International Journal of Advances in Mathematics
Volume 2019, Number 4, Pages 53-65, 2019
eISSN 2456-6098


# Fixed point result in Menger spaces of p-cyclic ciric type contraction mappings

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ABSTRACT. In the present paper we give a generalization of Ciric [9] type fixed point results. We have used $p$-cyclic contraction in Menger spaces. This result is an instance of the use of $\Phi$-function introduced by Choudhury and Das [4]. We have given some corollaries to our theorem which are generalization of some existing results. We also given an example in support one of the corollary to the main theorem.

## 1 Introduction

Fixed point theory and related problems hold an important position in mathematical analysis. The idea of fixed point was first flashed in the mind of the great mathematician Cauchy. Metric fixed point theory is widely recognized to have originated in the celebrated work of S. Banach in 1922. The result of Banach is known as Banach contraction Mapping Principle. This principle has been subsequently apply to prove many fundamental results in different branches of mathematics. Now this principle has been generalized in many directions.

One of the recent generalization came into the literature of fixed point theory due to Khan, Swaleh and Sessa [18]. In 1984 they have introduced the concept of 'altering distance function' in metric space as a generalization of

* Corresponding Author.
Received April 02, 2019; Revised June 15, 2019; Accepted June 16, 2019.
2010 Mathematics Subject Classification: 47H10,54H25,54E70.
Key words and phrases: Menger space, fixed point, $\Phi$-function, $\Psi$-function.

Banach contraction mapping principle. The idea of 'altering distance function' opens a gateway of separate line of research in fixed point theory. The concept of 'altering distance function' was further generalized to two and three variables. This concept also used to find fixed point results for fuzzy mappings.

In 1942, K. Menger introduced the concept of probabilistic metric spaces. This concept was further generalized by many authors. One such generalization is due to Schweizer and Sklar [24]. They have extended the idea to statistical metric spaces. Again by use of $t$-norm statistical metric spaces turned in to Menger spaces. The theory of $t$-norm is a developed one having many applications in different branches of probabilistic and fuzzy theories. Sehgal and Bharucha-Reid [25] proved the first fixed point theorem in probabilistic metric spaces. The contraction used by the Sehgal and Bharucha-Reid is known as B-contraction or q-contraction. This contraction has been generalized to many directions. A comprehensive survey of this line of research may be seen in [14] and [24].

A generalization of 'altering distance function' was proposed by Choudhury and Das in the context of Menger spaces in [4]. They have given the idea with the help of $\Phi$-function. This idea of control function in Menger spaces opened the possibilities of proving new fixed point and coincidence point results. It has been shown in [4] that a $\Phi$-function can generate altering distance function in metric spaces. A cyclic contraction principle in Menger spaces have been proved with the help of the $\Phi$-function in [7]. Use of the $\Phi$-functions in recent times may also be noted in [1, 2, 5, 6, 8, 11, 12, 13, 20, 21] and [22].

Kirk, Srinivasan and Veermani [17] gave a new idea in fixed point research in the context of metric spaces. The idea used by them is known as cyclic contraction. They have investigated the existence of fixed points for cyclic contraction mapping. Cyclic contractions have also used to find proximity points. After the introduction of cyclic contraction there is a generalization which is known as $p$-cyclic contraction. There are many generalizations of cyclic and $p$-cyclic contraction. Some results dealing with cyclic contractions, $p$-cyclic contractions and proximity point problems may be noted in [3, 10, 15, 16, 19, 23, 26] and [27].

Ciric [9] initiated another type of contraction in metric fixed point theory. This result has been generalized to probabilistic and fuzzy fixed point theory. In the present paper we have given a generalization of Ciric type results with the help of $p$-cyclic contraction. Some corollaries has been given to our theorem, which are generalization of some present interesting results. Also an example is given to support one of our corollary.

L. B. Ciric gave following result [9] in the context of metric spaces.

**Theorem: 1.1.[9]** Let $(X, d)$ be a complete metric space. Assume there is a map $T : X \to X$ such that there exists a constant $h, 0 \leq h < 1$, and for each $x, y \in X$,

$$d(Tx, Ty) \leq h\, max\, (d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)). \tag{1.1}$$

Then $T$ has a fixed point.

Above theorem play a very important role not only in analysis but also in other areas of science involving mathematics, especially in graph theory, dynamical systems and programming languages. These renowned theorem have several mathematical and real world illustrations.

In recent years cyclic contraction and cyclic mapping have appeared in several works.

