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Solute Transport in a Semi-Infinite Porous Media with Input through a Curved Line Source

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ABSTRACT.Analytical solutions of two-dimensional advection-dispersion equation (ADE) with variable dispersion coefficient and velocity are obtained using Laplace Transformation Technique. This paper describes the solute transport phenomena with non-point source of conservative solute in two-dimensional heterogeneous semi-infinite porous media. A rectangular form of hyperbolic curve is considered as non-point source. Since in real life situations sources may not always be point or straight line sources, analytical solution of present study can be taken as a step toward irregular sources. Groundwater velocity is considered spatially and temporally dependent. It depends linearly to space variable. Curved shape input source injects constant input and medium is assumed uniformly polluted initially. The dispersion is considered linearly and squarely proportional to the velocity in temporal and spatial measurement respectively. New independent variables are introduced through separate transformations to advection-dispersion equation into constant coefficient. The obtained solution has also been developed for curved surface input source in a two-dimensional flow. The derived results are illustrated graphically demonstrating interesting features of the transport phenomena. Retardation factor that occurs in the porous medium due to adsorption is also taken into account.

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1 Introduction

Analytical solutions of advection-dispersion equation (ADE) are very useful for a variety of applications such as: for solute transport in aquifer, describing similar phenomena in biophysics and biomedical sciences. In order to

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handle aquifer contamination, it is necessary to understand the mechanism of tracer transport in the aquifer. The rate of percolation of tracer in aquifer depends on the physical processes caused by dispersion. In subsurface, intrusion of salt water into the fresh water is a common example of hydrodynamic dispersion. Solute transport in aquifer are usually described by solutions of mathematical formulations in transport models obtained from numerical or analytical methods. [26] showed that the hydrodynamic dispersion coefficient depends on non-linear function of groundwater velocity. [28] developed a solute transport model with a hyperbolic distance-dependent dispersivity in porous media. [5] developed one-dimensional analytical solutions of advection-dispersion for solute transport with spatially variable coefficients accounting dispersion coefficient proportional to the square of the velocity and velocity proportional to the position. Variations in velocity generally is simulated by spatial variability in porous properties, mainly due to variations in hydraulic conductivity. In the real situations, subsurface groundwater flows are of multi-dimensional patterns. [30] presented solution of a generalized three-dimensional analytical solute transport model by using Fourier analysis and Laplace transform techniques for arbitrary source input conditions. Study is made with a unidirectional flow field from time- and space-dependent rectangular sources in a bounded homogeneous medium.

[24] recommended that dispersion parameter may be proportionate to the power of the velocity; where ranging between 1 and 2. [23] obtained analytical solutions for two-dimensional aquifer assuming a constant velocity profile. [18] developed two-dimensional analytical solutions along uniform flow with temporally dependent dispersion coefficients. Scale-dependent solute transport in porous media has been solved analytically by [16, 8, 17]. [6] examined the effect dispersion and groundwater velocity in their two dimensional solute transport study keeping dispersion squarely proportional to velocity which is directly proportional to space variable. [15] sought for an analytical for model including pulse type source and scale dependent dispersion along with first order decay in their study for a semi-infinite length domain. [4] organised dispersion coefficients and components in global and local coordinate system and in this study their values change with respect to their respective angles. [27] obtained analytical solutions for two-dimensional advection-diffusion equation with variable coefficients in semi-infinite heterogeneous porous medium. [7] analyzed a solution for solute transport in rivers and broadened the scope by considering the effect of transient storage and first order decay. [19] proposed an analytical solution for two-dimensional advection-dispersion equation by using Lapalce transform technique and Fourier-Bessel series in cylindrical coordinates for infinite domains. Proposed work comprises finite radial boundary condition along with first-inlet condition. Analytical solutions describing solute transport through two-dimensional media are obtained by [9] considering first and third type boundary conditions. [13] developed analytical solution for advection dispersion with constant dispersion coefficients using Generalized Integral Transform Technique (GITT). [1] emphasized the dependency of flow and transport processes on spatial heterogeneity and temporal variability and examined that it occurs because seasonal and variations in water levels.

In development of groundwater modeling sometimes limitations of the analytical solutions may be witnessed that lead the way to numerical solutions to address the real life problems. [2, 25, 12, 14] presented numerical solutions/techniques of sorting out real life problems. In recent years, several problems close to practical world problem were solved analytically. Two-dimensional solute dispersion in saturated porous media is considered and solved analytically by using the Laplace Transform Technique (LTT) and numerically with the help of Explicit Finite Difference (EFD) method [21]. [22] presented analytical and numerical solution model of advection-dispersion equation including transient boundary conditions. Laplace Integral Transform Technique (LITT) and explicit finite difference methods are used for analytical and numerical solution respectively. Use of Green's function in study of solute transport has been very effective tool. Green's Function Method and pertinent coordinate transformation method are employed to get analytical solutions for solute transport in groundwater and riverine flow [3] . [31] analyzed effect of space dependent of dispersion, velocity and sorption in two dimensional taking all three function of longitudinal as well as transversal space variables.

The objective of this paper is to develop a new mathematical model to describe contaminant transport in a heterogeneous porous formation. Heterogeneity of the medium is described by variable coefficients depending on space variables of the advection-dispersion equation. Key parameters dispersion coefficient and velocity are considered functions of both the independent variables. Deviating from [20] we introduced a curved line source with constant potential at all of its part. [20] conceptualized introduction of solute trough plane and straight line sources which may not be case while dealing the solute transport through real world aquifers of irregular shapes i.e. not like straight line or planes. Present problem deals with curved shape input source at one boundary and thus it represents at some extant real life situations. Solution is obtained using Laplace Transformation Technique after using certain number of transformations. Latter solution is extended for a curved plane input source in two-dimensional flow. The effect of parameters on the concentration pattern is discussed and demonstrated graphically

