International Journal of Advances in Mathematics
Volume 2020, Number 1, Pages 24-32, 2020
eISSN 2456-6098

A Generalization of Uniform Boundedness Principle

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ABSTRACT. In 1998, several generalizations of uniform boundedness principle (U.B.P) was given for functions from topological spaces to topological spaces by Joseph, Kwack, and Nayar. Recently, using the definition of u-closure operator and α-set, we introduced three classes of continuous functions, namely αu-continuous, strongly u-continuous and semi-αu-continuous functions. In this study the U.B.P. is extended using these generalized continuous functions.

1. Introduction

In a study of generalized continuous functions [3] the following concepts were introduced. They are strongly u-continuous functions, αu-continuous functions and semi-αu-continuous functions. A function $g : X \to Y$ is αu-continuous (strongly u-continuous, semi-αu-continuous) at $x \in X$, if for each α-set (α-set, open set) W which contains a closed neighborhood of $g(x)$, there exists some α-set (open set, α-set)V which contains a closed neighborhood of x which satisfies the condition $g(clV) \subseteq clW$. If g is αu-continuous (strongly u-continuous, semi-αu -continuous) at each $x \in X$, we say $g : X \to Y$ is αu-continuous (strongly u-continuous, semi-αu-continuous) on X.

In [17] generalizations of uniform boundedness principle were given. The functions considered there were from a class of Baire topological spaces to a countable union of its subspaces satisfying some generalized compactness properties. In [12] other generalizations of the uniform boundedness principle were given for functions from topological spaces to topological spaces as applications of u-continuous functions introduced there. The

* Corresponding Author.
Received November 26, 2019; accepted December 30, 2019.
2010 Mathematics Subject Classification: 54C08, 54C10,54A05, 54A20.
Key words and phrases: Uniform boundedness, Generalize continuous functions, Urysohn spaces.

topological spaces considered there were Urysohn closed spaces and Quasi Urysohn Closed spaces. Rhe definitions of these concepts are given in the section for preliminaries . In [3] we defined and studied the above mentioned three classes functions as generalizations of u-continuous function. Further, these classes of functions are used to give several characterizations of compact and exteremally disconnected spaces in [4]. The present article is a study of another application of these classes of functions to give generalizations of uniform boundedness principle in topological spaces.

In classical analysis uniform boundedness principle is stated as follow; [19] let $\mathcal{F}$ be a family of continuous real-valued functions on a complete metric space X such that for each $x \in X$, there is some constant M_x such that $|f(x)| \leq M_x$, for all $f \in \mathcal{F}$. Then there is some constant M and a non-empty open set U in X such that $|f(x)| \leq M$ for each $x \in U$ and each $f \in \mathcal{F}$. To provide a generalization of this principle we introduce a class of spaces called α-Quasi Urysohn Closed(α-QUC) space, a subclass of the well-known class of QUC spaces. We provide a second category type property for this class of spaces. We further used the class of function introduced in [3] to identify collection of subsets of α-QUC space X on which a family of functions from X to an arbitrary space Y is uniformly bounded. These concepts are defined in the section of preliminaries.

2. **Preliminaries**

The concept of α-sets are defined by Njastad in 1965 [16]. Sets for which $A \subseteq int(cl(intA))$, where $intA$ and clA represents the interior and closure respectively of A. The complement of each α-set is called α-closed set. One can easily see that every open set is an α-set, but every α-set need not be open as is demonstrated in [16]. The collection of α-sets of a space form a topology and it is denoted by τ_α. Let X be a topological space. For $A \subset X$, $\Sigma(A)$ represents the collection of open sets containing A. The family $\Gamma(A)$ is the collection of closed neighborhoods of A and let $\Lambda(A) = \bigcup_{\Gamma(A)} \Sigma(A)$, union of open sets containing closed neighborhood of the set. (if $A = \{x\}$ the notation $\Sigma(x)$, $\Gamma(x)$, or $\Lambda(x)$ will be used). We shall denote $\alpha\Lambda(A)$ and $\alpha\Lambda(x)$ as collection of α-sets containing a closed neighborhood of A and x respectively. The concept of θ-closure of a set was introduced by Veličko [18]. For a topological space X and for a filterbase Ω on X, the adherence of Ω is denoted by $ad\Omega$ and $ad\Omega = \cap_\Omega clF$.

