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On the Comparative study of Numerical Solution of Fourth-order Singularly Perturbed Boundary-value Problems by Initial Value Technique and Haar Wavelets

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ABSTRACT. The objective of this study is to obtain numerical solution of the fourth-order singularly perturbed boundary value problem using Haar wavelet approach. The purity of the presented method has been confirmed by considering two numerical examples and also compared with the existing method. It is evident that the considered processor delivers finer values as compared to the existing one.

1 Introduction

The current study is concerned to the discussion of Haar wavelet technique for obtaining the numerical solutions of fourth order singularly perturbed boundary value problems, given by

$$Lz \equiv \varepsilon z^{(4)}(t) + p(t)z'(t) + q(t)z(t) = f(t), t \in [0,1], \tag{1.1}$$

which satisfy the boundary conditions

$$z(0) = \alpha, z(1) = \beta, z'(0) = \gamma, z''(0) = \delta, \tag{1.2}$$

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where $\varepsilon > 0$ is a small positive perturbation parameter and constants α, β, γ and δ are known. It is presumed that $p(t), q(t)$ and $f(t)$ are consecutive differentiable function in the interval $[0, 1]$. Numerical solution of Singularly perturbed problems has received a great deal of attention in the current past. The problem in which a small parameter multiplies to the highest order derivative is called singular perturbation problem. Singularly perturbed problems occur very frequently in scientific problems such as fluid mechanics, quantum mechanics, elasticity, plasticity, semi-conductor device physics, control theory geophysics, aerodynamics and mathematical models of chemical reactions etc. (one can follow [1]-[10]). In the current years, the studies of singularly perturbed boundary value problems have been tackled by many researcher but the majority of these problems can not be solved analytically, so one would like ideally to use numerical methods available in the open literature such as finite difference method [[11]-[16]], finite element method [23] and spline approximation [[18]-[21]], reproducing kernel method [22] namely, finite-volume method [24], variational iteration method ([25], [26]), homotopy perturbation method [27], Septic B-spline method [28], Septic B-spline [29], Non-polynomial Quintic Spline [30], Quintic Spline [31], septic Spline [32] etc. for the standard approaches to solve singularly perturbed boundary value problems numerically the ideal equation (1.1).

We have used the technique of Haar wavelet method to approximate highest derivative appearing in the differential equation by Haar series and other derivatives are obtained through integration of Haar series. The integration of Haar wavelets is preferred because the differentiation of Haar wavelet always results impulse functions. Through integration we can expand differential equation into Haar Matrix H with Haar coefficient matrix of $2M \times 2M$ order on collocation points. The main idea of this technique is to convert a differential equation into algebraic one. In order to approximate the solution of differential equation, we collocate the algebraic equations at collocation points. The benefits of Haar wavelets transform are sparse matrix of representations than other existing method. In this article, the error analysis is mentioned that shows high order convergence can be achieved on increasing the value of M to obtain the required approximation. Rest of the discussion is summarized as follows: In section 2, we mentioned the short introduction of the Haar wavelet. The detailed methodology for solving fourth order singularly perturbed boundary value problem is discussed in the next section. Just before the final section, we presented two numerical examples for comparative study with available technique in the literature. At the end, we give the concluding remarks.

2 Haar Wavelets

Wavelets are a family of functions assembled from dilation and translation of a specific function know as the mother wavelet and denoted as $\psi(x)$. When the parameters u and v vary, then we obtain the family of continuous wavelets as follows [33] :

$$\psi_{u,v}(x) = \frac{1}{\mid u \mid^{1/2}} \psi\left(\frac{x-v}{u}\right), u, v \in R, \ u \neq 0, \tag{2.1}$$

where u dilation parameter and v translation parameter. If we bound the variables u and v to be discontinuous values as $u = u_0^{-k}$, $v = nv_0 u_0^{-k}$, $u_0 > 0$, $u_0 > 0$, for n and k are positive integers, then we get the family of discrete wavelets as follows:

$$\psi_{k,n}(x) = \mid u_0 \mid^{k/2} \psi(u_0^k x - nv_0), \tag{2.2}$$

where $\psi_{k,n}(x)$ forms a wavelet basis for $L^2(R)$.