2 Mathematical Formulation and Solution

The problem is formulated mathematically as a curved line source of a rectangular hyperbolic shape which produces concentration of constant strength at one end of the semi-infinite domain. The aquifer is initially uniformly polluted with solute and flux at $x \to \infty$ and $y \to \infty$is zero. The physical geometry of the problem is shown in Fig. 1. The advection–dispersion equation in two-dimension derived on basis of mass conservation and Fick's law of diffusion may be written as [10] ,

$$R\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left\{D_x\left(x,y,t\right)\frac{\partial c}{\partial x} - u_x\left(x,y,t\right)c\right\} + \frac{\partial}{\partial y}\left\{D_y\left(x,y,t\right)\frac{\partial c}{\partial y} - u_y\left(x,y,t\right)c\right\} \tag{1}$$

where $c\left[ML^{-3}\right]$ is the solute concentration at position $x[L]$,$y[L]$and time $t[T]$. $D_x\left[L^2T^{-1}\right]$, $D_y\left[L^2T^{-1}\right]$ are the longitudinal and lateral dispersion coefficients respectively. $u_x\left[LT^{-1}\right]$,$u_y\left[LT^{-1}\right]$are groundwater velocity in longitudinal and lateral directions respectively. R is retardation factor which is a dimensionless quantity. Component of dispersion coefficient along an axis is linearly proportional to component of groundwater velocity along same axis temporally and square of it spatially. Both the components of the groundwater velocity are the function of time in multiple of corresponding space variable. The dispersion and groundwater velocity components along the coordinate axes may be defined as:

$$D_x = D_{x0}x^2 f\left(mt\right),\ D_y = D_{y0}y^2 f\left(mt\right)\ \ and\ \ u_x = u_{x0}xf\left(mt\right),\ u_y = u_{y0}yf\left(mt\right) \tag{2}$$

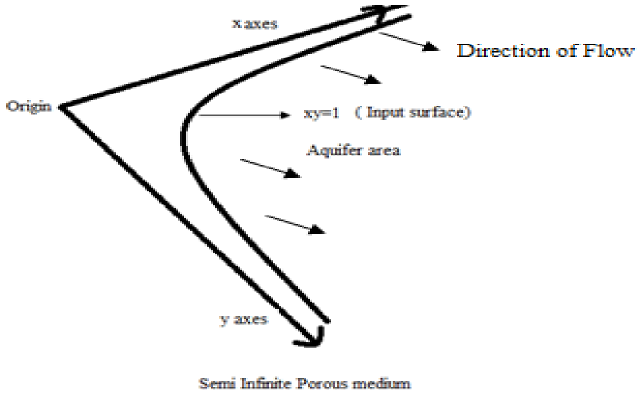

Figure 1: Geometry of the curved line source contamination

Using values of dispersion and groundwater velocity from Eq.(2) the Eq. (1) may be written as:

$$R\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left\{D_{x0}x^2f\left(mt\right)\frac{\partial c}{\partial x}-u_{x0}xf\left(mt\right)c\right\}+\frac{\partial}{\partial y}\left\{D_{y0}y^2f\left(mt\right)\frac{\partial c}{\partial y}-u_{y0}yf\left(mt\right)c\right\} \tag{3}$$

where, $m[T^{-1}]$ is an unsteady parameter that is used to regulate the time dependency of the dispersion and groundwater velocity. Since $f\left(mt\right)>0$ for $\forall t\geq 0$and also $x,y>0$ the dispersion coefficient D_x, $D_y$and groundwater velocity u_x,$u_y$never take negative values.

Mathematically, initial and boundary conditions may be written as:

$$c\left(x,y,t\right)=c_i;xy\geq 1 and x,y\geq 0,t=0 \tag{4}$$

$$c\left(x,y,t\right)=c_0;xy=1,t>0, \tag{5}$$

$$\frac{\partial c}{\partial x}=0\,;\;\frac{\partial c}{\partial y}=0\,;\;as\;x,y\rightarrow\infty,t\geq 0 \tag{6}$$

where, D_{x0},D_{x0} , u_{x0} and u_{y0} are constants . c_0 and c_i are the reference and resident concentrations. To make the Eq.(3) free from temporal coefficient following transformation is used [11] :

$$T=\int_0^t f(mt)dt \tag{7}$$

Using Eq. (7), the Eqs.(4–6) are reduced into following form:

$$R\frac{\partial c}{\partial T}=\frac{\partial}{\partial x}\left\{D_{x0}x^2\frac{\partial c}{\partial x}-u_{x0}xc\right\}+\frac{\partial}{\partial y}\left\{D_{y0}y^2\frac{\partial c}{\partial y}-u_{y0}yc\right\} \tag{8}$$

$$c\left(x,y,T\right)=c_i;xy\geq 1 and x,y\geq 0,T=0 \tag{9}$$

$$c\left(x,y,T\right)=c_0;xy=1,T>0 \tag{10}$$

$$\frac{\partial c}{\partial x}=0;\;\frac{\partial c}{\partial y}=0\;as\;x,y\rightarrow\infty,T\geq 0 \tag{11}$$

In order to reduce the Eq.(8) into one-dimensional i.e. in single space variable, we introduce a new transformation as:

$$Z=\left(\log lx+\log ly\right)/l \tag{12}$$

where $l = 1$ and dimension of the l is inverse of space. Thus dimension the Z is dimension of space. Here, when point is on the curved surface i.e. $xy = 1$ the variable Ztakes value zero i.e. $Z = 0$ and when the point inside the domain i.e., $xy > 1$ variable Ztakes the value greater than zero i.e. $Z > 0$.
With the transformation in Eq.(12), Esq. (8-11) take form:

$$R\frac{\partial c}{\partial T} = \left(D_{x0} + D_{y0}\right)\frac{\partial^2 c}{\partial Z^2} - \left(u_{x0} + u_{y0} - D_{x0} - D_{y0}\right)\frac{\partial c}{\partial Z} - \left(u_{x0} + u_{y0}\right)c \tag{13}$$

or

$$R\frac{\partial c}{\partial T} = D_0\frac{\partial^2 c}{\partial Z^2} - w_0\frac{\partial c}{\partial Z} - \mu c \tag{14}$$

where $D_0 = D_{x0} + D_{y0}$, $w_0 = u_{x0} + u_{y0} - D_{x0} - D_{y0}$and $\mu = u_{x0} + u_{y0}$

$$c\left(Z,T\right) = c_i; Z \geq 0, T = 0 \tag{15}$$

$$c\left(Z,T\right) = c_0; Z = 0, T > 0 \tag{16}$$

$$\frac{\partial c}{\partial Z} = 0;\ as\ Z \to \infty \tag{17}$$

In order to reduce the convective term from Eq. (14), following transformation is used:

$$c\left(Z,T\right) = k\left(Z,T\right)\exp\left[\frac{w_0}{2D_0}Z - \frac{1}{R}\left(\frac{{w_0}^2}{4D_0} + \mu\right)T\right] \tag{18}$$