In [10] the concept of u-closure of set was introduced by Joseph and was used to study, among others, compact spaces as well as Urysohn-closed spaces. Let $A \subseteq X$. The *u-closure* of A, denoted as $cl_u(A)$, = $\{x : clV \cap A \neq \emptyset, V \in \Lambda(x)\}$. A set A is u-closed if $A = cl_u A$. This concept was further used in the study of compact and exteremally disconnected spaces [4]. A filterbase Ω on a space X is said to *u-converge to* $x \in X$[12] ($\Omega \to_u x$) if for each $W \in \Lambda(x)$ there exists $F \in \Omega$ such that $F \subset clW$. A filter base Ω on a space X αu-converges to $x \in X$ ($\Omega \to_{\alpha u} x$) if for each α-set W which contains a closed neighborhood of x, there exists $F \in \Omega$ such that $F \subset clW$. A point $x \in X$ is in αu-closure of A, denoted as $cl_{\alpha u}A$ [3], if $cl_\alpha V \cap A \neq \emptyset$ for every α-set V containing a closed neighborhood of x. i.e. $cl_{\alpha u}A = \{x \in X : clV \cap A \neq \emptyset, V \in \tau_\alpha$, and V contains a closed neighborhood of $x\}$. A is called αu-closed if $cl_{\alpha u}A = A$. So, αu-closure of a set A is denoted as $cl_{\alpha u}A$and defined as $cl_{\alpha u}A = \cap_{\alpha\Lambda(A)} clV$. Since every open set is an α-set, the αu-closure of a set is contained in the u-closure of the set. Let Ω be a filter base on X. A point x is an αu- adherent point of a filter base Ω, denoted as $ad_{\alpha u}\Omega$ if x belongs to the αu-closure of every F in Ω and αu-adherence of Ω denoted by $ad_{\alpha u}\Omega = \bigcap_\Omega cl_{\alpha u}F$.

A space X is Urysohn if each singleton is u-closed [12]. A Urysohn filterbase on a space X is an open filterbase Ω such that whenever $x \in X - ad\Omega$ some $V \in \Sigma(x)$ and $F \in \Omega$ satisfies $clV \cap clF = \emptyset$. A Urysohn space is Urysohn-closed if each Urysohn filterbase on the space has non-empty adherence. An arbitrary space is Quasi Urysohn-closed (QUC) if each Urysohn filterbase on the space has nonempty intersection. Herrington [10] has shown that a space is QUC iff each filterbase on the space has nonempty u-adherence.

A function $g : X \to Y$ is *u-continuous at* $x \in X$ [12] if for each $W \in \Lambda(g(x))$ there exists a $V \in \Lambda(x)$ such that $g(clV) \subset clW$ and g is *u-continuous* on X if g is u-continuous at each $x \in X$. A filterbase Ω on a space X is said to *u-converge to* $x \in X$ [12] ($\Omega \to_u x$) if for each $W \in \Lambda(x)$ some $F \in \Omega$ satisfies $F \subset clW$. [12] For a filter base Ω on X, u-adherence of Ω is denoted by $ad_u\Omega$ and is defined as $\bigcap_\Omega cl_u F$. The concept of α-set is defined by Njastad in 1965 [14]. Sets for which $A \subseteq int(cl(intA))$ are α-sets, where $intA$ represents the interior of A. One can easily see that every open set is an α-set, but an α-set need not be open as is demonstrated in [14].