Haar wavelet appears in pairs of piecewise constant functions above and below the corresponding axis in its standard form. Such functions were introduced by Alfred Haar in 1910. Haar showed that certain square wave function could be translated and scaled to create a basis set that span $L^2(R)$. Years later, it was seen that the system of Haar is a particular wavelet system, and they have been used for solving problems in differential equations only from nineteen ninety seven [34]. The family of Haar wavelet for $t \in [0,1)$, defined as ([35]-[41])

$$H_l(t) = \begin{cases} 1 & for \;\; t \in [A,B), \\ -1 & for \;\; t \in [B,C), \\ 0 & elsewhere, \end{cases} \tag{2.3}$$

where

$$A = \frac{p}{m}, \; B = \frac{p+0.5}{m}, \; C = \frac{p+1}{m}$$
$$m = 2^j, j = 0,1,...,J, \;\; p = 0,1,...,m-1. \tag{2.4}$$

and

$$H_1(t) = \begin{cases} 1 & for \;\; t \in [0,1), \\ 0 & elsewhere. \end{cases} \tag{2.5}$$

Where $H_1(t)$ is scaling function for the family of Haar wavelets which is defined in (2.5). The relation between l, m and p is given by $l = m + p + 1$, the integer p is translation parameter and j indicates the level of the wavelet. Maximum level of resolution is J.

Family of Haar wavelets which is defined as the following notations are followed:

$$R_{l,1}(t) = \int_0^t H_l(z)dz, \tag{2.6}$$

$$R_{l,u+1}(t) = \int_0^t R_{l,u}(z)dz, \; u = 1,2,..., \tag{2.7}$$

and

$$D_{l,s} = \int_0^1 R_{l,s}(t)dx, \quad s = 1,2,...,. \tag{2.8}$$

Any square integrable function $g(t)$ in the interval (0,1) can be written in the form of an infinite sum of Haar wavelets as:

$$g(t) = \sum_{l=1}^{\infty} b_l H_l(t). \tag{2.9}$$

We also define the integer $M = 2^J$. With these notations any square integrable function $g(t)$ defined on $[0,1)$ can be approximated as a linear combination of finite members of the Haar wavelet family and is given as follows

$$g(t) = \sum_{l=1}^{2M} b_l H_l(t). \tag{2.10}$$

The following integrals can be calculated ([42],[43]):

$$R_{l,1}(t) = \begin{cases} t-A & for \;\; t \in [A,B), \\ C-t & for \;\; t \in [B,C), \\ 0 & elsewhere, \end{cases} \tag{2.11}$$

$$R_{l,2}(t) = \begin{cases} \frac{1}{2}(t-A)^2 & for \;\; t \in [A,B)\,, \\ \frac{1}{4m^2} - \frac{1}{2}(C-t)^2 & for \;\; t \in [B,C)\,, \\ \frac{1}{4m^2} & for \;\; t \in [C,1)\,, \\ 0 & elsewhere, \end{cases} \tag{2.12}$$

$$R_{l,3}(x) = \begin{cases} \frac{1}{6}(t-A)^3 & for \;\; t \in [A,B)\,, \\ \frac{1}{4m^2}(t-B) + \frac{1}{6}(C-t)^3 & for \;\; t \in [B,C)\,, \\ \frac{1}{4m^2}(t-B) & for \;\; t \in [C,1)\,, \\ 0 & elsewhere, \end{cases} \tag{2.13}$$

$$R_{l,4}(t) = \begin{cases} \frac{1}{24}(t-A)^4 & for \;\; t \in [A,B)\,, \\ \frac{1}{8m^2}(t-B)^2 - \frac{1}{24}(C-t)^4 + \frac{1}{192m^4} & for \;\; t \in [B,C)\,, \\ \frac{1}{8m^2}(t-B)^2 + \frac{1}{192m^4} & for \;\; t \in [C,1)\,, \\ 0 & elsewhere. \end{cases} \tag{2.14}$$