Hence, with transformation Eq.(18), Eqs. (14-17) get converted as:

$$R\frac{\partial k}{\partial T} = D_0\frac{\partial^2 k}{\partial Z^2} \tag{19}$$

$$k\left(Z,T\right) = c_i\exp\left[-\frac{w_0}{2D_0}Z\right]; T = 0, Z \geq 0 \tag{20}$$

$$k\left(x,T\right) = c_0\exp\left(\eta^2 T\right); 0 < T \leq T_0, Z = 0 \tag{21}$$

$$\frac{\partial k\left(Z,T\right)}{\partial Z} + \frac{w_0}{2D_0}k\left(Z,T\right) = 0\ as\ Z \to \infty, t \geq 0 \tag{22}$$

where, $\eta = \sqrt{\frac{1}{R}\left(\frac{{w_0}^2}{4D_0} + \mu\right)}$
Using the Laplace Transformation Technique, solution of Eqs. (19-22) may be written as:

$$\begin{aligned} k\left(Z,T\right) = \frac{c_0}{2}\Bigg\{ & exp\left(\eta^2 T - \frac{\eta\sqrt{R}Z}{\sqrt{D_0}}\right) erfc\left(\frac{Z\sqrt{R}}{2\sqrt{D_0 T}} - \eta\sqrt{T}\right) + exp\left(\eta^2 T + \frac{\eta\sqrt{R}Z}{\sqrt{D_0}}\right) \\ & erfc\left(\frac{Z\sqrt{R}}{2\sqrt{D_0 T}} + \eta\sqrt{T}\right) - \frac{c_i}{2}\Bigg\{ exp\left(\rho^2 T - \frac{\rho\sqrt{R}Z}{\sqrt{D_0}}\right) erfc\left(\frac{Z\sqrt{R}}{2\sqrt{D_0 T}} - \rho\sqrt{T}\right) \\ & + exp\left(\rho^2 T + \frac{\rho\sqrt{R}Z}{\sqrt{D_0}}\right) erfc\left(\frac{Z\sqrt{R}}{2\sqrt{D_0 T}} + \rho\sqrt{T}\right) + c_i exp\left[-\frac{w_0}{2D_0}Z + \rho^2 T\right] \end{aligned} \tag{23}$$

Using the transformation Eq. (18), the solution of the problem may be written as:

$$c\left(Z,T\right) = \left[\frac{c_0}{2}\left\{\exp\left(\eta^2 T - \frac{\eta\sqrt{R}Z}{\sqrt{D_0}}\right) erfc\left(\frac{Z\sqrt{R}}{2\sqrt{D_0 T}} - \eta\sqrt{T}\right) + \exp\left(\eta^2 T + \frac{\eta\sqrt{R}Z}{\sqrt{D_0}}\right)\right.\right.$$

$$\left. erfc\left(\frac{Z\sqrt{R}}{2\sqrt{D_0 T}} + \eta\sqrt{T}\right)\right\} - \frac{c_i}{2}\left\{\exp\left(\rho^2 T - \frac{\rho\sqrt{RZ}}{\sqrt{D_0}}\right) erfc\left(\frac{Z\sqrt{R}}{2\sqrt{D_0 T}} - \rho\sqrt{T}\right) + \right.$$

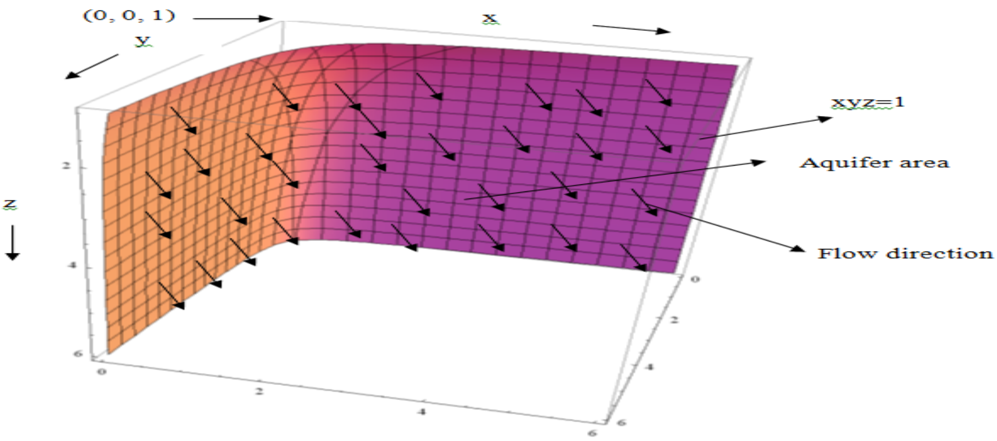

Figure 2: Geometry of the curved surface source contamination

$$+\exp\left(\rho^2 T+\frac{\rho\sqrt{R}Z}{\sqrt{D_0}}\right) erfc\left(\frac{Z\sqrt{R}}{2\sqrt{D_0 T}}+\rho\sqrt{T}\right)\Bigg\} +c_i \exp\left\{-\frac{w_0}{2D_0}Z+\rho^2 T\right\}\Bigg]\times$$
$$\exp\left[\frac{w_0}{2D_0}Z-\frac{1}{R}\left(\frac{w_0{}^2}{4D_0}+\mu\right)T\right] \tag{24}$$

where, $\rho=\sqrt{\frac{w_0{}^2}{4D_0 R}}$

2.1 Sub case: Solutions for curved surface source in three-dimensional solute transport

In present case, Input source is taken as a curved surface $xyz=1;(x>0;y>0;z\geq 1)$for two-dimensional flow. Here, z represents the depth and upper most horizontal surface of aquifer is taken at $z=1$. This curved surface source injects constant input. Porous medium i.e. $xyz\geq 1$is initially uniformly polluted. Concentration gradients along$x$, $y$and$z$axis at infinite is assumed to be zero. Thus problem of present case is an extension of the above problem. Since the study is made for horizontal flow the components of groundwater velocity along z axis vanish.

The curved surface as shown in Fig.(2) is infinitely spread and defined as $xyz=1$;$x>0;y>0$ and $z\geq 1$. This suggests upper most boundary of the input surface is the curve$xy=1$ and $z=1$.