**Definition: 1.2.[17]** Let A and B be two non-empty sets. A cyclic mapping is a mapping $T : A \bigcup B \to A \bigcup B$ which satisfies:

$$TA \subseteq B \text{ and } TB \subseteq A.$$

This line of research was initiated by Kirk, Srinivasan and Veermani [17], where they amongst other results established the following generalization of the contraction mapping principle.

**Theorem: 1.3.[17]** Let A and B be two non-empty closed subsets of a complete metric space X and suppose $f : X \to X$ satisfies :

$$fA \subseteq B \text{ and } fB \subseteq A \tag{1.2}$$

$$d(fx, fy) \leq kd(x, y) \tag{1.3}$$

for all $x \in A$ and $y \in B$, where $k \in (0, 1)$. Then $f$ has a unique fixed point in $A \bigcap B$.

The problems of cyclic contractions have been strongly associated with proximity point problems. After the introduction of cyclic mapping there is another type of generalization come to the literature which is known as $p$-cyclic mapping. The definition of $p$-cyclic mapping is as following:

**Definition: 1.4.** Let $\{A_i\}_{i=1}^{p}$ be non empty sets. A $p$-cyclic mapping is a mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ which satisfies the following conditions:

$$TA_i \subseteq A_{i+1} \text{ for } 1 \leq i < p,\ TA_p \subseteq A_1.$$

In the case when $p = 2$, this reduces to cyclic mappings. In this paper we are interested in the fixed point properties of $p$-cyclic mappings in probabilistic metric spaces. In the following we describe the space briefly and to the extent of our requirement. Several aspects of this space has been described comprehensively by Schweizer and Sklar [24].

**Definition: 1.5. [14, 24]** A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^+$ denotes the set of non-negative real numbers.

**Definition: 1.6.[14, 24]** A $t$-norm is a function $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions for all $a, b, c, d \in [0,1]$

(i) $\Delta(1, a) = a$,

(ii) $\Delta(a, b) = \Delta(b, a)$,

(iii) $\Delta(c, d) \geq \Delta(a, b)$ whenever $c \geq a$ and $d \geq b$,

(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

**Definition: 1.7. Menger space: [14, 24]**
A Menger space is a triplet $(X, F, \Delta)$ where $X$ is a non empty set, $F$ is a function defined on $X \times X$ to the set of distribution functions and $\Delta$ is a $t$-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ for all $x, y \in X$,

(ii) $F_{x,y}(s) = 1$ for all $s > 0$ and $x, y \in X$ if and only if $x = y$,

(iii) $F_{x,y}(s) = F_{y,x}(s)$ for all $x, y \in X, s > 0$ and

(iv) $F_{x,y}(u + v) \geq \Delta\left(F_{x,z}(u), F_{z,y}(v)\right)$ for all $u, v \geq 0$ and $x, y, z \in X$.

An interpretation of $F_{x,y}(t)$ is that is the probability of the event that the distance between the points $x$ and $y$ is less than $t$. A metric space becomes a Menger space if we write, $F_{x,y}(t) = H(t - d(x, y))$ where $H$ is the Heaviside function given by

$$\begin{aligned} H(t) &= 1, \text{ if } t > 0\,, \\ &= 0, \text{ if } t \leq 0\,. \end{aligned}$$

**Definition: 1.8.[14, 24]** A sequence $\{x_n\} \subset X$ is said to be converge to some point $x \in X$ if given $\epsilon > 0, \lambda > 0$ we find a positive integer $N_{\epsilon,\lambda}$ such that for all $n > N_{\epsilon,\lambda}$

$$F_{x_n,x}(\epsilon) \geq 1 - \lambda. \tag{1.4}$$

**Definition: 1.9.[14, 24]** A sequence $\{x_n\}$ is said to be Cauchy sequence in $X$ if given $\epsilon > 0, \lambda > 0$ there exists a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{x_n,x_m}(\epsilon) \geq 1 - \lambda. \text{ for all } m, n > N_{\epsilon,\lambda}. \tag{1.5}$$

Definition 1.8 and 1.9 can be equivalently written by replacing $(\geq)$ and with $(>)$ in 1.4 and 1.5 respectively. More often than not, they are written in that way. We have given them in the present form for our convenience in the proofs of our theorems.

**Definition: 1.10.[14, 24]** A Menger space $(X, F, \Delta)$ is said to be complete if every Cauchy sequence is convergent in $X$.