If X is a space, let $S(x) = \{V - \{x\}, V \in \Sigma(x)\}$. For spaces X and Y, a set valued function $f : X \to 2^Y$ is a function from X to $2^Y - \{\emptyset\}$, where 2^Y is the power set of Y. A set valued function $F \subset X \times 2^Y$ is subclosed (strongly subclosed)[u-strongly subclosed] if $adF(S(x)) \subset F(x)$ ($ad_\theta F(x) \subset F(x)$) [$ad_u F(x) \subset F(x)$], for each $x \in X$ for which $S(x)$ is a filterbase on X. The value function F is closed (strongly-closed)[u-strongly closed] if F(x) is closed (θ-closed) [u-closed] in Y for each $x \in X$. [17]

A subset A of a space X is Quasi Urysohn-closed (QUC) relative to the space X if each filterbase Ω on A satisfies $A \cap ad_u\Omega \neq \emptyset$.

3. **Generalizations of Uniform Boundedness Principle**

Uniform boundedness principle is defined as follow as stated in the introduction: [19] let $\mathcal{F}$ be a family of continuous real-valued functions on a complete metric space X such that for each $x \in X$, there is some constant M_x such that $|f(x)| \leq M_x$, for all $f \in \mathcal{F}$. Then there is some constant M and a non-empty open set U in X such that $|f(x)| \leq M$ for each $x \in U$ and each $f \in \mathcal{F}$.

Generalizations of uniform boundedness principle are presented here. As stated earlier, in [17] uniform boundedness principle was generalized for functions from Baire spaces satisfying certain properties. A topological space is a Baire space if the intersection of each sequence of dense open subsets of the space is dense in the space [12]. A subset Q of a topological space is of second category in X if Q is not the union of a countable number of nowhere dense subsets of X. It is clear that a topological space is a Baire space iff each non-empty open subset of the space is of second category in X [12].

Henceforth, we adopt the terminology and notations used in [12].

Definition 3.1. A family $\mathcal{F}$ of functions from a space X to a space Y is *uniformly bounded* on $A \subset X$ by $\Delta \subset 2^Y$ if there is a $C \in \Delta$ such that $g(A) \subset C$ for each $g \in \mathcal{F}$ [12]. Let $\mathcal{UB}[\mathcal{F}, X, Y, \Delta] = \{A \subset X : \mathcal{F}$ is uniformly bounded on A by $\Delta\}$.

Using the above terminology, Uniform Boundedness Principle is stated as follows[12]: let X be a complete metric space, let R be the Euclidean line, let $\mathcal{F}$ be the family of real-valued continuous functions on X and let Δ be

non-empty countable family of compact subsets of the reals. If $\mathcal{UB}[\mathcal{F}, X, R, \Delta]$ contains the collection of singletons then $\mathcal{UB}[\mathcal{F}, X, R, \Delta]$ has a non empty open subset as an element.

The following definitions are needed before we present our generalizations of uniform boundedness principle.

Definitions 3.2. [4]

a) A relation $R \subseteq X \times Y$ is αu-strongly subclosed if $ad_{\alpha u}R(\Omega) \subseteq R(x)$ for each $x \in X$ and each filterbase Ω on $X - \{x\}$ with $\Omega \to x$.

A function $f : X \to Y$, $f \subseteq X \times Y$ has αu-strongly subclosed inverse if $f^{-1} \subseteq f(X) \times X$is αu-strongly subclosed.

b) A relation $R \subseteq X \times Y$ is u-strongly subclosed if $ad_u R(\Omega) \subseteq R(x)$ for each $x \in X$ and each filterbase Ω on $X - \{x\}$ with $\Omega \to x$.

A function f from X to Y, $f \subseteq X \times Y$ has u-strongly subclosed inverse if $f^{-1} \subseteq f(X) \times X$ is u-strongly subclosed.

A u-strongly subclosed relation is an αu-strongly subclosed relation as well.