Mathematically advection-dispersion equation for the present sub case may be written as:

$$\begin{aligned} R\frac{\partial c}{\partial t}&=\frac{\partial}{\partial x}\left\{D_x(x,y,z,t)\frac{\partial c}{\partial x}-u_x(x,y,z,t)\,c\right\}\\ &+\frac{\partial}{\partial y}\left\{D_y(x,y,z,t)\frac{\partial c}{\partial y}-u_y(x,y,z,t)\,c\right\}\\ &+\frac{\partial}{\partial z}\left\{D_z(x,y,z,t)\frac{\partial c}{\partial z}-u_z(x,y,z,t)\,c\right\}\end{aligned} \tag{25}$$

In the sub case the flow is assumed horizontal and following this assumption velocity component along the depth vanishes i.e. $u_z=0$. The dispersion component along the depth may not be zero even though corresponding velocity component is absent [30]. All the components of velocity and dispersion may be described as:

$$\begin{aligned} &D_x=D_{x0}x^2 f(mt)\,,D_y=D_{y0}y^2 f(mt)\,,D_z=D_{z0}z^2 f(mt)\\ &and\ u_x=u_{x0}xf(mt),u_y=u_{y0}yf(mt)\,,\ u_z=0\end{aligned} \tag{26}$$

Mathematically, initial and boundary conditions may be written as:

$$c\,(x,y,z,t) = c_i; xyz \geq 1 and x, y \geq 0 and z \geq 1, t = 0 \tag{27}$$

$$c\,(x,y,z,t) = c_0; xyz = 1 and x, y \geq 0 and z \geq 1, t > 0 \tag{28}$$

$$\frac{\partial c}{\partial x} = 0;\ \frac{\partial c}{\partial y} = 0;\ \frac{\partial c}{\partial z} = 0\ as\ \ x,y,z \to \infty,\ t \geq 0 \tag{29}$$

Using transformation from Eq. (7), Eq. (25) and the conditions Eqs. (27–29) may be written as:

$$R\frac{\partial c}{\partial T} = \frac{\partial}{\partial x}\left\{D_{x0}x^2\frac{\partial c}{\partial x} - u_{x0}xc\right\} + \frac{\partial}{\partial y}\left\{D_{y0}y^2\frac{\partial c}{\partial y} - u_{y0}yc\right\} + \frac{\partial}{\partial z}\left\{D_{z0}z^2\frac{\partial c}{\partial z}\right\} \tag{30}$$

$$c\,(x,y,z,T) = c_i; xyz \geq 1 and x, y \geq 0, z \geq 1, T = 0 \tag{31}$$

$$c\,(x,y,z,T) = c_0; xyz = 1 and x, y \geq 0, z \geq 1, T > 0 \tag{32}$$

$$\frac{\partial c}{\partial x} = 0;\ \frac{\partial c}{\partial y} = 0;\ \frac{\partial c}{\partial z} = 0\ as\ x,y,z \to \infty,\ T \geq 0 \tag{33}$$

In order to reduce the Eqs.(30-33) into one-dimensional or single space variable, we introduce a new transformation as:

$$Z = (\log lx + \log ly + \log lz)\,/l \tag{34}$$

where $l = 1$and dimension of the l is inverse of space and therefore dimension of the $Z$is the dimension of space. It is obvious that for the points (x, y, z) on the curved surface i.e. $xyz = 1$, the value of variable Z is zero i.e. $Z = 0$ and for the points (x, y, z) inside the aquifer $xyz > 1$, the variable Z takes value greater than zero i.e. $Z > 0$. With the help of Eq. (34), Eqs. (30-33) may be converted into following form :

$$R\frac{\partial c}{\partial T} = \left(D_{x0} + D_{y0} + D_{z0}\right)\frac{\partial^2 c}{\partial Z^2} - \left(u_{x0} + u_{y0} - D_{x0} - D_{y0} - D_{z0}\right)\frac{\partial c}{\partial Z} - \left(u_{x0} + u_{y0}\right)c \tag{35}$$

Or

$$R\frac{\partial c}{\partial T} = D_0\frac{\partial^2 c}{\partial Z^2} - w_0\frac{\partial c}{\partial Z} - \mu c \tag{36}$$

where $D_0 = D_{x0} + D_{y0} + D_{z0}$, $w_0 = u_{x0} + u_{y0} - D_{x0} - D_{y0} - D_{z0}$and $\mu = u_{x0} + u_{y0}$

$$c\,(Z,T) = c_i; Z \geq 0, T = 0 \tag{37}$$

$$c\,(Z,T) = c_0; Z = 0, T > 0 \tag{38}$$

$$\frac{\partial c}{\partial Z} = 0;\ as\ Z \to \infty,\ T \geq 0 \tag{39}$$

Adopting similar steps as in Eq.(13-23) of previous case, we get the final solution as:

$$c\,(Z,T) = \left[\frac{c_0}{2}\left\{\exp\left(\eta^2 T - \frac{\eta\sqrt{R}Z}{\sqrt{D_0}}\right)\ erfc\left(\frac{Z\sqrt{R}}{2\sqrt{D_0 T}} - \eta\sqrt{T}\right) + \exp\left(\eta^2 T + \frac{\eta\sqrt{R}Z}{\sqrt{D_0}}\right)\right.\right.$$

$$\left. erfc\left(\tfrac{Z\sqrt{R}}{2\sqrt{D_0 T}} + \eta\sqrt{T}\right)\right\} - \tfrac{c_i}{2}\left\{\exp\left(\rho^2 T - \tfrac{\rho\sqrt{R}Z}{\sqrt{D_0}}\right)\ erfc\left(\tfrac{Z\sqrt{R}}{2\sqrt{D_0 T}} - \rho\sqrt{T}\right) + \right.$$

$$\left.\left. + \exp\left(\rho^2 T + \frac{\rho\sqrt{R}Z}{\sqrt{D_0}}\right)\ erfc\left(\frac{Z\sqrt{R}}{2\sqrt{D_0 T}} + \rho\sqrt{T}\right)\right\} + c_i\exp\left\{-\frac{w_0}{2D_0}Z + \rho^2 T\right\}\right] \times$$

