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Some Modular Equations and Lambert Series for a Continued Fraction of Ramanujan

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ABSTRACT. We give modular equations and Lambert Series for a continued fraction of Ramanujan.

1. Introduction

In Ramanujan's second notebook [16] there are several hundred modular equations. It is surprising that some of deepest results on modular equations appear only in first notebook. In chapter 36, B.C. Berndt [9] has established all the results on modular equations given in first notebook, but not in the second. In a fragment published with Ramanujan's notebook [17] there is a list of twenty identities involving lambert series. Then in another fragment on pages 356 & 357 there is an almost identical list of twenty one lambert series identities. Most of these can be found in Ramanujan's second notebook.

The celebrated Rogers-Ramanujan continued fraction

$$R(q) = 1 + \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \frac{q^4}{1+} \dots = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{1}$$

was extensively studied by Andrews [4, 5], Hirschhorn [12, 13]. Recently H. M. Srivastava et al.[21] using identities of Bowman and Maclaughlin [10] derived an associated continued fraction and using identities of Andrews and Berndt [6] derived q-identities. In [22] H. M. Srivastava et al. derived some new results which show the interrelationship between q-product identities and combinatorial partition identities. In [23] H. M. Srivastava et al.

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gave some q-identities using theta functions of Jacobi and Ramanujan and interestingly a relation between three theta type function arising from Jacobi's triple-product identity. In [1] Chandrashekar Adiga et al. established two q-series representations of Rogers-Ramanujan continued fraction. They also gave integral representations. In [2] Chandrashekar Adiga et al. gave some new modular relation for the Rogers-Ramanujan type function of order eleven.

We have earlier considered a continued fraction of Ramanujan [15] defined by

$$\begin{aligned} P(q) &= \frac{1}{1+}\frac{1-q^2}{1-q^3+}\frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)+}\frac{(q^{5/2}-q^{19/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)+}\cdots \\ &= \frac{(q^4,q^8;q^{12})_\infty}{(q^2,q^{10};q^{12})_\infty} \end{aligned} \tag{2}$$

Let

$$T(q) = \frac{1}{(q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty} \tag{3}$$

$$S(q) = \frac{1}{(q^2;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty} \tag{4}$$

So $P(q) = \frac{S(q)}{T(q)}$

The paper is organized as follows;

In section 3, we give two relations between $S(q)$ and $T(q)$ and with the help of these relations in section 4 we give three modular equations . In the section 5 we give some expressions for Lambert Series. In the last section 6 we prove an important identity.

We shall be using the following relations due to Slater[19, eq. (50) and eq.(51)] in the sequel:

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q;q)_{2n+1}} = (q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty \tag{5}$$

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q;q)_{2n+1}} = (q^2;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty \tag{6}$$

2. Notations and Definitions

We use the following standard notation throughout this paper.

For $|q| < 1$ and $x \neq 0$

$$j(x,q) = (x;q)_\infty(q/x;q)_\infty(q;q)_\infty \tag{7}$$

If m is a positive integer and a is an integer, then for $m > 1$

$$J_{a,m} = j(q^a,q^m), \tag{8}$$

$$J_{a,m} = j(-q^a,q^m), \tag{9}$$

$$J_m = j(q^m,q^{3m}) = (q^m;q^m)_\infty \tag{10}$$

Also from definitions

$$\begin{aligned}
&j(q/x,q) = j(x,q),\\
&j(x^{-1},q) = -x^{-1}j(x,q),\\
&j(x,q)j(-x,q) = J_{1,2}j(x^2,q^2),\\
&(a)_0 = (a;q)_0 = 1,\\
&(a)_n = (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ n \geq 1\\
&(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k),\ |q| < 1.
\end{aligned}$$

Using (7) we can write $S(q)$ and $T(q)$ as

$$S(q) = \frac{1}{j(q^2,q^{12})} \text{ and } T(q) = \frac{1}{j(q^4,q^{12})} \tag{11}$$

3. Identities Related to S(q) and T(q)

We prove the following two identities:

$$(i) \quad S(q^2)T^2(q) - S^2(q)T(q^2) = 2q^2 j(q^2,q^{24})j(q^{18},q^{24})j^{-1}(q^{12},q^{24})T(q)S(q)T(q^2)S(q^2) \tag{12}$$

$$(ii) \quad S(q^2)T^2(q) + S^2(q)T(q^2) = 2j(q^6,q^{24})j(q^{14},q^{24})j^{-1}(q^{12},q^{24})T(q)S(q)T(q^2)S(q^2) \tag{13}$$

Proof of (i).