In 1984 [18] Khan, Swaleh and Sessa introduced the concept of altering distance function in metric spaces to generalize the Banach contraction mapping principle. Choudhury and Das introduced the concept of altering distance function in the context of Menger spaces in 2008 [4]. The definition of $\Phi$-function is given here.

**Definition: 1.11. $\Phi$-function:** [4] A function $\phi : \mathbb{R} \to \mathbb{R}^+$ is said to be a $\phi$-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,

(ii) $\phi(t)$ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$,

(iii) $\phi$ is left continuous in $(0, \infty)$ and

(iv) $\phi$ is continuous at 0.

Our present result is also a instance of use of $\Phi$-function.

We now give the definition of $\Psi$-functions. This type of functions will be used in our result.

**Definition: 1.12. $\Psi$-function: [6]**

A function $\psi : [0,1]^5 \to [0,1]$ is said to be a $\Psi$-function if

(i) $\psi$ -is monotone increasing and continuous,

(ii) $\psi(x,x,x,x,x) > x$ for all $0 < x < 1$,

(iii) $\psi(1,1,1,1,1) = 1$ and $\psi(0,0,0,0,0) = 0$

An example of this type of $\Psi$-function is

$$\psi(x_1, x_2, x_3, x_4, x_5) = \frac{a\sqrt{x_1}+b\sqrt{x_2}+c\sqrt{x_3}+d\sqrt{x_4}+e\sqrt{x_5}}{a+b+c+d+e}$$ where $a, b, c, d$ and $e$ are positive real numbers.

## 2. Main Results:

Here now we give our main theorem. In this result we use $\Phi$-function as well as $\Psi$-function.

**Theorem : 2.1.** Let $(X, F, \Delta)$ be a complete Menger space where $\Delta$ is the minimum t-norm. Let $\{A_i\}_{i=1}^p$ be non-empty closed subsets of $X$ and the mapping $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a $p$-cyclic mapping, that is,

$$TA_i \subseteq A_{i+1} \text{ for } 1 \le i < p \text{ and } TA_p \subseteq A_1 \tag{2.1}$$

such that,

$$F_{Tx,Ty}(\phi(t)) \ge \psi(F_{x,y}(\phi(\tfrac{t_1}{a})), F_{x,Tx}(\phi(\tfrac{t_2}{b})), F_{y,Ty}(\phi(\tfrac{t_3}{c})), F_{y,Tx}(\phi(\tfrac{t_4}{d})), F_{x,Ty}(\phi(\tfrac{t_5}{e})) \tag{2.2}$$

whenever $x \in A_i$, $y \in A_j$ and $1 \le i, j \le p$, $i \ne j$, $t_1, t_2, t_3, t_4, t_5, t > 0$ where $t = t_1 + t_2 + t_3 + t_4 + t_5$ with $a, b, c, d, e > 0$ and $0 < a + b + c + d + e < 1$, $\psi$ is a $\Psi$-function and $\phi$ is a $\Phi$-function. Then $\bigcap_{i=1}^p A_i$ is a non-empty and $T$ has a unique fixed point in $\bigcap_{i=1}^p A_i$.

**Proof:** Let $x_0$ be any arbitrary point in $A_1$. Now we define the sequence $\{x_n\}_{n=0}^{\infty}$ in X by $x_n = Tx_{n-1},\ n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.

By (2.1), we have

$$x_0 \in A_1, x_1 \in A_2, x_2 \in A_3, ..., x_{p-1} \in A_p$$

and in general

$$x_{np} \in A_1, x_{np+1} \in A_2, ...., x_{np+(p-1)} \in A_p \text{ for all } n \geq 0.$$

Now for $t, t_1, t_2, t_3, t_4, t_5 > 0$ with $t = t_1 + t_2 + t_3 + t_4 + t_5$ we have,

$$\begin{aligned} F_{x_{n+1},x_n}(\phi(t)) &= F_{Tx_n,Tx_{n-1}}(\phi(t)) \\ &= F_{Tx_{n-1},Tx_n}(\phi(t)) \\ &\geq \psi(F_{x_{n-1},x_n}(\phi(\tfrac{t_1}{a})), F_{x_{n-1},x_n}(\phi(\tfrac{t_2}{b})), F_{x_n,x_{n+1}}(\phi(\tfrac{t_3}{c})), \\ &\qquad F_{x_n,x_n}(\phi(\tfrac{t_4}{d}))\, F_{x_{n-1},x_{n+1}}(\phi(\tfrac{t_5}{e}))). \end{aligned} \tag{2.3}$$

Taking $t_1 = \frac{at}{a+b+c+d+e}$ , $t_2 = \frac{bt}{a+b+c+d+e}$ , $t_3 = \frac{ct}{a+b+c+d+e}$ , $t_4 = \frac{dt}{a+b+c+d+e}$ and $t_5$ is so chosen that $t = t_1 + t_2 + t_3 + t_4 + t_5$ with,

$$\phi(\tfrac{t_5}{e}) > \phi(t) + \phi(\tfrac{t}{k}), \tag{2.4}$$

where $0 < a + b + c + d + e = k < 1$.

