gx_n\}$ is a Cauchy sequence in (X, d), and hence there exists x in X such that

$$\lim_{n \to \infty} gx_{n+1} = \lim_{n \to \infty} Tx_n = x.$$

Suppose that the assumption (i) holds. We show that $gx = Tx$. From the triangle equality, we have

$$d(Tx, gx) \leq d(Tx, Tgx_n) + d(Tgx_n, gTx_n) + d(gTx_n, gx), \tag{2.8}$$

also, from the continuity of T and g,

$$\lim_{n \longrightarrow \infty} T(g(x_n)) = T(\lim_{n \longrightarrow \infty} g(x_n)) = Tx. \tag{2.9}$$

Letting $n \to \infty$ in (2.8) and using (2.9), we get $d(Tx, gx) = 0$, that is, x is a coincidence point of T and g.
Finally, suppose that (ii) holds. Since $\{Tx_n\} \subseteq gX$ and gX is closed, there exists $z \in X$ such that $x = gz$. Also $\{gx_n\}$ is non-decreasing sequence and $gx_n \to x$ by assumption (ii), we have

$$x = gz = \sup\{gx_n\}. \tag{2.10}$$

Particularly, $gx_n \leq gz$ for all n. Now, we claim that z is a coincidence point of T and g. We have

$$d(gz, Tz) \leq d(gz, gx_{n+1})) + d(gx_{n+1}, Tz).$$

Taking $n \longrightarrow \infty$ in the above inequality, we have

$$d(gz, Tz) \leq \limsup_{n \longrightarrow \infty} d(Tx_n, Tz), \tag{2.11}$$

By property of ψ and using (2.11), we have

$$\begin{aligned}
\psi(d(gz, Tz)) &\leq \limsup_{n \longrightarrow \infty} \psi(d(Tx_n, Tz)) \\
&\leq \limsup_{n \longrightarrow \infty} [\psi(M(x_n, z)) - \phi(M(x_n, z))] \\
&= \psi(\limsup_{n \longrightarrow \infty} M(x_n, z)) - \liminf_{n \longrightarrow \infty} \phi(M(x_n, z)),
\end{aligned}$$

where

$$\limsup_{n \longrightarrow \infty} M(x_n, z) = \limsup_{n \longrightarrow \infty} [max\{d(gx_n, gz), d(gx_n, Tx_n)\varphi(d(gx_n, gz), d(gz, Tz))\}] = 0,$$

which is possible only when $d(gz, Tz) = 0$, which implies that $Tz = gz$, that is, z is a coincidence point of T and g.

Corollary 2.2 *Let $(X, d, \leq)$ be a partially ordered complete metric space. Suppose $T, g : X \longrightarrow X$ are such that T is a $g-$non-decreasing and for every two comparable gx and gy with $gx \neq gy$, we have*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

where

$$M(x, y) = max\{d(gx, gy), \frac{d(gx, Tx)\, d(gy, Ty)}{d(gx, gy)}\},$$

$\psi \in \Psi$, and $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$. Suppose $TX \subseteq gX$, g is continuous and also suppose either

(i) T is continuous and the pair (T, g) is compatible or

(ii) gX is closed and X has the property (1.2).

If there exists an $x_0 \in X$ with $g(x_0) \leq T(x_0)$ then T and g have a coincidence point.

Proof: In Theorem 2.1, taking $\varphi(0, 0) = 1$ and $\varphi(t, s) = \frac{s}{t}$, for all $t, s > 0$, we get Corollary 2.2. Corollary 2.2 is a generalization of Theorem 1.5.

Corollary 2.3 *Let $(X, d, \leq)$ be a partially ordered complete metric space. Suppose $T : X \longrightarrow X$ is a non-decreasing, and for every two comparable x and y, we have*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

where

$$M(x, y) = max\{d(x, y), d(x, Tx)\varphi(d(x, y), d(y, Ty))\},$$

$\psi \in \Psi$, and $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$, and $\varphi : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ is a continuous function such that $\varphi(t, t) = 1$ for all $t \in [0, \infty)$. Suppose either

(i) T is continuous or

(ii) *X has the property (1.2).*

If there exists an $x_0 \in X$ with $x_0 \leq T(x_0)$ then T has a fixed point.

Proof: In Theorem 2.1, taking $g = I$, we get Corollary 2.3.

Corollary 2.4 *Let $(X, d, \leq)$ be a partially ordered complete metric space. Suppose $T, g : X \longrightarrow X$ are such that T is a $g-$non-decreasing, commutes with g and for every two comparable gx and gy with $gx \neq gy$, we have*

$$d(Tx, Ty) \leq \lambda \; max\{d(gx, gy), \frac{d(gx, Tx)\; d(gy, Ty)}{d(gx, gy)}\}.$$

Suppose $TX \subseteq gX$, g is continuous and also suppose either

(i) *T is continuous and the pair (T, g) is compatible or*

(ii) *gX is closed and X has the property (1.2).*

If there exists an $x_0 \in X$ with $g(x_0) \leq T(x_0)$ then T and g have a coincidence point.