3 Haar wavelet method for solving fourth order differential equations

The following collocation points are considered:

$$t_j = \frac{j-0.5}{2M},\; j = 1,2,...,2M. \tag{3.1}$$

We approximate highest order derivative using Haar wavelet series as follows

$$z^{(4)}(t) = \sum_{i=1}^{2M} b_l H_l(t). \tag{3.2}$$

Integrate (3.2) with the boundary conditions (1.2), we get $z(t)$ $z^{'}(t)$, $z^{''}(t)$, $z^{'''}(t)$ and $z^{(4)}(t)$ and these derivatives can be expressed in terms of Haar functions and their integrals.

$$z^{'''}(t) = z^{'''}(0) + \sum_{l=1}^{2M} b_l R_{l,1}(t), \tag{3.3}$$

$$z^{''}(t) = \delta + tz^{'''}(0) + \sum_{l=1}^{2M} b_l R_{l,2}(t), \tag{3.4}$$

$$z^{'}(t) = \gamma + \delta t + \frac{t^2}{2} z^{'''}(0) + \sum_{l=1}^{2M} b_l R_{l,3}(t), \tag{3.5}$$

$$z(t) = \alpha + \gamma t + \delta \frac{t^2}{2} + \frac{t^3}{6} z^{'''}(0) + \sum_{l=1}^{2M} b_l R_{l,4}(t), \tag{3.6}$$

The value of unknown term $z^{'''}(0)$ can be calculated by integrating equation (3.5) from 0 to 1, we obtain

$$z^{'''}(0) = 6\left(\beta - \alpha - \gamma - \frac{1}{2}\delta - \sum_{l=1}^{2M} b_l D_{l,3}\right). \tag{3.7}$$

Put the value of $z^{'''}(0)$ in the equations (3.4) and (3.5) in order to obtain system of equations whose solution gives us the Haar coefficients.

The equations $(3.2) - (3.5)$ can be return as matrix form

$$z^{(4)} = \begin{bmatrix} H_1(t_1) & \ldots & H_{2M}(t_1) & 0 \\ H_1(t_2) & \ldots & H_{2M}(t_2) & 0 \\ \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \\ H_1(t_{2M}) & \ldots & H_{2M}(t_{2M}) & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_{2M} \\ z^{'''}(0) \end{bmatrix}, \tag{3.8}$$

$$z^{'''} = \begin{bmatrix} R_{1,1}(t_1) & \ldots & R_{2M,1}(t_1) & 1 \\ R_{1,1}(t_2) & \ldots & R_{2M,1}(t_2) & 1 \\ \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \\ R_{1,1}(t_{2M}) & \ldots & R_{2M,1}(t_{2M}) & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_{2M} \\ z^{'''}(0) \end{bmatrix}, \tag{3.9}$$

$$z^{''} = \begin{bmatrix} R_{1,2}(t_1) & \ldots & R_{2M,2}(t_1) & t_1 \\ R_{1,2}(t_2) & \ldots & R_{2M,2}(t_2) & t_2 \\ \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \\ R_{1,2}(t_{2M}) & \ldots & R_{2M,2}(t_{2M}) & t_{2M} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_{2M} \\ z^{'''}(0) \end{bmatrix} + \begin{bmatrix} \delta \\ \delta \\ . \\ . \\ . \\ \delta \end{bmatrix}, \tag{3.10}$$

$$z' = \begin{bmatrix} R_{1,3}(t_1) & ... & R_{2M,3}(t_1) & t_1^2/2 \\ R_{1,3}(t_2) & ... & R_{2M,3}(t_2) & t_2^2/2 \\ ... & ... & ... & \\ ... & ... & ... & \\ ... & ... & ... & \\ R_{1,3}(t_{2M}) & ... & R_{2M,3}(t_{2M}) & t_{2M}^2/2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_{2M} \\ z'''(0) \end{bmatrix} + \begin{bmatrix} \gamma + \delta t_1 \\ \gamma + \delta t_2 \\ . \\ . \\ . \\ \gamma + \delta t_{2M} \end{bmatrix}, \tag{3.11}$$