Therefore from (2.3) we can write,

$$\begin{aligned} F_{x_{n+1},x_n}(\phi(t)) \geq \psi(&F_{x_{n-1},x_n}(\phi(\tfrac{t}{k})), F_{x_{n-1},x_n}(\phi(\tfrac{t}{k})), F_{x_n,x_{n+1}}(\phi(\tfrac{t}{k})), \\ &F_{x_n,x_n}(\phi(\tfrac{t}{k})), \min\{F_{x_{n-1},x_n}(\phi(\tfrac{t}{k})), F_{x_n,x_{n+1}}(\phi(t))\}). \end{aligned} \tag{2.5}$$

We claim that for all $t > 0, n \geq 0$,

$$F_{x_n,x_{n+1}}(\phi(t)) \geq F_{x_{n-1},x_n}(\phi(\tfrac{t}{k})). \tag{2.6}$$

If possible, let for some $s > 0$, and some $n \geq 0$,

$$F_{x_n,x_{n+1}}(\phi(s)) < F_{x_{n-1},x_n}(\phi(\tfrac{s}{k})).$$

Then we can write,

$$\begin{aligned} \min\{F_{x_{n-1},x_n}(\phi(\tfrac{s}{k})), F_{x_n,x_{n+1}}(\phi(s))\} &= F_{x_n,x_{n+1}}(\phi(s)) \\ &= F_{x_{n+1},x_n}(\phi(s)). \end{aligned}$$

Then we have from (2.5), for $s > 0$,

$$\begin{aligned} F_{x_{n+1},x_n}(\phi(s)) &\geq \psi\{F_{x_{n+1},x_n}(\phi(s)), F_{x_{n+1},x_n}(\phi(s)), F_{x_{n+1},x_n}(\phi(s)), \\ &\qquad F_{x_{n+1},x_n}(\phi(s)), F_{x_{n+1},x_n}(\phi(s))\} \\ &> F_{x_n,x_{n+1}}(\phi(s)) \text{ [by the property of } \psi] \end{aligned}$$

which is a contradiction.

Therefore, for all $t > 0$ and $n \geq 1$, (2.6) hold.

Now, using (2.6) in (2.5) and by the properties of $\psi$, for all $t > 0, n \geq 1$, we have

$$
\begin{aligned}
F_{x_{n+1},x_n}(\phi(t)) &\geq \psi(F_{x_{n-1},x_n}(\phi(\tfrac{t}{k})), F_{x_{n-1},x_n}(\phi(\tfrac{t}{k})), F_{x_{n-1},x_n}(\phi(\tfrac{t}{k})), \\
&\qquad F_{x_{n-1},x_n}(\phi(\tfrac{t}{k})), F_{x_{n-1},x_n}(\phi(\tfrac{t}{k}))) \\
&= \psi(F_{x_n,x_{n-1}}(\phi(\tfrac{t}{k})), F_{x_n,x_{n-1}}(\phi(\tfrac{t}{k})), F_{x_n,x_{n-1}}(\phi(\tfrac{t}{k})), \\
&\qquad F_{x_n,x_{n-1}}(\phi(\tfrac{t}{k})), F_{x_n,x_{n-1}}(\phi(\tfrac{t}{k}))) \\
&> F_{x_n,x_{n-1}}(\phi(\tfrac{t}{k})).
\end{aligned}
$$

Therefore,

$$F_{x_{n+1},x_n}(\phi(t)) > (F_{x_n,x_{n-1}}(\phi(\tfrac{t}{k}))). \tag{2.7}$$

By repeated applications of (2.7), for all $t > 0, n \geq 1$ we get,

$$F_{x_{n+1},x_n}(\phi(t)) > F_{x_1,x_0}(\phi(\tfrac{t}{k^n})). \tag{2.8}$$

Taking limit as $n \to \infty$ on both sides of (2.8), for all $t > 0$, we obtain,

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(\phi(t)) = 1. \tag{2.9}$$

Again by virtue of a property of $\phi$, given $\epsilon > 0$, we can find $t > 0$ such that $\epsilon > \phi(t)$.