By a Theorem of Hickerson [14, Th. 1.2]. For $0 < |q| < 1,\ x \neq 0,\ y \neq 0$

$$j(-x,q)j(y,q) - j(x,q)j(-y,q) = 2xj(y/x,q^2)j(xyq,q^2) \tag{14}$$

Replacing q by q^{12}, x by q^2 and y by q^4 in (14), we have

$$j(-q^2,q^{12})j(q^4,q^{12}) - j(q^2,q^{12})j(-q^4,q^{12}) = 2q^2 j(q^2,q^{24})j(q^{18},q^{24}) \tag{15}$$

Multiplying both side by $\frac{1}{j(q^2,q^{12})j(q^4,q^{12})j(q^4,q^{24})j(q^8,q^{24})}$, we get

$$\begin{aligned}
&\frac{j(-q^2,q^{12})}{j(q^2,q^{12})j(q^4,q^{24})j(q^8,q^{24})} - \frac{j(-q^4,q^{12})}{j(q^4,q^{12})j(q^4,q^{24})j(q^8,q^{24})}\\
&\qquad = \frac{2q^2 j(q^2,q^{24})j(q^{18},q^{24})}{j(q^2,q^{12})j(q^4,q^{12})j(q^4,q^{24})j(q^8,q^{24})}
\end{aligned} \tag{16}$$

Using equation (11), we get

$$S^2(q)T(q^2) - T^2(q)S(q^2) = 2q^2 j(q^2,q^{24})j(q^{18},q^{24})j^{-1}(q^{12},q^{24})T(q)S(q)T(q^2)S(q^2) \tag{17}$$

which proves (i)

Proof of (ii).

Replacing q by q^{12}, x by $1/q^2$ and y by q^4 in (14), we have

$$j(-q^2,q^{12})j(q^4,q^{12})+j(q^2,q^{12})j(-q^4,q^{12})=2j(q^6,q^{24})j(q^{14},q^{24}) \tag{18}$$

Multiplying both side by $\frac{1}{j(q^2,q^{12})j(q^4,q^{12})j(q^4,q^{24})j(q^8,q^{24})}$, we get

$$\begin{aligned}\frac{j(-q^2,q^{12})}{j(q^2,q^{12})j(q^4,q^{12})j(q^4,q^{24})j(q^8,q^{24})}&+\frac{j(-q^4,q^{12})}{j(q^2,q^{12})j(q^4,q^{12})j(q^4,q^{24})j(q^8,q^{24})}\\ &=\frac{j(q^6,q^{24})j(q^{14},q^{24})}{j(q^2,q^{12})j(q^4,q^{12})j(q^4,q^{24})j(q^8,q^{24})}\end{aligned} \tag{19}$$

Again by equation (11), we get

$$S^2(q)T(q^2)+T^2(q)S(q^2)=j(q^6,q^{24})j(q^{14},q^{24})j^{-1}(q^{12},q^{24})S(q)T(q)S(q^2)T(q^2) \tag{20}$$

which proves (ii).

Now dividing the equation (17) by (20), we have

$$\frac{S^2(q)T(q^2)-T^2(q)S(q^2)}{S^2(q)T(q^2)+T^2(q)S(q^2)}=\frac{q^2j(q^2,q^{24})}{j(q^{14},q^{24})} \tag{21}$$

Writting the equation (21) in the form

$$H(q)\frac{S^2(q)T(q^2)-T^2(q)S(q^2)}{S^2(q)T(q^2)+T^2(q)S(q^2)}=\frac{P(q)}{P^2(q^2)} \tag{22}$$

where

$$H(q)=\frac{j^2(q^4,q^{24})(q^4,q^{20};q^{24})_\infty(q^{12};q^{12})_\infty}{q^2j^2(q^2,q^{24})(q^4;q^4)_\infty} \tag{23}$$

Using (12) and (13) we give modular equations.

