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Approximate Solution of Cubic Nonlinear Stochastic Oscillators Under Parametric Excitations

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ABSTRACT. In this paper an approximate solution of cubic nonlinear stochastic oscillators under parametric excitations were investigated using Picard iterative procedure. The results presents the existence of the solution, and the solution. This improves and extends some results in literature.

1 Introduction

Consider the second order cubic nonlinear stochastic oscillator of the form:

$$\ddot{u}(t,\omega) + \delta\dot{u} + w^2 qu + 2w^2 pu^2 + \epsilon w\gamma u^3 = \epsilon f(t,\omega) + g(t)N(t) \tag{1}$$

with the deterministic initial conditions;

$$u(0) = u_0 \text{ and } \dot{u}(0) = \dot{u}_0 \tag{2}$$

Where $N(t)$ is a one-dimensional white noise i.e. a Brownian motion and is the intensity of the noise could also be interpolated as the amplitude of the external force which may be policy interventions in the market.
δ is the extent of economic damping due to speculations, w is the natural frequency, representing the number

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of times it completes a market cycle from the time of purchase to the time of selling. γ and p represent the prices being bearish or bullish in the market. That is the pull or push in the market. q is the resonance term that represents the extent of the resistance to the deformation in response to the external force that causes the pull or push in the market. $\epsilon f(t, \omega)$ represents the excitations; ω, the frequency and ϵ, the amplitude of the external force which may be a policy intervention. $g(t)$ is the deterministic envelope function. $N(t)$ it is the white noise that causes random fluctuations in the market.

$\omega \in (\Omega, B, \rho)$ belongs to a triple probability space with Ω as the sample space, B is the σalgebra on events in Ω and ρ is the probability measure.

$n(x)$ is the white noise that has the following statistical properties;

$$En(x) = 0 \tag{3}$$

$$En(x).n(x) = \delta(x_1 - x_2) \tag{4}$$

where, $\delta(.)$ is the Dirac delta function and E denotes the ensemble average operator. The Duffing oscillator, named after a German Physicist, Georg Duffing is afamous damped and forced nonlinear dynamical systems Kovacic, [13]. From the last century, the Duffing type nonlinear dynamical systems have been investigated uninterruptedly in so many fields by many researchers in physics engineering, chemistry, economics and biology. See for instance Meekangvan [17], Chen et al [4], Li et al [14] Mathony et al [16], and their reference there in.

Increase in the study of nonlinear dynamics has observed in both financial press and academic literature since the crash of stock market in Hsieh [11], Hu et al [12], Okumura [19] and the references there in. Various techniques have been used by different researchers to obtain an approximate solution for nonlinear stochastic and or cubic stochastic oscillator with resounding results, see for instance, Farzaneh et al [7], Quian et al [21], Tripathi et al [25], He [10], Lai [14] etc and the references there in there Chowdhury [5], Caughey [3], Ahmadi et al [1] Spanos [23], Atkinson [2] Zhu [26] and Gaward et al [8].

The nonlinear oscillators have recovered remarkable attention in recent decades to the variety of their engineering. For example, Duffing oscillatior is used in the study of magneto-elastic mechanical system Guchenheimer et al [9], nonlinear vibration of beams, plates and vibrations induced by flow Srinil and Zanganeh [24], etc.

Motivated by the above literature and on-going research in this direction, the objectives of this paper therefore are, to obtain an approximate solution that will govern the nonlinearity term and deterministic equation that can solve effectively the problem of unpredictability and jumps in the capital market using the Picards successive approximation technique which has been in literature in solving this problem to the best of our knowledge. This paper is organised as follows; Section 2 dwells on derivations of equations (1.0) and section 3 presents the main results of the paper while section 4 is used for the conclusion.

2 Preliminaries

2.1 Derivation of equation (1)

The Duffing equation represents the interaction in the stock market as an external force with the stock cycle and how the market responds to the stock fluctuations.

The second order cubic nonlinear differential equation under some parametric stochastic excitations motivating this work is derived as follows;

The Newtons second law with the net or resultant, force of the restoring force and the weight is given as:

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg = -kx + mg - ks = kx, \tag{5}$$

For $mg - ks = 0$ (this is equilibrium condition where the weight $W = mg$ is equal to the restoring-force ks). Dividing equation (1) by the mass m, we obtain the second-order differential equation;

$$m\frac{d^2x}{dt^2} + F(x) = 0, \tag{6}$$

Where $F(x) = kx$ is Hooke's law Zil, [27]

The general form of equation (3) which may be interpreted as movement of a unit point mass under a spring force is given as Salle, [22];

$$\ddot{x} + g(x) = 0, \tag{7}$$

Is a linear combination of powers of the displacement such that given by the nonlinear function; $g(x) = kx + k_1x^3$, then a spring whose mathematical restorative force, such as;

$$\ddot{x} + kx + k_1x^3, \tag{8}$$

is called a nonlinear spring.