Definitions 3.3.

a) A set A is said to be αu-dense in X, if $cl_{\alpha u}A = X$.

b) A set A is αu-nowhere dense in X, if $int_\alpha(cl_{\alpha u}A) = \emptyset$, where a point $x \in A$ is an α-interior point of A, if there exists an α-set containing x contained in A.

c) A subset A is of αu-first category in X if $A = \cup_{F \in \mathcal{A}} F$, where $\mathcal{A}$ is a countable family of subsets of X such that $int_\alpha(cl_{\alpha u}(F)) = \emptyset$ for $F \in \mathcal{A}$.

d) A subset A is of αu-second category in X if it is not of αu-first category, i.e. A is of αu-second category if A is not the union of countable family $\mathcal{A}$ of subsets of X such that $int_\alpha(cl_{\alpha u}(F)) = \emptyset$ for $F \in \mathcal{A}$.

e)A space is αu-Baire if $cl_{\alpha u}W$ is of the αu-second category for each non-empty α-subset of X.

Result 3.4.

If A is an open αu-dense subset of X, then $X - A$ is αu-nowhere dense set.

Proof. Let A be an open αu-dense, then $cl_{\alpha u}A = X$. Therefore $int(cl_{\alpha u}A) = int(X) = X$.

Therefore, $int(cl_{\alpha u}(X - A)) = cl(X - (X - cl_{\alpha u}(X - A))) = cl(cl_{\alpha u}(X - A)) = X - int(cl_{\alpha u}A) = X - X = \emptyset$.

Definition 3.5. A subset A of a space X is α-Quasi Urysohn Closed (α-QUC) relative to the space X if each filterbase Ω on A satisfies $A \cap ad_{\alpha u}\Omega \neq \emptyset$.

Note $ad_{\alpha u}\Omega \subset ad_u\Omega$, therefore every α-QUC space is a QUC space but not every $cl_u A$ for $A \subset X$ is $cl_{\alpha u}A$, so an α-QUC space might not be a QUC space.

Theorem 3.6.

Let X and Y be spaces and let $g : X \to Y$ have a αu-subclosed graph. Then $g^{-1}(A)$ is αu-closed in X for each α-QUC relative to Y subset A.

Proof. For $v \in cl_{\alpha u}(g^{-1}(A) - \{v\})$, there is a filterbase Ω on $g^{-1}(A) - \{v\}$ such that $\Omega \to_{\alpha u} v$[3]. Then $g(\Omega)$ is a filterbase on A and since $\emptyset \neq A \cap ad_{\alpha u}g(\Omega)$ as A is α-QUC relative to Y. Also, $A \cap ad_{\alpha u}g(\Omega) \subset \{g(v)\}$ since g has αu-subclosed graph. It follows that $\emptyset \neq A \cap ad_{\alpha u}g(\Omega) \subset \{g(v)\}$. Hence $v \in g^{-1}(A)$. Thus $g^{-1}(A)$ is αu-closed.

Theorem 3.7.

Let X and Y be spaces and let A be α-QUC relative to Y. If $g : X \to Y$ has a αu-subclosed graph and $g(X) \subset A$ then g is αu-continuous.

Proof. Let $v \in X$. If $V = \{v\}$, and $clV = \{v\}$ for some α-set V containing a closed neighborhood of v, then g is αu-continuous at v. Otherwise, $\Omega = \{clV - \{v\} : V$ is an α-set containing a closed neighborhood of $v\}$ is a filterbase on X, and $ad_{\alpha u}g(\Omega) \subset \{g(v)\}$ since g has αu-subclosed graph. Let W be an α-set containing closed neighborhood of $g(v)$. Then there exists a $F \in \Omega$ such that $g(F) \subset clW$. Since $g(X) \subseteq A$, an α-QUC relative to Y, $\{g(F) - clW\} \subseteq A$. If not then $\gamma = \{g(F) - clW : F \in \Omega\}$ is a filterbase on A, $g(v) \notin ad_{\alpha u}\gamma \subset ad_{\alpha u}g(\Omega)$, which is a contradiction since we found a filterbase with empty αu-adherence in an αQUC set A.

Lemma 3.8. If $\{A_\beta; \beta \in \}$ is a collection of αu-closed subsets of a space X, then $\bigcap_{\beta\in} A_\beta$ is an αu-closed set.