Thus the above limit implies that for all $\epsilon > 0$,

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(\epsilon) = 1. \tag{2.10}$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{x_{m(r)}\}$ and $\{x_{n(r)}\}$ of $\{x_n\}$ with $n(r) > m(r) > r$ such that

$$F_{x_{m(r)},x_{n(r)}}(\epsilon) < 1 - \lambda. \tag{2.11}$$

We take $n(r)$ corresponding to $m(r)$ to be the smallest integer satisfying (2.11) so that,

$$F_{x_{m(r)},x_{n(r)-1}}(\epsilon) \geq 1 - \lambda. \tag{2.12}$$

If $\epsilon_1 < \epsilon$, then we have,

$$F_{x_{m(r)},x_{n(r)}}(\epsilon_1) \leq F_{x_{m(r)},x_{n(r)}}(\epsilon).$$

We conclude that it is possible to construct $\{x_{m(r)}\}$ and $\{x_{n(r)}\}$ with $n(r) > m(r) > r$ and satisfying (2.11) and (2.12) whenever $\epsilon$ is replaced by a smaller positive value. As $\phi$ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain

$\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.

Then by the above argument, it is possible to obtain an increasing sequence of integers $\{x_{m(r)}\}$ and $\{x_{n(r)}\}$ with $n(r) > m(r) > r$ such that

$$F_{x_{m(r)},x_{n(r)}}(\phi(\epsilon_2)) < 1 - \lambda. \tag{2.13}$$

and

$$F_{x_{m(r)},x_{n(r)-1}}(\phi(\epsilon_2)) \geq 1-\lambda. \tag{2.14}$$

By (2.13), we can get,

$$\begin{aligned}
1-\lambda &> F_{x_{m(r)},x_{n(r)}}(\phi(t)) \\
&= F_{Tx_{m(r)-1},Tx_{n(r)-1}}(\phi(t)) \\
&\geq \psi(F_{x_{m(r)-1},x_{n(r)-1}}(\phi(\tfrac{t_1}{a})), F_{x_{m(r)-1},x_{m(r)}}(\phi(\tfrac{t_2}{b}))), F_{x_{n(r)-1},x_{n(r)}}(\phi(\tfrac{t_3}{c})), \\
&\qquad F_{x_{n(r)-1},x_{m(r)}}(\phi(\tfrac{t_4}{d})), F_{x_{m(r)-1},x_{n(r)}}(\phi(\tfrac{t_5}{e}))).
\end{aligned} \tag{2.15}$$

[ since $x_{m(r)-1} \in A_{m(r)}$ and $x_{n(r)-1} \in A_{n(r)}, m(r) \neq n(r))$ ].

By (2.4) that is, taking $t_1 = \frac{at}{a+b+c+d+e}$, $t_2 = \frac{bt}{a+b+c+d+e}$, $t_3 = \frac{ct}{a+b+c+d+e}$ ,$t_4 = \frac{dt}{a+b+c+d+e}$ and $t_5$ is so chosen that $t = t_1+t_2+t_3+t_4+t_5$ with $\phi(\frac{t_5}{e}) > \phi(t) + \phi(\frac{t}{k})$, where $a+b+c+d+e = k$, $0<k<1$, we have from (2.15),

$$\begin{aligned}
1-\lambda > \psi(&F_{x_{m(r)-1},x_{n(r)-1}}(\phi(\tfrac{t}{k})), F_{x_{m(r)-1},x_{m(r)}}(\phi(\tfrac{t}{k})), F_{x_{n(r)-1},x_{n(r)}}(\phi(\tfrac{t}{k})), \\
&F_{x_{n(r)-1},x_{m(r)}}(\phi(\tfrac{t}{k})), \min\{F_{x_{m(r)-1},x_{n(r)-1}}(\phi(\tfrac{t}{k})), F_{x_{n(r)-1},x_{n(r)}}(\phi(t))\}).
\end{aligned} \tag{2.16}$$

We have $t > 0$ and $0 < k < 1$ and $\phi$ is strictly monotone increasing function,

therefore, $\phi(\frac{t}{k}) > \phi(t)$.