Another interesting application of this type of equation with nonlinear elements is the standard closed electric LRC circuit, given as,

$$L\frac{d^2x}{dt^2} + R\frac{dq}{dt} + \frac{1}{c}q + g(x) = 0, \tag{9}$$

where, R is the resistance, L the inductance and C the capacitance. The general second-order equation of electricity is the form;

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{10}$$

Equation (11) has been extensively investigated and on which many papers have been written and continue to be written. In these general investigations the varied assumptions are made. However, as great generality is not our purpose, we shall consider frankly simple economic situation and restrict most sharply the functions f and g. For this purpose, we shall assume $f(x) = \delta$ and $g(x)$ a polynomial.

For example, the quadratic nonlinear equation of the form:

$$\ddot{x} + x + \epsilon x^2 = 0, \;\; x(0) = a, \;\; \dot{x}(0) = 0, \tag{11}$$

Solved by Mondal et al, [18] using a new analytical approach based on harmonic balance method (HBM) is a special case of equation (10) when $f(x) = 0$ and $g(x) = k_1x + k_2x^2$ with k_1 and k_2 being 1 and ϵ respectively.
The motive of this work is to derive and obtain an approximation solution of cubic (instead of quadratic) nonlinear stochastic oscillator via WHEP. In what follows therefore, we write the inhomogeneous form of the equation (11);

$$\ddot{u} + \delta\dot{u} + k_1u + k_2u^2 + k_3u^3 = F(t), \tag{12}$$

Where $F(t)$ is the forced motion or potential source $u = u(t, \omega)$. We further subject the potential source environment noise and describe it by Mao, [15]:

$$F(t) = \epsilon f(t, \omega) + g(t, x)B(x) \tag{13}$$

Where $B(x)$ is a 1-dimensional white noise (i.e. $B(x)$ is a Brownian motion) and ϵ is the intensity of the noise. This is further interpreted as the amplitude of the external force which may be a policy intervention. Substituting (14) into (13) gives;

$$\ddot{u} + \delta\dot{u} + k_1u + k_2u^2 + k_3u^3 = \epsilon f(t, \omega) + g(t, x)B(x), \tag{14}$$

Here δ is the extent of the economic damping due to speculations. There are other governing factors that affect the market. These include;

1. The prices being bearish-bullish in the market in order words the pull or the push in the market.
2. There is also the extent of the resistance to the deformation in response to the external force that causes the push or pull in the market.
3. There are also the natural frequencies; the number of times it takes to complete a market cycle from the time of purchase to the time if sale.

We therefore write equation (11) as;

$$\ddot{u}(t, \omega) + \delta\dot{u} + w^2qu + 2\epsilon w_2pu^2 + \epsilon w\gamma u^3 = f(t, \omega) + g(t)N(t), \tag{15}$$

with the deterministic initial condition;

$$u(0) = u_0, \; \dot{u}(0) = \dot{u}_0 \tag{16}$$

where, $k_1 = w^2q, k_2 = 2\epsilon w^2p, k_3 = \epsilon w\gamma$ and $B(x) = N(t)$.
w is the natural frequency, the number of times it takes to complete a market cycle from the time of purchase to the time if sale. γ and p are the prices being bearish- bullish in the market, q the extent of resistance to the deformation in response to the external force that causes the push or pull in the market. $\omega \in (\Omega, \beta, \rho)$ belongs to a triple probability space with ω as the sample space, β is the σ algebra on events in Ω and ρ is the probability measure.
$g(t)$ is the deterministic envelope function $\epsilon f(t, \omega)$ represents the excitations: ω, the frequency, and ϵ the amplitude of the external force which may be a policy intervention.
The parameters in the equation reflect the market situation in terms of traders psychology and speculation.

2.2 The Picards Local Existence Theorems

A simple proof of existence of the solution is obtained by successive approximation. In this context, the method is known as Picard iteration. It can then be shown, by using Banach fixed point theorem that the sequence of Picard iterates φ_k is convergent and that the limit is a solution to the problem.

Theorem 1. Consider the initial value problem $y'(t) = f(t, y(t)), y(t_0) = y_0$

Suppose f is uniformly Lipschitz continuous in y (meaning: the Lipschitz constant can be taken independent of t) and continuous in t, then for some value $\epsilon > 0$, there exist a unique $y(t)$ to the initial value problem on the interval, $[t_0 - \epsilon, t_0 + \epsilon]$.