and

$$z = \begin{bmatrix} R_{1,4}(t_1) & ... & R_{2M,4}(t_1) & t_1^3/6 \\ R_{1,4}(t_2) & ... & R_{2M,4}(t_2) & t_2^3/6 \\ ... & ... & ... & \\ ... & ... & ... & \\ ... & ... & ... & \\ R_{1,4}(t_{2M}) & ... & R_{2M,4}(t_{2M}) & t_{2M}^3/6 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_{2M} \\ z'''(0) \end{bmatrix} + \begin{bmatrix} \alpha + \gamma t_1 + \delta t_1^2/2 \\ \alpha + \gamma t_2 + \delta t_2^2/2 \\ . \\ . \\ . \\ \alpha + \gamma t_{2M} + \delta t_{2M}^2/2 \end{bmatrix}. \tag{3.12}$$

whose solution gives us the Haar coefficients.

Now, let $z(t)$ is a function which is differentiable and having first derivative bounded on $(0, 1)$ that is

$$|z(t)| \leq M;\ \exists\ M > 0; \forall\ t \in (0,1) :\ |z'(t)| \leq M. \tag{3.13}$$

where M is a positive constant. Suppose $z_M(t)$ is Haar wavelet approximation of the given function $z(t)$

$$z_M(t) = \sum_{l}^{2M} b_l H_l(t). \tag{3.14}$$

Babolian and Shahsavaran [48], have proved that for wavelet approximation the square of error norm

$$||z(t) - z_K(t)||^2 = \frac{M^3}{12K^2}. \tag{3.15}$$

Hence,

$$||z(t) - z_K(t)||_E = O\left(\frac{1}{K}\right). \tag{3.16}$$

Equation (3.16) shows that the error bound is inversely proportional to the level of resolution of Haar wavelet. This relation make sure when K is increased then Haar wavelet approximation is convergent .

4 Numerical Examples

To establish the superiority of the method, we deal with the two linear singular perturbed examples, which have been widely discussed in the numerical and exact solutions are available for comparison. The computer characteristic is Microsoft Windows 10 Intel(R) Core(TM) i3 CPU M 380@ 2.53 GHz with 3.00 GB of RAM, 64-bit operating system throughout this paper. Here we use the software MATLAB R2014a, for numerical computing.

Example 4.1: Solve the 4^{th} order singular perturbation boundary value problem [49]:

$$\begin{aligned} \epsilon z^{(4)}(t) \quad + \quad & \left(1+\frac{t}{2}\right) z'(t) + z(t) = \epsilon\left(-t^2-7t-8\right) \\ & + e^t\left(-t^2-t+1\right)\left(1+\frac{t}{2}\right) + \frac{2}{3}e(1-3t^2)\left(1+\frac{t}{2}\right) \\ & + (t-t^2)e^t + \frac{2}{3}e(t-t^3), \end{aligned} \tag{4.1}$$

which satisfy the boundary conditions

$$z(0) = z(1) = 0,\ \ z'(0) = 1 + \frac{2}{3}e, \quad z''(0) = 0. \tag{4.2}$$

which has the exact solution

$$z(t) = \left(t-t^2\right)e^t + \frac{2}{3}e\left(t-t^3\right). \tag{4.3}$$

The mathematical solutions of the Example 4.1 are presented in Tables 1, 2, 5 and 6. Table 1 and Table 2, contains the absolute error obtained by presented method along with the method discussed by Mishra and Saini in [49] for $J = 3$; $\epsilon = 10^{-6}$ and 10^{-9}, respectively. Table 5 and Table 6 contains the maximum absolute errors (MAE) and maximum relative errors (MRE) at collocation points for different values of J and $\epsilon = 10^{-6}$ and $\epsilon = 10^{-9}$, respectively. Figure 1 and Figure 2, represents the physical behavior of numerical solution and exact solution for $J = 3$; $\epsilon = 10^{-6}$ and $\epsilon = 10^{-9}$, respectively at grid points and for collocation points in Figure 5 and Figure 6. The above figure show that the mathematical and exact solution almost coinciding.