We make a choice of the positive number $\eta$ such that $\eta < \phi(\frac{t}{k}) - \phi(t)$,
that is, $\phi(\frac{t}{k}) - \eta > \phi(t)$.
In view of (2.9) we may choose $r$ large enough so that,

$F_{x_{m(r)},x_{m(r)-1}}(\eta) > 1-\lambda_1$ for given $0 < \lambda_1 < \lambda$. With the above choice of $\eta$ and $r$ we can write,

$$\begin{aligned}
F_{x_{m(r)-1},x_{n(r)-1}}(\phi(\tfrac{t}{k})) &\geq \min\{F_{x_{m(r)},x_{n(r)-1}}(\phi(\tfrac{t}{k})-\eta), F_{x_{m(r)-1},x_{m(r)}}(\eta)\} \\
&\geq \min\{F_{x_{m(r)},x_{n(r)-1}}(\phi(t)), F_{x_{m(r)-1},x_{m(r)}}(\eta)\}.
\end{aligned}$$

Therefore,

$$F_{x_{m(r)-1},x_{n(r)-1}}(\phi(\tfrac{t}{k})) \geq \min\{1-\lambda, 1-\lambda_1\} = 1-\lambda. \text{ [ as } 0<\lambda_1<\lambda \text{ ]}$$

$$F_{x_{m(r)-1},x_{n(r)-1}}(\phi(\tfrac{t}{k})) \geq 1-\lambda \tag{2.17}$$

Using (2.14) and (2.17) in (2.16) we can write,

$$\begin{aligned}
1-\lambda &> \psi(1-\lambda, 1-\lambda, 1-\lambda, 1-\lambda, \min\{1-\lambda, 1-\lambda\}) \\
&> 1-\lambda, \text{ and we arrived at a contradiction.}
\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence.

Since X is complete, we have

$$\lim_{n\to\infty} x_n = z. \tag{2.18}$$

By the construction of the sequence $\{x_n\}$, we have $x_p \in A_1, x_{2p} \in A_1, ...x_{np} \in A_1$. As the $\{x_n\}$ converges to $z$ we have all the subsequences $\{x_{np}\}, \{x_{np+1}\}....., \{x_{np+p-1}\}$ also converges to $z$. As the subsequences $\{x_{np}\}$ of $\{x_n\}$ belongs to $A_1$ and $A_1$ is closed subset of $X$, $\{x_{np}\}$ converges to $z$. We have $z \in A_1$. By the same arguments we say that $z \in A_2, A_3, ....A_p$.

Therefore, $z \in A_1 \bigcap A_2 \bigcap A_3..... \bigcap A_p$, and we see that $\bigcap_{i=1}^{p} A_i$ is non-empty set.

Now, we prove that $Tz = z$.

Putting $x = x_n, y = z$ in the inequality (2.2), for all $t > 0$, we have,

$$F_{Tx_n,Tz}(\phi(t)) \geq \psi(F_{x_n,z}(\phi(\tfrac{t_1}{a})), F_{x_n,x_{n+1}}(\phi(\tfrac{t_2}{b})), F_{z,Tz}(\phi(\tfrac{t_3}{c})), F_{z,x_{n+1}}(\phi(\tfrac{t_4}{d})),$$
$$F_{x_n,Tz}(\phi(\tfrac{t_5}{e}))),$$

and taking $t_1 = \frac{at}{a+b+c+d+e}$ , $t_2 = \frac{bt}{a+b+c+d+e}$ , $t_3 = \frac{ct}{a+b+c+d+e}$ , $t_4 = \frac{dt}{a+b+c+d+e}$,
$t_5 = \frac{et}{a+b+c+d+e}$ and $k = a + b + c + d + e$ we have,

$$F_{x_{n+1},Tz}(\phi(t)) \geq \psi(F_{x_n,z}(\phi(\tfrac{t}{k})), F_{x_n,x_{n+1}}(\phi(\tfrac{t}{k})), F_{z,Tz}(\phi(\tfrac{t}{k})),$$
$$F_{z,x_{n+1}}(\phi(\tfrac{t}{k})), F_{x_n,Tz}(\phi(\tfrac{t}{k}))).$$