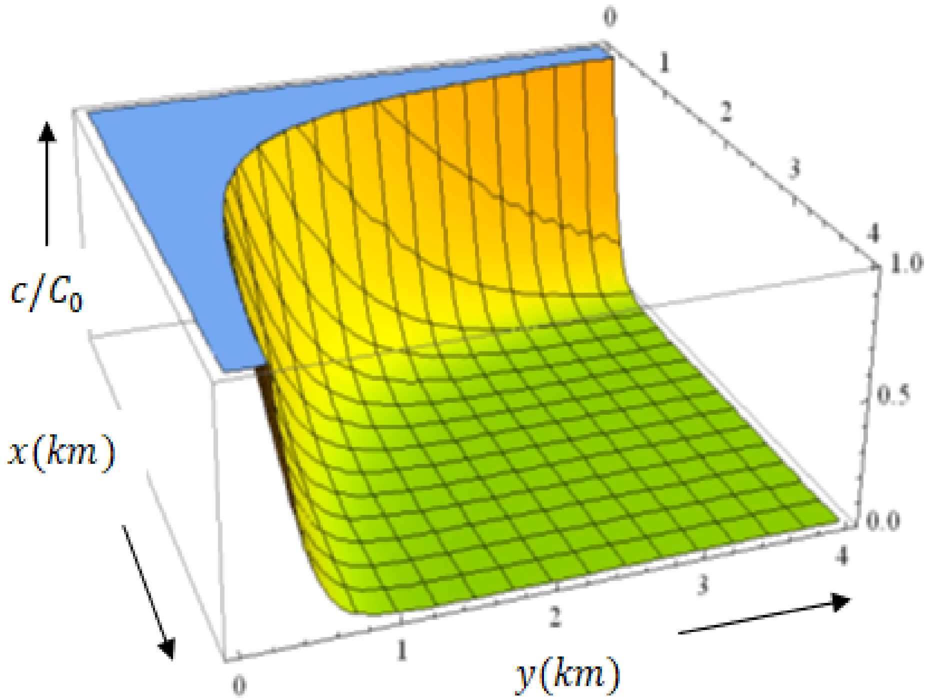

Figure 3: Dimensionless concentration distribution in $x-y$ plane obtained from solution Eq.(24) at $t\,(year) = 2$

$$\exp\left[\frac{w_0}{2D_0}Z - \frac{1}{R}\left(\frac{{w_0}^2}{4D_0}+\mu\right)T\right] \tag{40}$$

where $\rho = \sqrt{\frac{{w_0}^2}{4D_0R}}$ and $\eta = \sqrt{\frac{1}{R}\left(\frac{{w_0}^2}{4D_0}+\mu\right)}$

3.Results and Discussions

The solute concentration is anticipated with a set of values for input parameters based on published literature [20]Singh M. K. (2016). The analytical solution obtained in equation (24) is illustrated with figures in a longitudinal domain $0 \leq x(km) \leq 4$and lateral domain $0 \leq y(km) \leq 4$. The numerical values of parameters and constants are defined as: the value the dispersion and groundwater velocity is taken proportional to time function $f\,(mt) = \exp(-mt)$with value of unsteady parameter $m\,(year^{-1}) = 0.1$ and reference and resident concentrations c_0 and c_i are taken 1 and 0.01respectively. Longitudinal and lateral dispersion coefficients are taken$D_{x0}\,(kmyear^{-2}) = 0.001$;$D_{y0}\,(kmyear^{-2}) = 0.0525$ respectively. Similarly, values components of groundwater velocity are assumed as $u_{x0} = 0.10\,(kmyear^{-1})$;$u_{y0} = 0.15\,(kmyear^{-1})$. The range of groundwater velocity is taken from $2m/day$ to $2m/year$ depending upon geometrical conditions of porous media [29]. The retardation factor R is 1.15. The distribution of concentration profiles demonstrated by surface plots with distance along x and y axes are shown in Fig. 3, 4 and 5. Three plots in Fig. 3, 4 and 5 are obtained from the solutions Eq. (24) at times $t(year) = 2, 5 and 8$ respectively. In all three following figures green shaded parts are depicted at low concentration in comparison to yellow shaded parts.

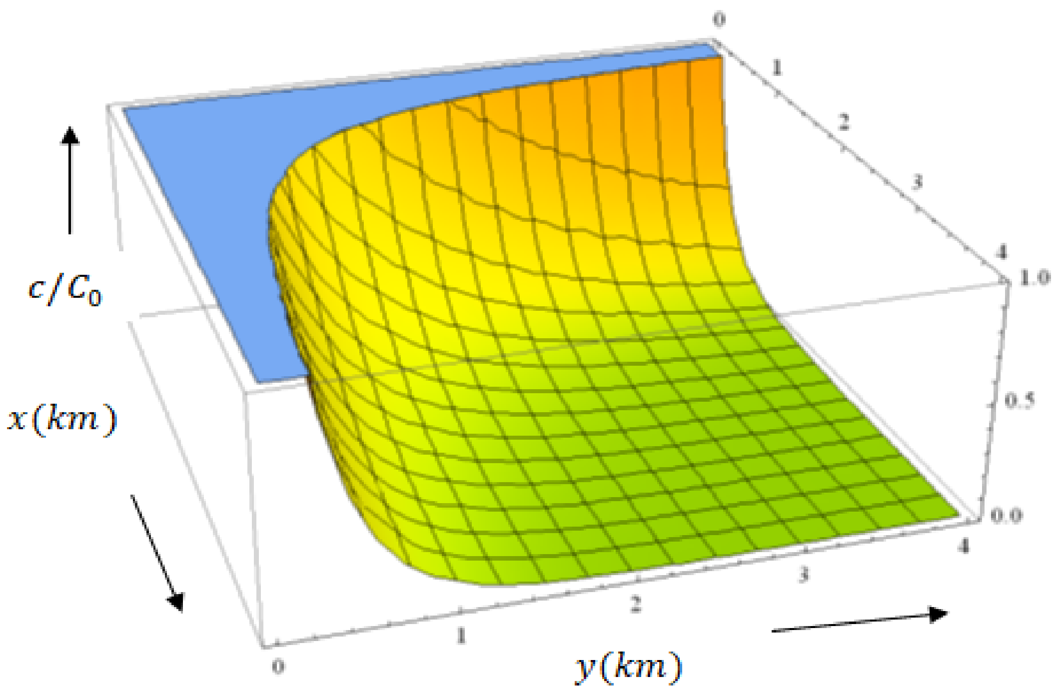

Figure 4: Dimensionless concentration distribution in x-y plane obtained from solution Eq.(24) at $t\,(year) = 5$