2.3 Picard Fundamental Theorem

Theorem 2. Consider the initial value problem(IVP)

$$y' = f(x, y), y(x) = y_0 \tag{17}$$

Suppose $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ continuous functions in the same open rectangle, $R = \{(x, y) : a < x < b,\ c < y < d\}$ that contains the point (x_0, y_0). Then the IVP has a unique solution in the same closed interval $I = \{x_0 - h, x_0 + h\}$ where $h > 0$. Moreover the Picard iteration defined by

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))dt \tag{18}$$

Produces a sequence of functions $\{y_0(x)\}$ that converges to this solution uniformly on I. **Convergence.** To carry out a rigorous test for convergence which is especially necessary when we dont recognize the sequence of Picard iterates, we need some idea of distance between functions. The distance measured used in the proof of the norm of a function as given in the following definition.

Definition 1. Let $C[a, b]$ denote the set of all functions that are continuous on $[a, b]$. If $y \in C[a, b]$, then the norm of y is

$$||y|| := max_{x \in [a,b]} |y(x)| \tag{19}$$

Then the norm of a function $y(x)$ may be regarded as the distance between $y(x)$ and $y = 0$, the function that is identically zero. With this in mind we may define the distance between two functions, $y, zC[a, b]$ to be the norm of $y - z$, or $||y - z|| := max_{x \in [a,b]} |y(x) - z(x)|$.

Using this measure of distance, we can define the convergence of a sequence of functions to a limiting function.

Definition 2. A sequence $\{y_n(x)\}$ of functions in $C[a, b]$ converges uniformly to the function

$$y(x) \in C[a, b] \;\; iff \;\; \lim_{n \to \infty} ||y_n - y|| = 0 \tag{20}$$

Illustration of Uniform Convergence:
If $y_n(x) = \sum_{i=0}^{n} \frac{(2x)^i}{i!}$ and $y(x) = e^{2x}$ on the interval $[0,1]$. Then,

$$\begin{aligned} \lim_{n\to\infty} ||y_n - y|| &= \lim_{n\to\infty} \max_{x\in[0,1]} |\sum_{i=0}^{n} \frac{(2x)^i}{i!} - e^{2x}| \\ &= \lim_{n\to\infty} \max_{x\in[0,1]} |\sum_{i=0}^{n} \frac{(2x)^i}{i!}| \\ &= \lim_{n\to\infty} \sum_{i=0}^{n} \frac{(2x)^i}{i!} = 0 \end{aligned} \tag{21}$$

Therefore $y_n \to y$. One important detail to note in the above example is that the uniform convergence of this sequence $\{y_n(x)\}$ to $y(x) = e^{2x}$ on $[a,b]$ occurs only when the interval is bounded on the right. In order words, b must be finite. If we were to take b to be finite, and interval under consideration to be the whole real line, then the sequence would not converge uniformly. The reason is that, if the value of x is unbounded, then for any finite n, the norm of $\sum_{i=n+1}^{\infty} \frac{(2x)^i}{i!}$ is infinite. Uniform convergence is particularly useful in that if a sequence of differentiable (and therefore continuous) functions is uniformly convergent then the function to which it converges is also continuous.

Properties:

1. If a function f defined on the real line with real values is Lipschitz continuous with Lipschitz constant, $L < 1$, then this function has precisely one fixed point and the fixed point iteration converges towards that fixed point for any initial guess x_0.
2. The speed of the convergence of the iteration sequence can be increased by using a convergence acceleration method such as Aitkens delta-squared process. The application of Aitkens method to fixed-point iteration is known as Steffensens method yields a rate of convergence that is at least quadratic. ['infinite compositions of analytic functions', May, 2010 from Wikipedia, the free encyclopaedia.

2.4 Assumptions underlying Picards Existence and Uniqueness Theorem

1. The Continuity Assumption: suppose Theorem3: Suppose the functions $z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$ are continuous on $(a,b) \times (c,d)$ and the point $(x_0, y_0) \in (a,b) \times (c,d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a,b)$ such that there exists a unique function $y = \emptyset(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfied the following initial value problem $y' = f(t,y), y(t_0) = y_0$.
2. The Lipschitz Assumption: In order to develop the Picard iterative method, we need the Lipschitz condition for the function $f(t,u) \in C[(t_0, t_0 + T) \times R, R]$ is said to be a Lipschitz function in u if for any u_1, u_2, there exists an $L > 0$ such that $|f(t,u_1 - f(t,u_2)| \leq L|u_1 - u_2|$.
   The above is a global Lipschitz condition, but if u_1, and u_2 are in a known closed interval containing u_0, then we say that $f(t,u)$ is locally Lipschitzian.