Proof. Since each A_β is αu-closed for each $\beta \in$, if $x \in X - A_\beta$, there exists an α-set U containing a closed neighborhood of x such that $cl_\alpha U \cap A_\beta = \emptyset$. Hence there exists an α-set containing a closed neighborhood of x such that $cl_\alpha U \subseteq X - A_\beta$. This is true for each A_β, therefore $cl_\alpha U \subseteq \bigcup_\beta X - A_\beta = X - \bigcap_\beta A_\beta$. Hence $cl_\alpha U \cap (\bigcap_\beta A_\beta) = \emptyset$.

Theorem 3.9. Let X be a αu-Baire space, Let Y be a space, Let Δ be a non-empty countable family of α-QUC relative to Y subsets and $\mathcal{F}$ be a family of functions from X to Y with αu-subclosed graphs such that $\mathcal{UB}[\mathcal{F}, X, Y, \Delta]$ contains the collection of singletons of X. Then

(1) There is a non-empty α-set V of X such that $cl_{\alpha u}V \in \mathcal{UB}[\mathcal{F}, X, Y, \Delta]$.

(2) For each $g \in \mathcal{F}$ the restriction of g to $cl_{\alpha u}V$ of part (1) is αu-continuous.

(3) Each $g \in \mathcal{F}$ is αu-continuous at each point of V of part (1).

(4) There is a W, an α-set in X such that each $g \in \mathcal{F}$ is αu-continuous at $x \in W$ and $cl_{\alpha u}W = X$.

Proof.

(1) For each $C \in \Delta$ let $I(C) = \bigcap g^{-1}(C)$. Then $I(C)$ is αu-closed, in view of Theorem 3.6 and Lemma 3.8. If $x \in X$ there is a $C(x) \in \Delta$ such that $g(x) \in C(x)$ for all $g \in \mathcal{F}$. Hence $\{I(C); C \in \Delta\}$ is a cover of X by a countable family of αu-closed sets. Since X is of αu-second category it is not contained in the union of a countable family $\mathcal{A}$ of subsets of space such that $int_\alpha(cl_{\alpha u}(F)) = \emptyset$ for $F \in \mathcal{A}$. Therefore there exists a $C_0 \in \Delta$ such that $int_\alpha(I(C_0)) \neq \emptyset$. Let $V = int_\alpha(I(C_0))$, so $cl_{\alpha u}V \in I(C_0)$. Hence for each $g \in \mathcal{F}$, $g(cl_{\alpha u}V) \subset g(I(C_0)) \subset g(g^{-1}(C_0)) \subset C_0$. Hence $cl_{\alpha u}V \in \mathcal{UB}[\mathcal{F}, X, Y, \Delta]$.

(2) Let V be α-set described in part (1), and let $g \in \mathcal{F}$. Since C_0 is α-QUC relative to Y, and the restriction of g to $cl_{\alpha u}V$ has αu-subclosed graph, by the theorem 3.7, g_{clV} is αu-continuous.

(3) Let V be an α-set of the type guaranteed in part (1). Let $x \in V$, and let $g \in \mathcal{F}$ and let W be an α-set containing a closed neighborhood of g(x), since restriction of g to $cl_{\alpha u}V$ is αu-continuous, there is an α-set Q containing a closed neighborhood of x such that $g|_{cl_{\alpha u}V}(cl(Q \cap cl_{\alpha u}V)) \subset cl(W)$. Since Q is an α-set and V is an α-set, then $Q \cap V$ is an α-set containing a closed neighborhood of x, and $cl(Q \cap V) \subseteq cl(Q \cap cl_{\alpha u}V)$ therefore $g(cl(Q \cap V)) \subset g(cl(Q \cap cl_{\alpha u}V) \subset clW$. Therefore $g(cl(Q \cap V)) \subset cl(W)$. Hence g is αu-continuous.