4. Some Modular Equations

We prove the following modular equations:

$$(i)\quad H(q)\frac{v^2-u}{v^2+u}=\frac{v}{u^2} \tag{24}$$

$$(ii)\quad k\left(\frac{1-k/H(q)}{1+k/H(q)}\right)^2=\frac{1}{P^3(q)} \tag{25}$$

$$(iii)\quad \frac{H(q)}{v/u^2}-\frac{v/u^2}{H(q)}=\frac{\phi(q^2)\phi(q^6)}{q^2\chi^2(q^2)(\phi^2(q^2)-\phi(q^6))} \tag{26}$$

where $u=P(q^2)$, $v=P(q)$ and $k=\frac{P(q)}{P^2(q^2)}$

Proof of (i).

Multiplying and dividing by $T^2(q)T(q^2)$ in left-hand side of (22) and using definitions of u and v we get (i).

Proof of (ii).

From equation (22)

$$\frac{S^2(q)T(q^2) - T^2(q)S(q^2)}{S^2(q)T(q^2) + T^2(q)S(q^2)} = \frac{k}{H(q)}$$

$$k\left(\frac{1-k/H(q)}{1+k/H(q)}\right)^2 = k\left(\frac{1-\dfrac{S^2(q)T(q^2) - T^2(q)S(q^2)}{S^2(q)T(q^2) + T^2(q)S(q^2)}}{1+\dfrac{S^2(q)T(q^2) - T^2(q)S(q^2)}{S^2(q)T(q^2) + T^2(q)S(q^2)}}\right)^2$$

$$= \frac{P(q)}{P^2(q^2)}\left(\frac{2T^2(q)S(q^2)}{2S^2(q)T(q^2)}\right)^2$$

$$= \frac{P(q)}{P^2(q^2)}\frac{P^2(q^2)}{P^4(q)} = \frac{1}{P^3(q)}$$

Proof of (iii)

$$\frac{H(q)}{v/u^2} - \frac{v/u^2}{H(q)} = \frac{S^2(q)T(q^2) + T^2(q)S(q^2)}{S^2(q)T(q^2) - T^2(q)S(q^2)} - \frac{S^2(q)T(q^2) - T^2(q)S(q^2)}{S^2(q)T(q^2) + T^2(q)S(q^2)}$$

$$= \frac{j(q^{14},q^{24})}{q^2 j(q^2,q^{24})} - \frac{q^2 j(q^2,q^{24})}{j(q^{14},q^{24})}, \quad \text{From [18, Lemma 3.1]}$$

$$= \frac{\phi^2(q^2)+\phi(q^6)}{\phi^2(q^2)-\phi(q^6)} - \frac{\phi^2(q^2)-\phi(q^6)}{\phi^2(q^2)+\phi(q^6)}$$

$$= \frac{\phi(q^2)\phi(q^6)}{q^2\chi^2(q^2)(\phi^2(q^2)-\phi(q^6))}$$

5. Lambert Series for Ramanujan's Continued Fraction

We have proved the following Lambert series in [15]

$$(q^{12},q^{12})_\infty^2 S(q) = \sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{12n+10}} = \sum_{n=-\infty}^{\infty}\frac{q^{10n}}{1-q^{12n+1}} \tag{27}$$

$$(q^{12},q^{12})_\infty^2 S(q) = \sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{12n+5}} = \sum_{n=-\infty}^{\infty}\frac{q^{5n}}{1-q^{12n+2}} \tag{28}$$

$$(q^{12},q^{12})_\infty^2 T(q) = \sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{12n+8}} = \sum_{n=-\infty}^{\infty}\frac{q^{8n}}{1-q^{12n+2}} \tag{29}$$

$$\frac{(q^{12},q^{12})_\infty^2}{(q^6,q^{12})_\infty^2}\frac{S(q)}{T(q)} = \sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{12n+6}} = \sum_{n=-\infty}^{\infty}\frac{q^{6n}}{1-q^{12n+2}} \tag{30}$$

$$\frac{(q^{12},q^{12})_\infty^2}{(q^6,q^{12})_\infty^2}\frac{T(q)}{S(q)} = \sum_{n=-\infty}^{\infty}\frac{q^{4n}}{1-q^{12n+6}} = \sum_{n=-\infty}^{\infty}\frac{q^{6n}}{1-q^{12n+4}} \tag{31}$$

$$(q^{12},q^{12})_\infty^2 T(q) = \sum_{n=-\infty}^{\infty}\frac{q^{4n}}{1-q^{12n+4}} \tag{32}$$

$$(q^{12},q^{12})_\infty^2 \frac{S^2(q)}{T(q)} = \sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{12n+2}} \tag{33}$$