Example 4.2: Consider the another 4^{th} order singular perturbation boundary value problem [49]:

$$\begin{aligned} \epsilon z^{(4)}(t) \quad + \quad & 4z'(t) - 4z(t) = \epsilon e^t(4+t) + 4e^t + \left(\frac{8}{3}-2e\right) \\ & -\frac{32}{3}t + 2et + (8-6e)t^2 - 4\left(\frac{1}{3}-\frac{1}{2}e\right)t^3, \end{aligned} \tag{4.4}$$

which satisfy the boundary conditions

$$z(0) = z(1) = 0,\ \ z'(0) = 0.307525752, \quad z''(0) = 0. \tag{4.5}$$

which has the exact solution

$$z(t) = te^t + \left(\frac{2}{3}-\frac{1}{2}e\right)t - t^2 + \left(\frac{1}{3}-\frac{1}{2}e\right)t^3. \tag{4.6}$$

The mathematical solutions of the Example 4.2 are presented in Tables 3, 4, 7 and 8. Table 3 and Table 4, consists of the absolute error calculated by considered method along with the method discussed by Mishra and Saini in [49]. In Table 7 and Table 8 contains the MAE and MRE at collocation points for different values of J and $\epsilon = 10^{-6}$ and 10^{-9}, respectively. Figure 3 and Figure 4, shows the the comparative study of numerical solution and exact solution for $J = 3$; $\epsilon = 10^{-6}$ and $\epsilon = 10^{-9}$ at grid points, respectively. Similarly, Figure 7 and Figure 8 for collocation points. The above figure show that the mathematical and exact solution almost coinciding.

Table 1: Comparison of the absolute error for Example 4.1 with $\varepsilon = 10^{-6}$ and $J = 3$

t	Mishra and Saini [49]	Present method
0.000000	0.000000000	0.000000000E-00
0.000001	0.010224821	0.000000000E-00
0.000010	0.022771053	4.811147140E-19
0.000100	0.022264367	4.824699668E-16
0.001000	0.017181948	4.672490693E-13
0.010000	0.022602155	3.254673059E-10
0.020000	0.022384267	1.535032351E-09
0.030000	0.022169650	2.259745502E-09
0.040000	0.021927438	7.150202652E-11
0.050000	0.021666869	6.955776150E-09
0.100000	0.020382255	4.148213012E-08
0.300000	0.017875194	8.346872782E-08
0.500000	0.018984437	1.769933125E-07
0.700000	0.016054987	3.150548811E-07
0.900000	0.007220685	3.875424965E-07
1.000000	0.000000000	0.000000000E-00

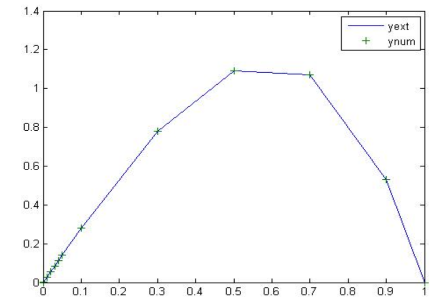

Figure 1: Analytical and approximate solutions for Example 4.1 at grid points with $\varepsilon = 10^{-6}$ and $J = 3$.

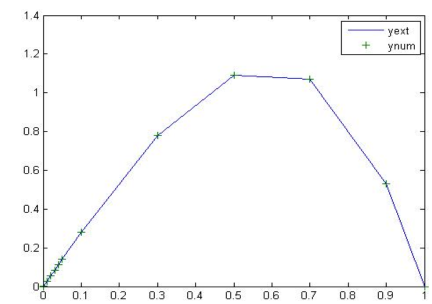

Figure 2: Analytical and approximate solutions for Example 4.1 at grid points with $\varepsilon = 10^{-9}$ and $J = 3$.