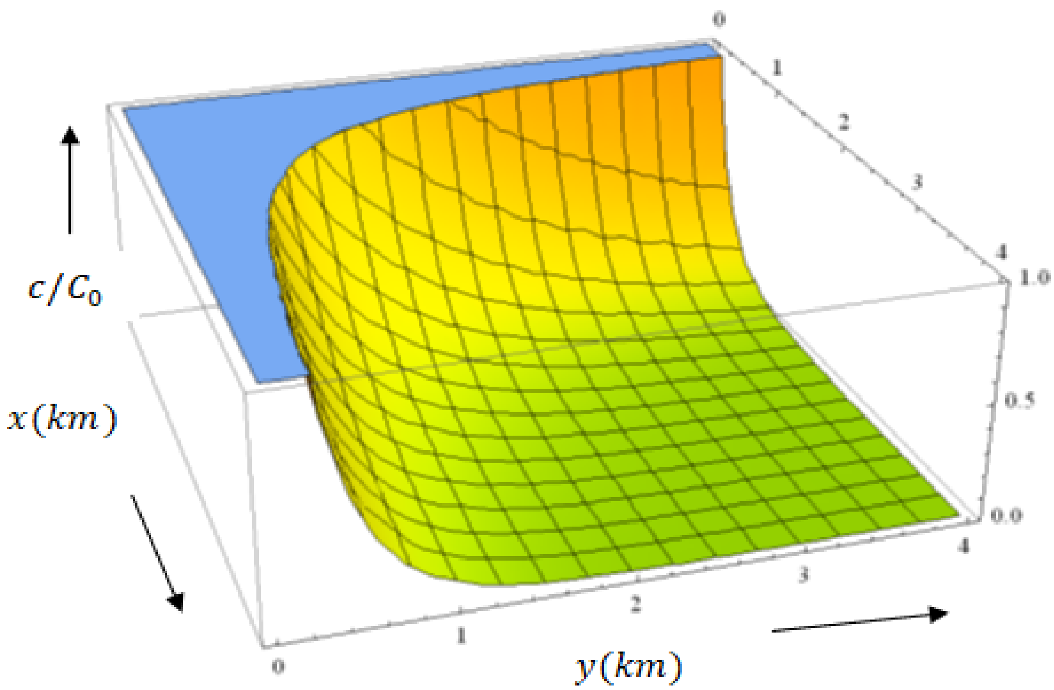

Figure 5: Dimensionless concentration distribution in x-y plane obtained from solution Eq.(24) at time $t = 8\,(year)$

Dist.along x axis (km) / Dist. along y axis (km)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
0.0	1	0.650	0.463	0.344	0.262	0.203	0.160	0.127	0.102
0.5	0.650	0.398	0.262	0.180	0.127	0.092	0.068	0.051	0.039
1.0	0.463	0.262	0.160	0.102	0.068	0.047	0.033	0.024	0.018
1.5	0.344	0.180	0.102	0.062	0.039	0.026	0.018	0.013	0.010
2.0	0.262	0.127	0.068	0.039	0.024	0.016	0.011	0.008	0.007
2.5	0.203	0.092	0.047	0.026	0.016	0.011	0.008	0.006	0.005
3.0	0.160	0.068	0.033	0.018	0.011	0.008	0.006	0.005	0.004
3.5	0.127	0.051	0.024	0.013	0.008	0.006	0.005	0.004	0.004
4.0	0.102	0.039	0.018	0.010	0.007	0.005	0.004	0.004	0.003

Figure 3, 4 and 5 are drawn at different times $t(year) = 2, 5 and 8$.It may be observed that as the time elapses the input concentration disperse in plane and level of concentration increases in the aquifer. It may also be observed that concentration c/c_0 at curved $xy = 1$ remains constant and equal to 1. It is noticed that as time increases the green shaded part which represents the low concentration compared to yellow part reduces. It decodes that concentration inside the domain increases with time.

Table : Concentration distribution patterns of solute in two-dimensional aquifer at time $t = 8\,(year)$ with distance along x and y axes from the point $(1, 1)$ on curved line source.

Concentration distribution patterns of solute in aquifer at particular time $t(year) = 8$ are tabulated. It is observed from table that concentration level decreases with distance on moving away from the input curve $xy = 1$. Level of concentration maintains symmetry from the point $(1, 1)$ on the input curve in the $x$and y axis.

Figure.6 assists us to understand the pattern of concentration demonstrated in following figure.7. Two points A$(0.5, 2.0)$ and B$(2.0, 0.5)$ are on the curve input source $xy = 1$. From each of these two points the concentration is plotted along three direction ratios $(1, 0)$, $(1, 1)$ and $(0, 1)$.

The point A and B which are two points in a symmetrical positions on the curve $xy = 1$. In figure 7, from the concentration patterns on moving along the mentioned directions it is observed that concentration decreases same way for both points for a fixed direction..

Sub case: Solutions for curved surface source in three-dimensional solute transport

Concentration values are evaluated from analytical Eq.(40) for curved surface input source $xyz = 1$. The solute dispersion parameter is considered proportional to the velocity$f\,(mt) = \exp(-mt)$. The concentration values

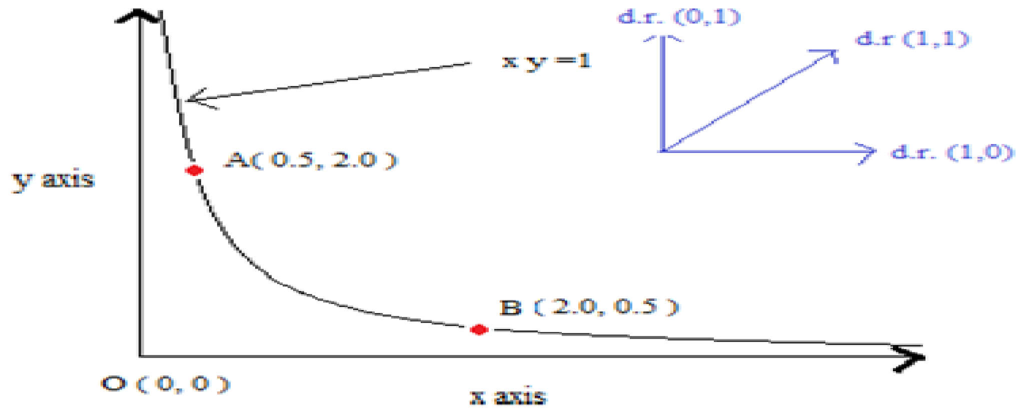

Figure 6: Figure to demonstrate the evaluation of concentration pattern in Fig.7

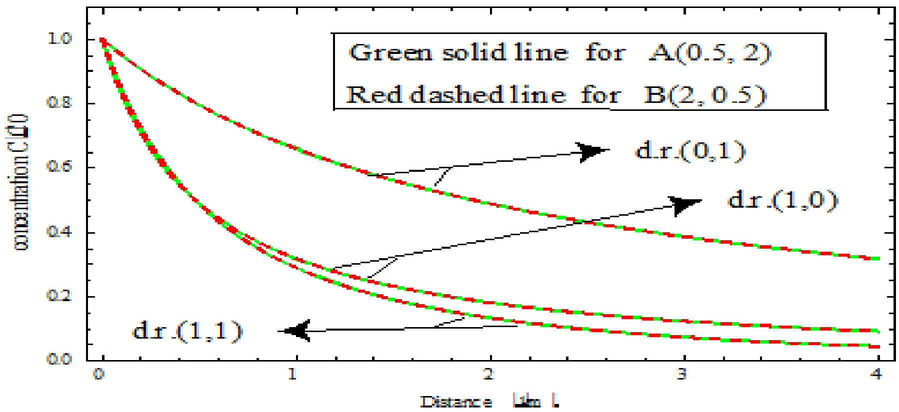

Figure 7: Dimensionless concentration distribution along the lines obtained from solution Eq.(24) at time $t\ (year) = 15$