2.5 Picards successive approximation method

Let us consider the first order differential equation

$$\frac{dy}{dx} = f(x,y) \;\; and \;\; y = y_0 \;\; for \;\; x = x_0. \tag{22}$$

The method, Integrating (1) between the limits x_0 and x, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)dx \;\; or \;\; y - y_0 = \int_{x_0}^{x} f(x,y)dx \;\; or \;\; y = y_0 + \int_{x_0}^{x} f(x,y)dx \tag{23}$$

Equation (2) is the solution of (1). But (2) contains the unknown y under the integral sign on the right hand side. On putting y_0 for y on the R.H.S. of (2), we get a first approximation

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y)dx \tag{24}$$

From (3) we get the value of y_1 and we put y_1 for y on the R.H.S. of (2) to get second approximation y_2.
Thus

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)dx$$

Similarly third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x,y_{2_1})dx$$

and so on.
In this way we get a better approximation each time than the preceding one [32].

3 Result

3.1 The Existence of the Approximate Solution

On the existence of the solution of the problem under study;

$$L(u(t,x)) = -\epsilon(2w^2pu^2 + w\gamma u^3) + f(t,x) + g(t,x)N(x); \;\; (t,x) \in ((0,T] \times R) \tag{25}$$

Theorem 1 is used.
Theorem 1; It states that the solution to the above equation (3.1) exists if

$$u(t,x) = \sum_{i=0}^{\infty} \epsilon^i u_i(t,x), \tag{26}$$

a power series in ϵ converges. It converges if;

$$\epsilon \leq \frac{u_i}{u_{i+1}} \tag{27}$$

Proof. Using Picards method, the solution obtained is;

$$u(t,x) = \lim_{k\to\infty} u^{(k)}(t,x) \tag{28}$$

Where the k^th approximation is computed as;

$$L(u^k(t,x)) = -\epsilon[2w^2p(u^{k-1})^2 + w\gamma(u^{k-1})^3] + f(t,x)g(t,x)N(x); \tag{29}$$

Use the inverse operator L^{-1} on both sides and get;

$$(u^k(t,x)) = -\epsilon L^{-1}[2w^2 p(u^{k-1})^2 + w\gamma(u^{k-1})^3] + L^{-1}f(t,x) + L^{-1}g(t,x)N(x); \tag{30}$$

On Picards iteration, let

$$(u^k(t,x)) = L^{-1}[f(t,x) + g(t,x)N(x)]; \tag{31}$$

Which is the solution at $\epsilon = 0$.

So the Picards $k^t h$ approximation (iteration) is;

$$(u^k(t,x)) = u^0(t,x) - \epsilon L^{-1}[2w^2 p(u^{k-1})^2 + w\gamma(u^{k-1})^3]; \tag{32}$$

The $k^t h$ approximation as a power series of ϵ is written as;

$$(u^k(t,x)) = \sum_{i=0}^{A_k} \epsilon^i u_i^k(t,k) \tag{33}$$

Where A_{k+1} is the number of terms in the series, $u_i^{(k)}(t,x)$ is the i^{th} approximation of $u^k(t,x)$. For $k = 0, A_0 = 0, k = 1, A_1$; but for $k \geq 2$ the value of A_k depends on n.

There is need to show that a power series solution will be obtained at the first approximation i.e. at $(k = 1)$ and at $(k+1)^{th}$ approximation. At $k = 1$ the Picards first approximation will be;

$$(u^1(t,x)) = u^0(t,x) - \epsilon L^{-1}[2w^2 p(u^0)^2 + w\gamma(u^0)^3] \tag{34}$$

which is a power series in ϵ. Then $u^{k+1}(t,x)$ is again shown to be a power series in ϵ by computing the $(k+1)^{th}$ approximation thus;

$$(u^{k+1}(t,x)) = u^0(t,x) - \epsilon L^{-1}[2w^2 p(u^k(t,x))^2 + w\gamma(u^k(t,x))^3] \tag{35}$$

Now substitute the power series in (3.8) and get;

$$(u^k(t,x)) = u^0(t,x) - \epsilon L^{-1}[2w^2 p\Sigma_{i=0}^{A_k}\epsilon^i(u_i^k(t,x))^2 + w\gamma(\Sigma_{i=0}^{A_k}\epsilon^i(u_i^k(t,x))^3)] \tag{36}$$

The second term of the right hand side can be expanded using the multinomial theorem to get;

$$\Sigma_{i=0}^{A_k}\epsilon^i(u_i^k(t,x))^2 = \Sigma_h C_h \prod_{i=0}^{A_k} \epsilon^i (u_i^k(t,x))^{j_h^i} \tag{37}$$

Where $C_h = \frac{2!}{\prod_{i=0}^{A_k} j_h^i}$ the counter h runs over all $\binom{2+A_k}{2}$ combinations of the positive integers $j_h^0, j_h^1, j_h^2, ..., j_h^{A_k}$ such that $\Sigma_{i=0}^{A_k} j_h^i = 2$.