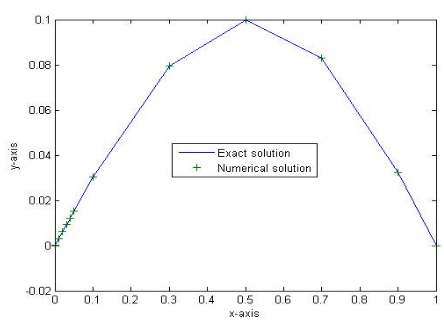

Figure 3: Analytical and approximate solutions for Example 4.2 at grid points with $\varepsilon = 10^{-6}$ and $J = 3$.

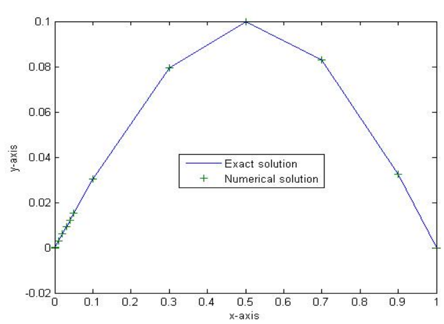

Figure 4: Analytical and approximate solutions for Example 4.2 at grid points with $\varepsilon = 10^{-9}$ and $J = 3$.

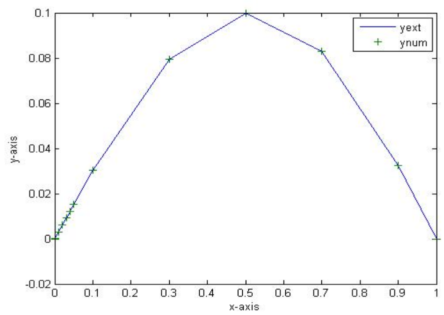

Figure 5: Analytical and approximate solutions for Example 4.1 at collocation points with $\varepsilon = 10^{-6}$ and $J = 3$.

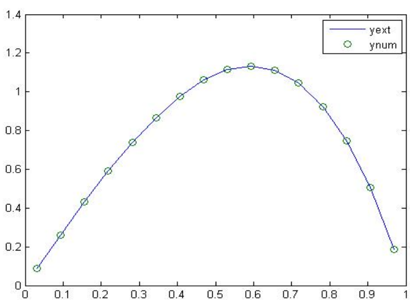

Figure 6: Analytical and approximate solutions for Example 4.1 at collocation points with $\varepsilon = 10^{-9}$ and $J = 3$.

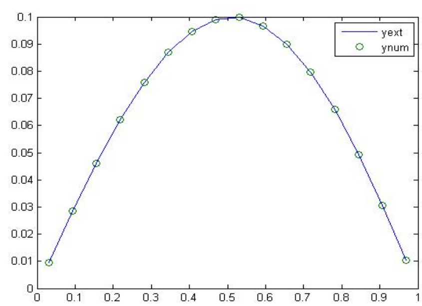

Figure 7: Analytical and approximate solutions for Example 4.2 at collocation points with $\varepsilon = 10^{-6}$ and $J = 3$.

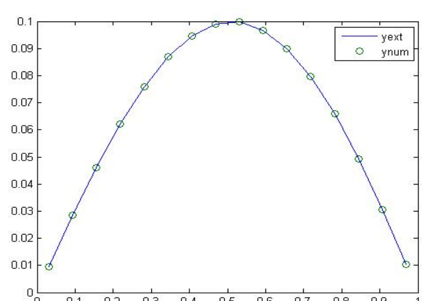

Figure 8: Analytical and approximate solutions for Example 4.2 at collocation points with $\varepsilon = 10^{-9}$ and $J = 3$.