With the help of these identities we can prove the following relations.

$$(i)\quad \frac{S(q)}{T(q)} = \frac{\sum_{n=0}^{\infty} \frac{q^{12n^2+11n}(1-q^{24n+11})}{(1-q^{12n+10})(1-q^{12n+1})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+13n+1}(1-q^{24n+13})}{(1-q^{12n+2})(1-q^{12n+11})}}{\sum_{n=0}^{\infty} \frac{q^{12n^2+10n}(1-q^{24n+10})}{(1-q^{12n+2})(1-q^{12n+8})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+14n+2}(1-q^{24n+14})}{(1-q^{12n+4})(1-q^{12n+10})}}$$

$$= \frac{1}{1+}\,\frac{1-q^2}{1-q^3+}\,\frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)+}\,\frac{(q^{5/2}-q^{19/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)+}\cdots \tag{34}$$

$$(ii)\quad \frac{S(q)}{T(q)} = \frac{\sum_{n=0}^{\infty} \frac{q^{12n^2+11n}(1-q^{24n+11})}{(1-q^{12n+10})(1-q^{12n+1})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+13n+1}(1-q^{24n+13})}{(1-q^{12n+2})(1-q^{12n+11})}}{\sum_{n=0}^{\infty} \frac{q^{12n^2+8n}(1+q^{12n+4})}{(1-q^{12n+4})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+16n+4}(1+q^{12n+8})}{(1-q^{12n+8})}}$$

$$= \frac{1}{1+}\,\frac{1-q^2}{1-q^3+}\,\frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)+}\,\frac{(q^{5/2}-q^{19/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)+}\cdots \tag{35}$$

$$(iii)\quad \frac{S(q)}{T(q)} = \frac{\sum_{n=0}^{\infty} \frac{q^{12n^2+7n}(1-q^{24n+7})}{(1-q^{12n+2})(1-q^{12n+5})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+17n+5}(1-q^{24n+17})}{(1-q^{12n+7})(1-q^{12n+10})}}{\sum_{n=0}^{\infty} \frac{q^{12n^2+10n}(1-q^{24n+10})}{(1-q^{12n+2})(1-q^{12n+8})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+14n+2}(1-q^{24n+14})}{(1-q^{12n+4})(1-q^{12n+10})}}$$

$$= \frac{1}{1+}\,\frac{1-q^2}{1-q^3+}\,\frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)+}\,\frac{(q^{5/2}-q^{19/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)+}\cdots \tag{36}$$

$$(iv)\quad \frac{S(q)}{T(q)} = \frac{\sum_{n=0}^{\infty} \frac{q^{12n^2+7n}(1-q^{24n+7})}{(1-q^{12n+2})(1-q^{12n+5})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+17n+5}(1-q^{24n+17})}{(1-q^{12n+7})(1-q^{12n+10})}}{\sum_{n=0}^{\infty} \frac{q^{12n^2+8n}(1+q^{12n+4})}{(1-q^{12n+4})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+16n+4}(1-q^{12n+8})}{(1-q^{12n+8})}}$$

$$= \frac{1}{1+}\,\frac{1-q^2}{1-q^3+}\,\frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)+}\,\frac{(q^{5/2}-q^{19/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)+}\cdots \tag{37}$$

$$(v)\quad \frac{S^2(q)}{T^2(q)} = \frac{\sum_{n=0}^{\infty} \frac{q^{12n^2+4n}(1+q^{124n+2})}{(1-q^{12n+2})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+20n+8}(1+q^{124n+10})}{(1-q^{12n+10})}}{\sum_{n=0}^{\infty} \frac{q^{12n^2+10n}(1-q^{24n+10})}{(1-q^{12n+2})(1-q^{12n+8})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+14n+2}(1-q^{24n+14})}{(1-q^{12n+4})(1-q^{12n+10})}}$$

$$= \left\{\frac{1}{1+}\,\frac{1-q^2}{1-q^3+}\,\frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)+}\,\frac{(q^{5/2}-q^{19/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)+}\cdots\right\}^2 \tag{38}$$