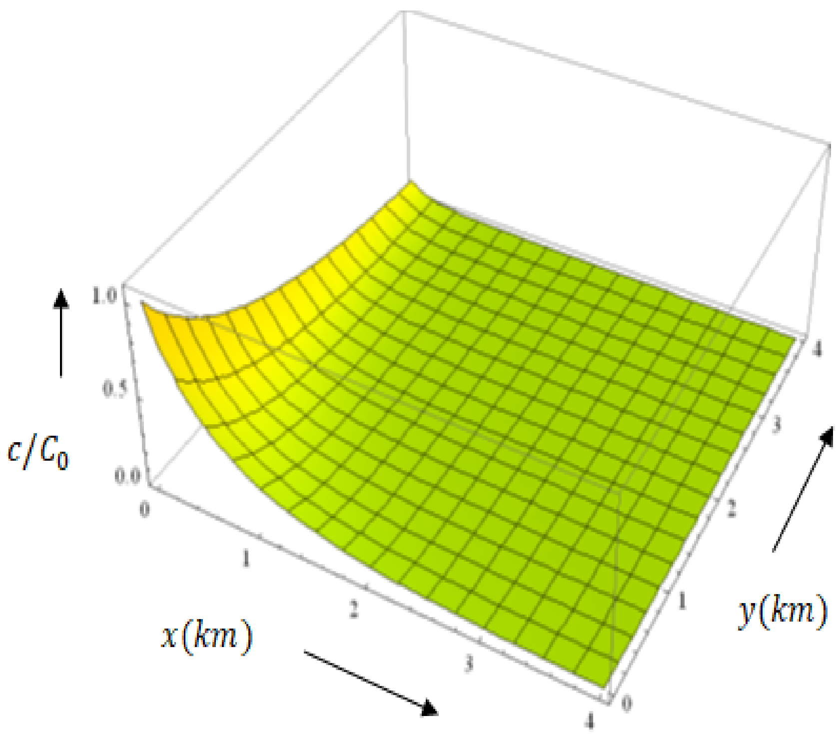

Figure 8: Dimensionless concentration distribution in x-y plane obtained from solution Eq.(40) at fixed $z(km) = 1.60$ and time $t = 8\ (year)$

c/c_0 are evaluated with input data unsteady parameter $m\ (year^{-1}) = 0.1$ and reference and resident concentrations c_0 and c_i are taken 1 and 0.01respectively. Components of dispersion coefficient and groundwater velocity are taken $D_{x0}\ (kmyear^{-2}) = 0.002$; $D_{y0}\ (kmyear^{-2}) = 0.0024$; $D_{z0}\ (kmyear^{-2}) = 0.038$and $u_{x0}\ (kmyear^{-1}) = 0.10$;$u_{y0}\ (kmyear^{-1}) = 0.15$. The surface plots in figures 7, 8 and 9 are obtained to illustrate the concentration pattern for (a) fixed z and varying x,y (b) fixed x and varying y, z and (c) fixed y and varying x, z respectively . These graphs are drawn for distance $4km$ along x, y and z axes from point $(0.50, 1.25, 1.6)$lying on input source plane $xyz = 1$. The retardation factor is $R = 1.15$. All these figures are drawn at time $t(year) = 8$.

Figures 8,9 and 10 show the surface plots to demonstrate the concentration entered through curved surface into a three-dimensional porous medium at time $t(year) = 8$. The comparison results reveal that yellow shaded part shows greater concentration in comparison to green shaded part of the concentration surface. The fig. 8 is drawn for fixed z axis at $z(km) = 1.6$. Concentration profiles c/c_0 decreases faster along x axis in comparison to same distance along y axis from the point$(0.50, 1.25, 1.6)$ on the surface $xyz = 1$. The Fig. 9 analyses the trend of concentration profile for fixed x coordinate at $x(km) = 0.5$ and for a varying y and z . In Fig. 9. Concentration decreases faster along y axis in comparison to z axis from the point$(0.50, 1.25, 1.6)$. In Fig. 10 the concentration pattern is demonstrated for fixed y at $y(km) = 1.25$. Concentration decreases faster along x axis in comparison to along z axis.

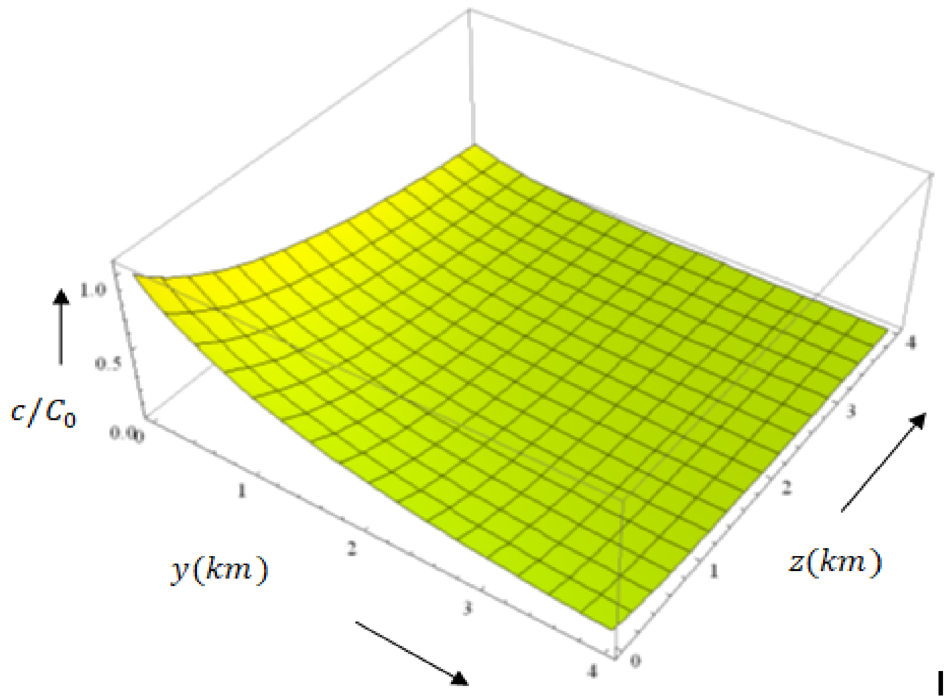

Figure 9: Dimensionless concentration distribution in y-z plane obtained from solution Eq.(40) at fixed $x(km) = 0.5$ and time $t = 8\,(year)$