Taking limit as $\lim n \to \infty$ on both sides of the above inequality, for all $t > 0$ and by the properties of $\phi$ we have,

$$\lim_{n\to\infty} F_{x_{n+1},Tz}(\phi(t)) \geq \lim_{n\to\infty}(\psi\ (F_{x_n,z}(\phi(\frac{t}{k})), F_{x_n,x_{n+1}}(\phi(\frac{t}{k})), F_{z,Tz}(\phi(\frac{t}{k})),$$
$$F_{z,x_{n+1}}(\phi(\frac{t}{k})), F_{x_n,Tz}(\phi(\frac{t}{k})))).$$

that is,

$$\begin{aligned} F_{z,Tz}(\phi(t)) &\geq \psi(F_{z,z}(\phi(\tfrac{t}{k})), F_{z,z}(\phi(\tfrac{t}{k})), F_{z,Tz}(\phi(\tfrac{t}{k})), F_{z,z}(\phi(\tfrac{t}{k})), F_{z,Tz}(\phi(\tfrac{t}{k}))). \\ &= \psi(1, 1, F_{z,Tz}(\phi(\tfrac{t}{k})), 1, F_{z,Tz}(\phi(\tfrac{t}{k}))). \\ &\geq \psi(F_{z,Tz}(\phi(\tfrac{t}{k})), F_{z,Tz}(\phi(\tfrac{t}{k})), F_{z,Tz}(\phi(\tfrac{t}{k})), F_{z,Tz}(\phi(\tfrac{t}{k})), F_{z,Tz}(\phi(\tfrac{t}{k}))). \\ &> (F_{z,Tz}(\phi(\tfrac{t}{k})) \end{aligned}$$

Therefore,

$$F_{z,Tz}(\phi(t)) > F_{z,Tz}(\phi(\tfrac{t}{k})) \geq F_{z,Tz}(\phi(t)). \tag{2.19}$$

Which is a contradiction.

Since $0 < k < 1$ and F is a non-decreasing . By the property of $\phi$ and for all $t > 0$, we can write,

$$z = Tz. \tag{2.20}$$

That is $z$ is a fixed point of $T$.

To prove the uniqueness of the fixed point, let us take $v \in A_1 \bigcap A_2 \bigcap A_3... \bigcap A_p$ be the another fixed point of $T$, that is $Tv = v$.

Now,

$$\begin{aligned}
F_{z,v}(\phi(t)) &= F_{Tz,Tv}(\phi(t)) \\
&\geq \psi(F_{z,v}(\phi(\tfrac{t_1}{a})), F_{z,Tz}(\phi(\tfrac{t_2}{b})), F_{v,Tv}(\phi(\tfrac{t_3}{c})), F_{v,Tz}(\phi(\tfrac{t_4}{d}))\ F_{z,Tv}(\phi(\tfrac{t_5}{e}))). \\
&= \psi(F_{z,v}(\phi(\tfrac{t}{k})), F_{z,Tz}(\phi(\tfrac{t}{k})), F_{v,Tv}(\phi(\tfrac{t}{k})), F_{v,Tz}(\phi(\tfrac{t}{k})), F_{z,Tv}(\phi(\tfrac{t}{k}))). \\
&= \psi(F_{z,v}(\phi(\tfrac{t}{k})), F_{z,z}(\phi(\tfrac{t}{k})), F_{v,v}(\phi(\tfrac{t}{k})), F_{v,z}(\phi(\tfrac{t}{k})), F_{z,v}(\phi(\tfrac{t}{k}))). \\
&= \psi(F_{z,v}(\phi(\tfrac{t}{k})), 1, 1, F_{v,z}(\phi(\tfrac{t}{k})), F_{z,v}(\phi(\tfrac{t}{k}))). \\
&\geq \psi(F_{z,v}(\phi(\tfrac{t}{k})), F_{z,v}(\phi(\tfrac{t}{k})), F_{z,v}(\phi(\tfrac{t}{k})), F_{v,z}(\phi(\tfrac{t}{k})), F_{z,v}(\phi(\tfrac{t}{k}))). \\
&> F_{z,v}(\phi(\tfrac{t}{k})) \text{ [by the property of } \psi].
\end{aligned}$$

Therefore,

$$F_{z,v}(\phi(t)) > F_{z,v}(\phi(\tfrac{t}{k})) \geq F_{z,v}(\phi(t)), \text{ which is a contradiction.}$$

Since $0 < k < 1$ and F is a non-decreasing. By the property of $\phi$ and for all $t > 0$, we can conclude that,

$$z = v.$$

Therefore,

$$z \in A_1 \bigcap A_2 \bigcap A_3 ..... \bigcap A_p \text{ is a unique fixed point of T.}$$

Taking two non-empty sets $A$ and $B$ of $X$ we get the following corollary from above theorem.