Proof: In Corollary 2.2, taking $\psi = I$ and $\phi(t) = (1 - \lambda)t$, for all $t \geq 0$, we get Corollary 2.4. For $a, b \geq 0$, $a + b < 1$ and for all $x, y \in X$, $x \neq y$, we have

$$\begin{aligned} d(Tx, Ty) &\leq ad(gx, gy) + b\frac{d(gx, Tx)\; d(gy, Ty)}{d(gx, gy)} \\ &\leq (a + b)max\{d(gx, gy), \frac{d(gx, Tx)\; d(gy, Ty)}{d(gx, gy)}\} \\ &= \lambda \; max\{d(gx, gy), \frac{d(gx, Tx)\; d(gy, Ty)}{d(gx, gy)}\}. \end{aligned}$$

where $\lambda = a + b \in [0, 1)$. Therefore, Corollary 2.4 is a generalization of Theorem 1.4.

Corollary 2.5 *Let $(X, d, \leq)$ be a partially ordered complete metric space. Suppose $T, g : X \longrightarrow X$ are such that T is a $g-$non-decreasing and for every two comparable gx and gy with $gx \neq gy$, we have*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

where

$$M(x, y) = max\{d(gx, gy), \frac{d(gx, Tx)(1 + d(gy, Ty))}{1 + d(gx, gy)}\},$$

$\psi \in \Psi$, and $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$. Suppose $TX \subseteq gX$, g is continuous and also suppose either

(i) *T is continuous and the pair (T, g) is compatible or*

(ii) *gX is closed and X has the property (1.2).*

If there exists an $x_0 \in X$ with $g(x_0) \leq T(x_0)$ then T and g have a coincidence point.

3 Uniqueness of common fixed point

In this section we shall provide some sufficient conditions under which T and g have a unique common fixed point. From Theorem 2.1, it follows that the set $C(T, g)$ of coincidences is nonempty.

Theorem 3.1 *By adding to the hypotheses of Theorem 2.1, the condition:*
for every x and y in X, there exists a $u \in X$ such that Tu is comparable to Tx and to Ty, and T and g are weakly compatible. Then T and g have a unique common fixed point.

Proof: We know, from Theorem 2.1, that there exists at least a coincidence point. Suppose that x and y are coincidence points of T and g, that is, $Tx = gx$ and $Ty = gy$. We shall show that $gx = gy$. By the assumptions, there exists $u \in X$ such that Tu is comparable to Tx and to Ty. Without any restriction of the generality, we can assume that

$$Tx \le Tu \text{ and } Ty \le Tu.$$

Put $u_0 = u$ and choose $u_1 \in X$ such that $gu_1 = Tu_0$. For $n \ge 1$, continuing this process we can construct sequence $\{gu_n\}$ such that

$$gu_{n+1} = Tu_n \text{ for all } n.$$

Further, set $x_0 = x$ and $y_0 = y$ and on the same way define sequences $\{gx_n\}$ and $\{gy_n\}$. Since $gx = Tx = gx_1$ and $Tu = gu_1$ are comparable, $gx \le gu$. One can show, by induction, that

$$gx_n \le gu_n \text{ for all } n.$$

Thus from (2.1), we have

$$\begin{aligned} \psi(d(gx, gu_{n+1})) &= \psi(d(Tx, Tu_n)) \\ &\le \psi(M(x, u_n)) - \phi(M(x, u_n)), \end{aligned}$$

where

$$\begin{aligned} M(x, u_n) &= max\{d(gx, gu_n), d(gx, Tx).\varphi(d(gx, gu_n), d(gu_n, Tu_n)) \\ &= d(gx, gu_n). \end{aligned}$$

Hence

$$\begin{aligned} \psi(d(gx, gu_{n+1})) &\le \psi(d(gx, gu_n)) - \phi(d(gx, gu_n)) \\ &\le \psi(d(gx, gu_n)). \end{aligned} \tag{3.1}$$