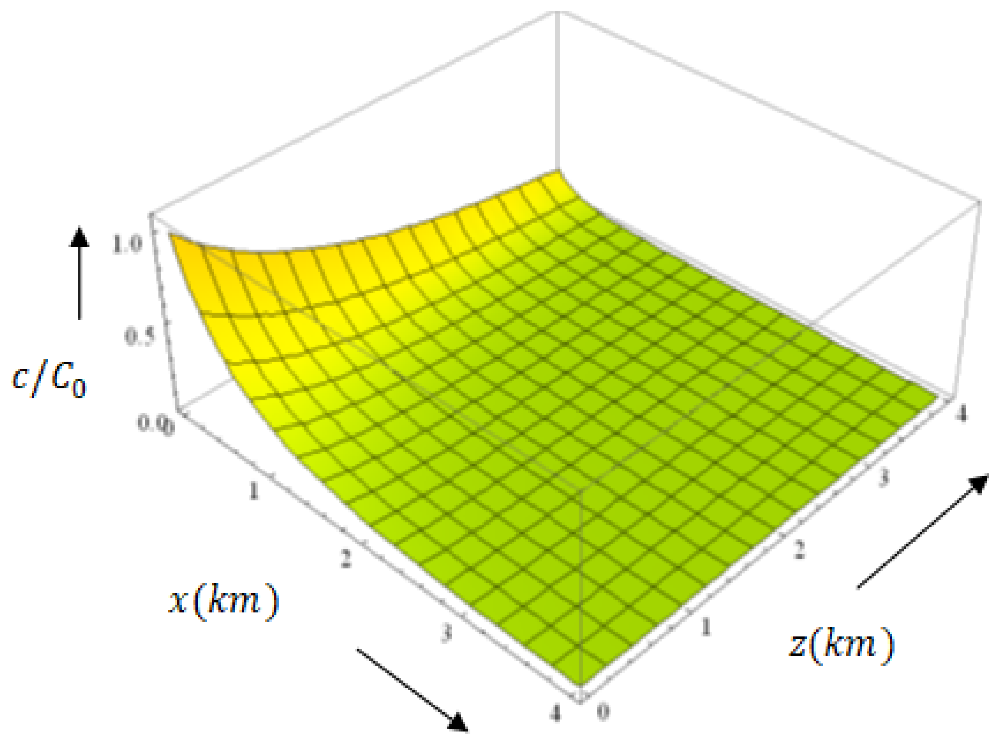

Figure 10: Dimensionless concentration distribution in x-z plane obtained from solution Eq.(40) at fixed $y(km) = 1.25$ and time $t\ (year) = 8$

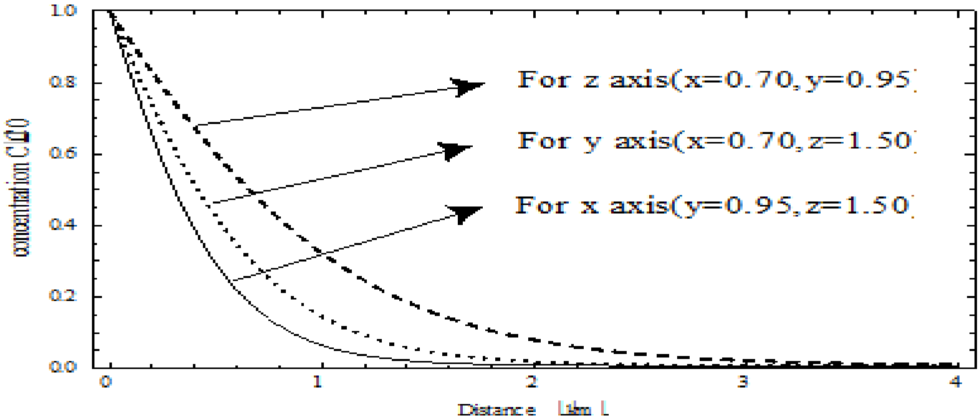

Figure 11: Dimensionless concentration distribution along the axes obtained from solution Eq.(40) along the axes at time $t = 2\,(year)$

The figure.11 demonstrates the concentration pattern along x, y and z axis from the point $(0.70, 0.95, 1.50)$ on the curved surface $xyz = 1$ at time $t = 2\,(year)$. The value parameters are same as it were for the figures 8, 9,10.
At the point $(0.70, 0.95, 1.50)$is at concentration level $c/c_0 = 1$. As we move along x ,$y$and$z$ axes from this point equal distances $4(km)$from point $(0.70, 0.95, 1.50)$ on plane it is observed that attenuation rate of concentration is fastest along x axes and slowest along z direction. Solid, dotted and dashed line represent concentration pattern along$x$, y and$z$ respectively.

4.Conclusions

Two-dimensional solute dispersion problem is discussed using the Laplace Transformation Technique subjected to hyperbolic curved source. Model includes time and space dependent dispersion and velocity where dispersion is square of the velocity spatially along the axes. A constant concentration is injected through a hyperbolic shape curved source. The aquifer is assumed to be heterogeneous and semi-infinite in nature. Initially the aquifer is supposed to be solute free. Solute concentration increases with time at all position of the region and at curved input surface it is constant with strength $c/c_0 = 1$. Concentration distribution may be noticed symmetrical i.e., concentration at points (a, b) and (b, a) with in the region of aquifer are recorded equal. The problem for curved line input source extended to the curved surface $xyz = 1$ ($x \geq 0, y \geq 0, z \geq 1$).The derived analytical results that can be used as a benchmark check for numerical solutions and complex time varying boundary conditions. It enhances the worth of the present article.

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