**Corollary: 2.2.** Let $(X, F, \Delta)$ be a complete Menger space where $\Delta$ is the minimum t-norm and let there exists two closed subsets $A$ and $B$ of $X$ such that the mapping

$T : A \bigcup B \to A \bigcup B$ satisfies the following conditions:

(i) $TA \subseteq B$ and $TB \subseteq A$, (2.21)

(ii) $$F_{Tx,Ty}(\phi(t)) \geq \psi(F_{x,y}(\phi(\tfrac{t_1}{a})), F_{x,Tx}(\phi(\tfrac{t_2}{b})), F_{y,Ty}(\phi(\tfrac{t_3}{c})), F_{y,Tx}(\phi(\tfrac{t_4}{d})), F_{x,Ty}(\phi(\tfrac{t_5}{e}))), \quad (2.22)$$

for all $x \in A$ and $y \in B, t_1, t_2, t_3, t_4, t_5, t > 0$ with $t = t_1 + t_2 + t_3 +, t_4 + t_5$ here $a, b, c, d, e > 0$ and $0 < a + b + c + d + e < 1, \psi$ is a $\Psi$-function and $\phi$ is a $\Phi$ -function. Then $A \bigcap B$ is a non-empty and $T$ has a unique fixed point in $A \bigcap B$.

Taking $p = 2$, $t_1 = \frac{at}{a+b+c+d+e}$ , $t_2 = \frac{bt}{a+b+c+d+e}$ , $t_3 = \frac{ct}{a+b+c+d+e}$ , $t_4 = \frac{dt}{a+b+c+d+e}$, $t_5 = \frac{et}{a+b+c+d+e}$ and $k = a + b + c + d + e$ in theorem (2.1) we get the following corollary.

**Corollary :2.3.** Let $(X, F, \Delta)$ be a complete Menger space where $\Delta$ is the minimum t-norm and let there exists two closed subsets $A$ and $B$ of $X$ such that the mapping

$T : A \bigcup B \to A \bigcup B$ satisfies the following conditions:

(i) $TA \subseteq B$ and $TB \subseteq A$, (2.23)

(ii)
$$F_{Tx,Ty}(\phi(t)) \geq \psi(F_{x,y}(\phi(\tfrac{t}{k})), F_{x,Tx}(\phi(\tfrac{t}{k})), F_{y,Ty}(\phi(\tfrac{t}{k})), F_{y,Tx}(\phi(\tfrac{t}{k})), F_{x,Ty}(\phi(\tfrac{t}{k}))), \tag{2.24}$$

for all $x \in A$ and $y \in B, t > 0$ with $0 < k < 1$, $\psi$ is $\Psi$-function and $\phi$ is a $\Phi$-function. Then $A \bigcap B$ is a non-empty and T has a unique fixed point in $A \bigcap B$.

Now we give an example to Corollary :2.3.

**Example: 2.4.** Let $(X, F, \Delta)$ be a complete Menger space where, $X = \{x_1, x_2, x_3\}$, $\Delta(a, b) = \min\{a, b\}$ and $F_{x,y}(t)$ be defined as follows:

$$F_{x_1,x_2}(t) = F_{x_2,x_1}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.9, & \text{if} \ 0 < t \leq 1, \\ 1, & \text{if} \quad t > 1. \end{cases}$$

$$F_{x_2,x_3}(t) = F_{x_3,x_2}(t) = F_{x_1,x_3}(t) = F_{x_3,x_1}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ 0.75, & \text{if} \ 0 < t \leq 2, \\ 1, & \text{if} \quad t > 2. \end{cases}$$

Let $\phi(t) = t^2, A = \{x_1, x_3\}, B = \{x_1, x_2\}$ and $k = 0.8$ with $Tx_1 = x_1, Tx_2 = x_1$ and $Tx_3 = x_2$. Then this satisfies all the condition of Corollary: 2.3 and $x_1$ is the unique fixed point of $T$.

**3. Conclusion:** Fixed point theory has many applications in different branches of sciences such as nonlinear programming, economics, game theory, theory of differential equations and many more. The structure of Menger spaces allows us to develop fixed point results in several ways which are not possible in ordinary metric spaces. In this paper we have proved a fixed point result for $p$-cyclic mapping satisfying Ciric-type contraction in Menger spaces. We have deduce some corollaries to our theorem which are generalizations of some existing results in the literature and have given an example to validate our result. We have used two control functions ($\Phi$-function and $\Psi$-function) to prove our theorem. This paper is also an instance of the use of control functions in fixed point theory in Menger spaces. This control function appears to be helpful in exploring the geometric aspects of Menger spaces which is also relevant to the study of geometry at the quantum level.

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