$$(vi)\quad \frac{S^2(q)}{T^2(q)}=\frac{\sum_{n=0}^{\infty}\frac{q^{12n^2+4n}(1+q^{12n+2})}{(1-q^{12n+2})}-\sum_{n=0}^{\infty}\frac{q^{12n^2+20n+8}(1+q^{12n+10})}{(1-q^{12n+10})}}{\sum_{n=0}^{\infty}\frac{q^{12n^2+8n}(1+q^{12n+4})}{(1-q^{12n+4})}-\sum_{n=0}^{\infty}\frac{q^{12n^2+16n+4}(1+q^{12n+8})}{(1-q^{12n+8})}}$$

$$=\left\{\frac{1}{1+}\frac{1-q^2}{1-q^3+}\frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)+}\frac{(q^{5/2}-q^{19/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)+}\cdots\right\}^2 \tag{39}$$

$$(vii)\quad \frac{S^2(q)}{T^2(q)}=\frac{\sum_{n=0}^{\infty}\frac{q^{12n^2+8n}(1-q^{24n+8})}{(1-q^{12n+2})(1-q^{12n+6})}-\sum_{n=0}^{\infty}\frac{q^{12n^2+16n+4}(1-q^{24n+16})}{(1-q^{12n+10})(1-q^{12n+6})}}{\sum_{n=0}^{\infty}\frac{q^{12n^2+10n}(1-q^{24n+10})}{(1-q^{12n+4})(1-q^{12n+6})}-\sum_{n=0}^{\infty}\frac{q^{12n^2+14n+2}(1-q^{24n+14})}{(1-q^{12n+6})(1-q^{12n+8})}}$$

$$=\left\{\frac{1}{1+}\frac{1-q^2}{1-q^3+}\frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)+}\frac{(q^{5/2}-q^{19/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)+}\cdots\right\}^2 \tag{40}$$

Proof of (i)

Before proving (i) recall the Rogers-Fine identity [3, p.564].

$$\sum_{n=0}^{\infty}\frac{(a;q)_n t^n}{(b;q)_{n+1}}=\sum_{n=0}^{\infty}\frac{(a;q)_n(at/b;q)_n b^n t^n q^{n^2}(1-atq^{2n})}{(b;q)_{n+1}(t;q)_{n+1}} \tag{41}$$

Put $b=aq$ we have

$$\sum_{n=0}^{\infty}\frac{t^n}{(1-aq^n)}=\sum_{n=0}^{\infty}\frac{(at)^n q^{n^2}(1-atq^{2n})}{(1-aq^n)(1-tq^n)} \tag{42}$$

Replace q by q^k, put $t=q^j$ and $a=q^i$ we have

$$\sum_{n=0}^{\infty}\frac{q^{nj}}{(1-q^{kn+i})}=\sum_{n=0}^{\infty}\frac{q^{kn^2+n(i+j)}(1-q^{2kn+i+j})}{(1-q^{kn+i})(1-q^{kn+j})} \tag{43}$$

Now let $i=j$, we have

$$\sum_{n=0}^{\infty}\frac{q^{ni}}{(1-q^{kn+i})}=\sum_{n=0}^{\infty}\frac{q^{kn^2+2ni}(1+q^{kn+i})}{(1-q^{kn+i})} \tag{44}$$

Now dividing the equation (27) by (28), we have

$$\frac{S(q)}{T(q)}=\frac{\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{12n+10}}}{\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{12n+8}}}$$

$$=\frac{\sum_{n=0}^{\infty}\frac{q^n}{1-q^{12n+10}}+\sum_{n=-1}^{-\infty}\frac{q^n}{1-q^{12n+10}}}{\sum_{n=0}^{\infty}\frac{q^{2n}}{1-q^{12n+8}}+\sum_{n=-1}^{-\infty}\frac{q^{2n}}{1-q^{12n+8}}}$$

Replace $n\to-(n+1)$ in the second term of numerator and denominator we get

$$=\frac{\sum_{n=0}^{\infty}\frac{q^n}{1-q^{12n+10}}-\sum_{n=0}^{\infty}\frac{q^{11n+1}}{1-q^{12n+2}}}{\sum_{n=0}^{\infty}\frac{q^{2n}}{1-q^{12n+8}}-\sum_{n=0}^{\infty}\frac{q^{10n+2}}{1-q^{12n+4}}}$$