(4) Let W be union of all α-sets V such that each $g \in \mathcal{F}$ is αu-continuos at each $x \in V$. Then each $g \in \mathcal{F}$ is αu-continuous in each $x \in W$. If A is a non-empty α-subset of the αu-Baire space X, then $cl_{\alpha u}A$ is of αu-second category and $\mathcal{F}_{cl_{\alpha u}VF}$ satisfies the conditions on $\mathcal{F}$ in the hypothesis (relative to X). Hence from part (1) and (2) there is a non-empty α-set B of $cl_{\alpha u}V$ with each $h \in \mathcal{F}_{cl_{\alpha u}V}$ αu-continuous at each point of the non-empty subset $A \cap H$ where $B = H \cap cl_{\alpha u}V$ and H is α-set in X. It follows as in the proof of part (3) that each $g \in \mathcal{F}$ is α u-continuous at each $x \in A \cap H$ Hence $A \cap H \subset W$, $A \cap W \neq \emptyset$, and $cl_{\alpha u}W = X$.

The following results lead us to the final Main Theorem.

Theorem 3.10. The following statements are equivalent for the space X and $A \subset X$.

(1) The relation $int_\alpha(cl_{\alpha u}(A)) = \emptyset$ holds.

(2) If V is a nonempty α-subset of X and W is an α-set containing a closed neighborhood of V, there is an $x \in V$ and Q, an α-set containing closed neighborhood of x such that $Q \subset W$ and $A \cap clQ = \emptyset$.

(3) For each nonempty α-set V in X there is a nonempty α-set $Q \subset V$ such that $Q \cap cl_{\alpha u}(A) = \emptyset$

Proof.

(1) $\Rightarrow$ (2) If (1) holds and V is a nonempty α-subset of X, then $V - cl_{\alpha u}(A) \neq \emptyset$. Let $x \in V - cl_{\alpha u}(A)$. Then there exists an α-set P containing a closed neighborhood of x such that $A \cap clP = \emptyset$. If W is an α-set containing a closed neighborhood of V and if Q is an α-set containing a closed neighborhood of $x \in V$, then $Q \subset W$, and $A \cap clQ = \emptyset$.

(2) $\Rightarrow$ (3) Suppose (2) holds and let V be a nonempty α-set in X, From (2) there is an $x \in V$ and α-set Q containing a closed neighborhood of x satisfying $A \cap clQ = \emptyset$. Let T be an α-set containing x with $Q \in \alpha\Lambda(T)$. Then $X - cl_{\alpha u}V \in \alpha\Lambda(A)$. Since $Q \subset V$, $clQ \subset clV$, therefore $X - clV \subset X - clQ$, so $A \subset X - clQ$. Also from (2), $A \cap clQ = \emptyset$, since $clT \subset Q$, then $A \cap clT = \emptyset$. Therefore $A \subset X - clQ \subset X - clT$, and $cl_{\alpha u}A \subset cl_{\alpha u}(X - clT)$. Consequently, $V \cap T \cap cl_{\alpha u}(A) \subset T \cap cl(X - clT) = \emptyset$. So $V \cap T = T$ is the desired α-set satisfying (3).

(3) $\Rightarrow$ (1) Assume (3) holds, so for each α-set V in X there is a nonempty α-set $Q \subset V$ such that $Q \cap cl_{\alpha u}A = \emptyset$, therefore $int_\alpha(cl_{\alpha u}A) = \emptyset$.

Theorem 3.11. If X is a space $A \subset X$ satisfies $int_\alpha(cl_{\alpha u}(A)) = \emptyset$, then for each nonempty α-set $V \subset X$ there is a nonempty α-set $Q \subset V$ such that $A \cap cl_{\alpha u}Q = \emptyset$.

Proof. If $int_\alpha(cl_{\alpha u}(A)) = \emptyset$, and V is a nonempty α-set, subset of X, there is an α-set W containing a closed neighborhood of A such that $V - clW \neq \emptyset$. Then $Q = V - clW$ satisfies $Q \subset V$ and $A \cap cl_{\alpha u}(Q) = \emptyset$.

Theorem 3.12. If X is an α-QUC space and $V \subset X$ is a nonempty α-set, then $cl_{\alpha u}(V)$ is not contained in the union of countable family, Ω, of subsets of X such that $int_\alpha(cl_{\alpha u}(F)) = \emptyset$ for all $F \in \Omega$.