In a simple form;

$$\Sigma_{i=0}^{A_k}\epsilon^i(u_i^k(t,x))^2 = \Sigma_h C_h \epsilon^{w_h} \prod_{i=0}^{A_k} (u_i^k(t,x))^{j_h^i} \tag{38}$$

where $w_h = \Sigma_{i=0}^{A_k} i j_h^i$.

If $\prod_{i=0}^{A_k}(u_i^k(t,x))^{j_h^i} = V_h(t,x)$, then the expansion takes the form;

$$\Sigma_{i=0}^{A_k}\epsilon^i(u_i^k(t,x))^2 = \Sigma_h C_h \epsilon^{w_h} V_h(t,x) \tag{39}$$

Use l instead of h for the nonlinear exponent 3 instead of 2 respectively and substituting in the Picards $(k+1)^{th}$ approximation to get;

$$u^{k+1}(t,k) = u^0(t,x) - \epsilon L^{-1}[2w^2 \sum_h C_h \epsilon^{w_h} V_h(t,x) + w\gamma \Sigma_l C_l \epsilon^{w_l} V_l(t,x)] \tag{40}$$

or

$$u^{k+1}(t,k) = u^0(t,x) - 2w^2 \rho \sum_h C_h \epsilon^{1+w_h} L^{-1} V_h(t,x) - w\gamma \Sigma_l C_l \epsilon^{1+w_l} L^{-1} V_l(t,x)] \tag{41}$$

The equation (3.1.18) is the power series in ϵ

3.2 Picards Successive Approximation Method (with Maple software) for Equation (1)

The Picards successive approximation method (with Maple software) is used here to find the approximate solution to model equation (1) without the white noise.

Let $x = u_1$ and $y = u_2$, so for $\dot{u}_1 = u_2, \Leftrightarrow \dot{x} = y$.

Therefore, $\ddot{x} = \ddot{u} = \dot{y} = -\delta y - w^2 qx - 2\epsilon w^2 x^2 - w\gamma x^3 + \epsilon x cos(\omega t)$.

The function $f(x,y) = y\prime = (x,y) \to -\delta y - w^2 qx - 2\epsilon w^2 x^2 - w\gamma x^3 + \epsilon x cos(\omega t)$

$$y(x_0) = y(0)$$

set $a = x_0, a = 0, \varnothing_0 = y(x_0) = y_0$

The Picard Iterates:

- For k from 0 to $N-1$ do
- $\varnothing_{k+1} := unapply\ (\varnothing_0 + \int_a^x f(t, \varnothing_k(t))dt, x)$;
- print $(unprint\ f(''iteratenumber\%d : k+1))$
- print$(\varnothing_{k+1}(x))$

end do: **Iterate 1**

$$\varnothing_0 + \int_a^x (-\delta \varnothing_0(t) - w^2 qx - 2\epsilon w^2 x^2 - w\gamma x^3 + \epsilon x cos(\omega t))dt$$

Iterate 2

$$\varnothing_0 + \int_a^x (-\delta(\varnothing_0 + \int_a^t (-\delta \varnothing_0(t) - w^2 qx - 2\epsilon w^2 x^2 - w\gamma cos(\omega t))dt) - w^2 qt - 2\epsilon w^2 t^2 - w\gamma x^3 + \epsilon x cos(\omega t)$$

Iterate 3

$$\varnothing_0 + \int_a^x (-\delta(\varnothing_0 + \int_a^t (-\delta \varnothing_0(t) - w^2 qx - 2\epsilon w^2 x^2 - w\gamma cos(\omega t))dt) - w^2 qt - 2\epsilon w^2 t^2 - w\gamma x^3 + \epsilon x cos(\omega t)dt - w^2 qx - 2\epsilon w^2 x^2 - w\gamma x^3$$

4 Conclusion

It is important to examine nonlinearity in order to ascertain whether the process governing fluctuation is deterministic or stochastic. A deterministic process facilitates prediction of the future by economic agents which solution has provided. Picard method is studied extensively. The method is used to examine the existence of equation (1) and also used to find the approximate solution of the same equation (1).

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