Using the non-decreasing property of ψ, we get

$$d(gx, gu_{n+1}) \le d(gx, gu_n),$$

implies that $\{d(gx, gu_n)\}$ is a non-increasing sequence. Hence, there exists $r \ge 0$ such that

$$\lim_{n \longrightarrow \infty} d(gx, gu_n) = r.$$

Passing the upper limit in (3.1) as $n \longrightarrow \infty$, we obtain

$$\psi(r) \le \psi(r) - \phi(r),$$

which implies that $\phi(r) = 0$ and then, $r = 0$. We deduce that

$$\lim_{n \longrightarrow \infty} d(gx, gu_n) = 0. \tag{3.2}$$

Similarly, one can prove that

$$\lim_{n \longrightarrow \infty} d(gy, gu_n) = 0. \tag{3.3}$$

From (3.2) and (3.3), we have $gx = gy$. Since $gx = Tx$ and $gy = Ty$, by w-compatible of T and g , we have

$$g(gx) = g(Tx) = T(gx). \tag{3.4}$$

Denote $gx = a$, then from (3.4),

$$ga = Ta. \tag{3.5}$$

Thus, a is a coincidence point, it follows that $ga = gx$, that is,

$$ga = a. \tag{3.6}$$

From (3.5) and (3.6) ,

$$a = ga = Ta. \tag{3.7}$$

Therefore, a is a common fixed point of T and g. To prove the uniqueness of the point a, assume that b is another common fixed point of T and g. Then we have

$$b = gb = Tb.$$

Since b is a coincidence point of T and g, we have $gb = gx = a$. Thus $b = gb = ga = a$, which is the desired result. Since every commuting pair of functions is a w-compatible, we have the following corollary.

Corollary 3.2 *By adding to the hypotheses of Theorem 2.1, the condition:*
for every x and y in X, there exists a $u \in X$ such that Tu is comparable to Tx and to Ty, and T and g are commuting. Then T and g have a unique common fixed point.

Set $\mathbf{I} = \{f : R^+ \longrightarrow R^+;\ f$ is a Lebesgue integrable mapping which is summable and nonnegative and satisfies $\int_0^\epsilon f(t)dt > 0$, for each $\epsilon > 0\}$.

Corollary 3.3 *Let $(X, d, \leq)$ be a partially ordered complete metric space. Suppose $T, g : X \longrightarrow X$ are such that T is a $g-$non-decreasing and for every two comparable gx and gy, we have*

$$\int_0^{\psi_1(d(Tx,Ty))} f(s)ds \leq \int_0^{\psi_1(M(x,y))} f(s)ds - \int_0^{\psi_2(M(x,y))} f(s)ds, \tag{3.8}$$

where $\psi_1 \in \Psi$, and $\psi_2 : [0, \infty) \longrightarrow [0, \infty)$ is a lower semi-continuous function with $\psi_2(t) = 0$ if and only if $t = 0$, and $\varphi : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ is a continuous function such that $\varphi(t, t) = 1$ for all $t \in [0, \infty)$, $M(x, y)$ is given by (2.1) and $f \in \mathbf{I}$. Suppose $TX \subseteq gX$, g is continuous and also suppose either

(i) *T is continuous and the pair (T, g) is compatible or*

(ii) *gX is closed and X has the property (1.2).*

If there exists an $x_0 \in X$ with $g(x_0) \leq T(x_0)$ then T and g have a coincidence point. Further, if T and g commute at their coincidence points, then T and g have a unique common fixed point.

Proof: Define $F : R^+ \longrightarrow R^+$ by $F(t) = \int_0^t f(s)ds$, then $F \in \Psi$ and (3.8) becomes

$$F(\psi_1(d(Tx, Ty))) \leq F(\psi_1(M(x, y))) - F(\psi_2(M(x, y))),$$

which further can be written as

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

where $\psi = Fo\psi_1 \in \Psi$ and $\phi = Fo\psi_2$ is lower semi continuous with $\phi(t) = 0$ for $t = 0$. Hence T and g have a unique common fixed point.

Let $X = [0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ then it is clear that (X, d) is a complete metric space. Also, this space can also be equipped with a partial order given by

$x, y \in X, \qquad x \preceq y \Longleftrightarrow x \leq y.$

Let $T, g : X \longrightarrow X$, and $\psi, \phi : [0, \infty) \longrightarrow [0, \infty)$ be given by $Tx = \frac{1}{3}x$, $gx = \frac{1}{2}x$, $\psi(t) = 6t$, $\phi(t) = 2t$. Clearly ψ is continuous and nondecreasing, ϕ is a lower semi continuous, and $\phi(t) = 0 = \psi(t)$ if and only if $t = 0$. We show that condition (2.1) is satisfied.

If $x, y \in X$, $x \leq y$, then we have

$$\begin{aligned}\psi(p(Tx, Ty)) &= \psi(|\frac{1}{3}x - \frac{1}{3}y|) = 2|x - y| = 4d(gx.gy)\\ &\leq 4M(x, y)\\ &= \psi(M(x, y)) - \phi(M(x, y)).\end{aligned}$$

Note that, T and g satisfy all the conditions given in Theorem 3.1. Moreover, 0 is a unique common fixed point of T and g.

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