Proof. Let $\Omega = \{F(n) : n \in N\}$ such that $int_\alpha(cl_{\alpha u}(F(n))) = \emptyset$ for each $F(n) \in \Omega$. Using Theorem 3.11, a decreasing sequence $\{V(n) : n \in N\}$ of nonempty α-sets is constructed inductively such that $V(n+1) \subseteq V(n) \subseteq .... \subseteq V(1) \subseteq V$ and $F(n) \cap cl_{\alpha u}(V(n)) = \emptyset$ for each n. Therefore, $cl_{\alpha u}V(n) \cap (\bigcup_{i=1}^{n} F(i)) = \emptyset$. Since X is α-QUC, there is an $x \in \bigcap_n cl_{\alpha u}(V(n))$. So, $cl_{\alpha u}V \neq \bigcup_n F(n)$ and $int_\alpha(cl_{\alpha u}(F(n))) = \emptyset$, Since, $x \in cl_{\alpha u}V - \bigcup_n F(n)$.

Corollary 3.13. Every nonempty α-subset of an α-QUC space is of αu-second category.

Proof.

If X is α-QUC and $V \subset X$ is a nonempty α-set, then $cl_{\alpha u}(V)$ is not contained in the union of countable family of αu-closed subsets with empty α-interiors.

Corollary 3.14. If X is α-QUC and W is an α-set containing a closed neighborhood of A for some nonempty subset A, then clW is not contained in the union of countable family, Ω, of subsets of X such that $int_\alpha(cl_{\alpha u}(F)) = \emptyset$ for $F \in \Omega$.

Proof. Let V ben an α-set containing a closed neighborhood of x, for $x \in A$. such that $cl_{\alpha u}V \subset clW$. In view of 3.12, $cl_{\alpha u}V$ is not contained in the union of countable family of αu-closed subsets with empty α-interiors so neither could clW.

Theorem 3.15. Let X be α-QUC space, let Y be a space, let Δ be a nonempty countable family of α-QUC relative to Y subsets. Let $\mathcal{F}$ be a family of functions from X to Y with αu-subclosed graphs such that $\mathcal{UB}[\mathcal{F}, X, R, \Delta]$ contains the collection of singletons of X. Then

(1) There is a nonempty α-set $V \subset X$ such that $cl_{\alpha u}(V) \in \mathcal{UB}[\mathcal{F}, X, R, \Delta]$.

(2) For each $g \in \mathcal{F}$ the restriction of g to $cl_{\alpha u}(V)$ for V as stated in part (1) is αu-continuous.

(3) For each V satisfying (1), each $g \in \mathcal{F}$ is αu-continuous at each point x of V with a closed neighborhood of x is contained in V.

Proof.

(1) It can be proved using Theorems 3.6, 3.7, 3.8, 3.9 part (1) and Theorem 3.12 and corollary 3.13.

(2) Let V be α-set described in part (1), and let $g \in \mathcal{F}$. Since C_0 is α-QUC relative to Y, and the restriction of g to $cl_{\alpha u}V$ has αu-subclosed graph, by the theorem 3.7, $g|_{cl_{\alpha u}V}$ is αu-continuous.

(3) Let $g \in \mathcal{F}$ and $W \in \alpha\Lambda(x)$ satisfying (1), let $Q \in \alpha\Lambda(g(x))$. From (2) there exists an α-set A in X such that $A \cap cl_{\alpha u}W \in \alpha\Lambda(x)$ in $cl_{\alpha u}W$, and $g(cl(A \cap cl_{\alpha u}W)) \subset clQ$. Choose B, an α-set containing x in X such that $B \cap cl_{\alpha u}W \subset cl(B \cap cl_{\alpha u}W) \subset A \cap cl_{\alpha u}W$. Let P be an open set containing x in X such that $clP \subset W$. Then $P \cap B \cap W$ is an α-set containing x in X and $cl(P \cap B \cap W) \subset clP \cap cl(B \cap W) \subset W \cap A$. Hence $W \cap A \in \alpha\Lambda(x)$ in X and $g(cl(A \cap W) \subset g(cl(A \cap cl_{\alpha u}W)) \subset clQ$.