Table 2: Comparison of the absolute error for Example 4.1 with $\varepsilon = 10^{-9}$ and $J = 3$

t	Mishra and Saini [49]	Present method
0.000000	0.000000000	0.000000000E-00
0.000001	0.010224821	0.000000000E-00
0.000010	0.022771053	5.183841637E-19
0.000100	0.022264367	5.203628327E-16
0.001000	0.017181948	5.047871843E-13
0.010000	0.022602155	3.598001480E-10
0.020000	0.022384267	1.781201443E-09
0.030000	0.022169650	2.994400089E-09
0.040000	0.021927438	1.584956236E-09
0.050000	0.021666869	4.445024265E-09
0.100000	0.020382255	3.746916871E-08
0.300000	0.017875194	8.280514607E-08
0.500000	0.018984437	1.762710065E-07
0.700000	0.016054987	2.982621719E-07
0.900000	0.007220685	4.088074923E-07
1.000000	0.000000000	0.000000000E-00

Table 3: Comparison of the absolute error for Example 4.2 with $\varepsilon = 10^{-6}$ and $J = 3$

t	Mishra and Saini [49]	Present method
0.000000	0.000000000	0.000000000E-00
0.000001	0.010224821	4.371439632E-16
0.000010	0.022771053	4.371609039E-15
0.000100	0.022264367	4.388512936E-14
0.001000	0.017181948	6.027040846E-13
0.010000	0.022602155	1.219690915E-10
0.020000	0.022384267	5.869169413E-10
0.030000	0.022169650	9.683714577E-10
0.040000	0.021927438	4.637514903E-10
0.050000	0.021666869	1.590397232E-09
0.100000	0.020382255	1.329541376E-08
0.300000	0.017875194	3.499319674E-08
0.500000	0.018984437	8.028486888E-08
0.700000	0.016054987	1.444113880E-07
0.900000	0.007220685	2.178326952E-07
1.000000	0.000000000	0.000000000E-00

Table 4: Comparison of the absolute error for Example 4.2 with $\varepsilon = 10^{-9}$ and $J = 3$

t	Mishra and Saini [49]	Present method
0.000000	0.000000000	0.000000000E-00
0.000001	0.010224821	4.371440161E-16
0.000010	0.022771053	4.371612003E-15
0.000100	0.022264367	4.388798217E-14
0.001000	0.017181948	6.055326054E-13
0.010000	0.022602155	1.245591590E-10
0.020000	0.022384267	6.055179951E-10
0.030000	0.022169650	1.023996745E-09
0.040000	0.021927438	5.786481312E-10
0.050000	0.021666869	1.399106712E-09
0.100000	0.020382255	1.296298506E-08
0.300000	0.017875194	3.484848805E-08
0.500000	0.018984437	7.994026433E-08
0.700000	0.016054987	1.431595509E-07
0.900000	0.007220685	2.184725910E-07
1.000000	0.000000000	0.000000000E-00

Table 5: MAE and MRE at collocation point for Example 4.1 with $\varepsilon = 10^{-6}$

J	$2M$	MAE	MRE
1	4	1.379384955E-04	1.222795162E-04
2	8	9.550307063E-06	8.461084887E-06
3	16	6.894024730E-07	6.082412824E-07

Table 6: MAE and MRE at collocation point for Example 4.1 with $\varepsilon = 10^{-9}$

J	2M	MAE	MRE
1	4	1.376366789E-04	1.220119622E-04
2	8	9.415020152E-06	8.341227585E-06
3	16	6.101027572E-07	5.382772734E-07

Table 7: MAE and MRE at collocation point for Example 4.2 with $\varepsilon = 10^{-6}$

J	2M	MAE	MRE
1	4	7.127347485E-05	7.599179415E-04
2	8	5.046387369E-06	5.111636399E-05
3	16	3.412558811E-07	3.420051483E-06

Table 8: MAE and MRE at collocation point for Example 4.2 with $\varepsilon = 10^{-6}$

J	2M	MAE	MRE
1	4	7.121346551E-05	7.592781218E-04
2	8	5.021222830E-06	5.086146487E-05
3	16	3.295280909E-07	3.302516084E-06

5 Conclusion

The aim of present work is to discuss an efficient and more accurate method for solving the Haar wavelet technique is use to solve numerical solution of fourth order singularly perturbed boundary value problems. The proposed method is computationally efficient and the algorithm can be easily implemented on computer. It is due to the sparsity of the transformation matrix and the small number of the wavelets coefficients. The obtained numerical result display that the present method gives better results as compared withe existing method.This idea may be extended for other types of wavelets also.

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