4. **Conclusion.** We conclude by pointing out the following;

There have been studies on uniform boundedness principle in different topological spaces using different types of functions. In classical analysis uniform boundedness principle is defined for real-valued functions on complete metric spaces [19]. In [17] generalizations of uniform boundedness principle were given from a class of Baire topological spaces to a countable union of its subspaces satisfying some generalized compactness properties. In [12] other generalizations of the uniform boundedness principle were given for functions from topological spaces to topological spaces as applications of u-continuous functions introduced there. The topological spaces considered there were Urysohn Closed Spaces and Quasi Urysohn Closed Spaces. What makes the results in this paper unique is that the generalization of uniform boundedness principle given here is not for real-valued functions nor for continuous functions but for classes of functions which generalizes continuous functions and functions which satisfy certain graph conditions. An approach like the one presented here opens a path for new investigative avenues while seeking classes of spaces which form uniform bounds for more general classes of functions under more general conditions than usually seen in classical approaches.

REFERENCES

[1]. Arya, S. P. and Bhamini, M. P.: On some sub-classes of the class of semi-continuous functions, National Acad. Sci. Letters, 5, No. 1, 341-343 (1982) .

[2]. Arya, S. P. and Bhamini, M. P.: Some weaker forms of semi-continuous functions, Ganita 33, 124-134 (1982).

[3]. Z. Bandpey and B. M. P. Nayar, *A study of generalized Continuous functions*, International Journal of Advances in Mathematics, 6(2018), 16-26.

[4]. Bandpey, Z., Nayar, B. M. P. (2019).:*Compact and extremally disconnected spaces via generalized continuous functions*, Journal of Advanced Studies in Topology, 10(1), 28-34.

[5]. Crossley, S. G. and Hildebrand, S. K.: Semi-topological properties, Fund. Math. Vol 74, 233-254 (1972).

[6]. Espelie, M. S. and Joseph, J. E .: Remarks on two weak forms of continuity,Canad. Math. Bull. 25, No. 1, 59-63 (1982).

[7]. Espelie, M. S. and Joseph, M. H. Kwack .: Aplications of the u-closure operator, Proc. AMS 83 No. 1, 167-174 (1981).

[8]. Espelie, M. S. and Joseph, J. E .: Some properties of θ-closure, Can. J. Math., Vol. XXXIII No. 1, 142-149 (1981).

[9]. Ganster, M.: On strogly s-regular spaces, Glas. Mat. Ser.III Vol. 25 No.1, 195-201 (1990).

[10]. L. L. Herrington.: Characterizations of Urysohn-closed spaces, Proc. AMS 55 (1976), 435-439.

[11]. Joseph, J. E. : On Urysohn-closed and minimal Urysohn spaces, Proc. Amer. Math. Soc. 68, 235-242 (1978).

[12]. Joseph, J. E. , Kwack, M. H. and Nayar, B. M. P.: Weak continuity forms, graph conditions and applications, Scientiae Mathematicae Vol.2, No. 1, 65-88 (1999).

[13] T. Husain : Topology and maps, Plenum, New York, 1977.

[14]. Levine, N.: Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly Vol. 70, 36-41 (1963).

[15]. Nayar, B. M. P. and Arya, S. P. : Semi-topological properties, Internat. J. Math. and Math. Sci., Vol. 15, No.2, 267-272 (1992).

[16]. Njastad, O.: On some classes of nearly open sets, Pacific Journal of mathematics Vol. 15 No. 3, 961-970 (1965).

[17]. J. Joseph, Uniform Boundedness generalizations, Math. Chron. 7(1978), 77-83.

[18]. Veličko, N. V. : H-closed topological spaces, Math.Sb (N.S) 70 (112), 98 - 112 (1966) (Russian), (Amer. Math Soc. Transl. 78(2), 103-118 (1968)).

[19]. Stephen. Willard (1998). : general topology, University of Alberta, Dover publications, INC. Mineola, New York.