With the help of equation (43), we have

$$= \frac{\sum_{n=0}^{\infty} \frac{q^{12n^2+11n}(1-q^{24n+11})}{(1-q^{12n+10})(1-q^{12n+1})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+13n+1}(1-q^{24n+13})}{(1-q^{12n+2})(1-q^{12n+11})}}{\sum_{n=0}^{\infty} \frac{q^{12n^2+10n}(1-q^{24n+10})}{(1-q^{12n+2})(1-q^{12n+8})} - \sum_{n=0}^{\infty} \frac{q^{12n^2+14n+2}(1-q^{24n+14})}{(1-q^{12n+4})(1-q^{12n+10})}}$$

By equation (2) we have

$$= \frac{1}{1_+} \frac{1-q^2}{1-q^3}{}_+ \frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)}{}_+ \frac{(q^{5/2}-q^{19/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)}{}_+ \cdots$$

which proves (i). The proof of (ii)-(vii) are similar so are omitted here.

6. An important q-identity

$$\left[\frac{1}{1_+} \frac{1-q^2}{1-q^3}{}_+ \frac{(q^{5/2}-q^{7/2})(q^{1/2}-q^{11/2})}{(1-q^3)(q^6+1)}{}_+ \frac{(q^{5/2}-q^{11/2})(q^{1/2}-q^{23/2})}{(1-q^3)(q^{12}+1)}{}_+ \cdots\right]^2$$

$$= \frac{(q^4;q^4)_\infty}{(q^{12};q^{12})_\infty^3} \left[\sum_{n=0}^{\infty} q^{12n^2+4n} \frac{1+q^{12n+2}}{1-q^{12n+2}} - \sum_{n=0}^{\infty} q^{12n^2+20n+8} \frac{1+q^{12n+10}}{1-q^{12n+10}}\right]$$

Proof.

From [15, eqn. 2.4]

$$\sum_{n=0}^{\infty} \frac{q^{in}}{1-q^{12n+i}} = \frac{(q^{12};q^{12})_\infty (q^{2i};q^{12})_\infty (q^{12-2i};q^{12})_\infty}{(q^i;q^{12})_\infty^2 (q^{12-i};q^{12})_\infty^2} \tag{45}$$

Also from [7, p.58]

$$\sum_{n=-\infty}^{\infty} \frac{q^{kn}}{1-q^{jn+k}} = \sum_{n=0}^{\infty} q^{jn^2+2kn} \frac{1+q^{jn+k}}{1-q^{jn+k}} \tag{46}$$

for $j=12,\ k=2$ and $k=10$, we have

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{1-q^{12n+2}} = \sum_{n=0}^{\infty} q^{12n^2+4n} \frac{1+q^{12n+2}}{1-q^{12n+2}} \tag{47}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{10n}}{1-q^{12n+10}} = \sum_{n=0}^{\infty} q^{12n^2+20n} \frac{1+q^{12n+10}}{1-q^{12n+10}}$$

$$\sum_{n=0}^{\infty} \frac{q^{10n+8}}{1-q^{12n+10}} = \sum_{n=0}^{\infty} q^{12n^2+20n+8} \frac{1+q^{12n+10}}{1-q^{12n+10}} \tag{48}$$

Now

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{12n+2}} = \sum_{n=-\infty}^{-1} \frac{q^{2n}}{1-q^{12n+2}} + \sum_{n=0}^{\infty} \frac{q^{2n}}{1-q^{12n+2}} \tag{49}$$

Replace $n=-(n+1)$ in the first summation of right hand side, we have

$$\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{12n+2}} = \sum_{n=0}^{\infty} \frac{q^{2n}}{1-q^{12n+2}} - \sum_{n=0}^{\infty} \frac{q^{10n+8}}{1-q^{12n+10}} \tag{50}$$

Substracting eqn.(48) from eqn.(47), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{12n^2+4n}\frac{1+q^{12n+2}}{1-q^{12n+2}} - \sum_{n=0}^{\infty} q^{12n^2+20n+8}\frac{1+q^{12n+10}}{1-q^{12n+10}} \\
&= \sum_{n=0}^{\infty}\frac{q^{2n}}{1-q^{12n+2}} - \sum_{n=0}^{\infty}\frac{q^{10n+8}}{1-q^{12n+10}} \\
&= \sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{12n+2}} \\
&= \frac{(q^{12};q^{12})_\infty^2 (q^4;q^{12})_\infty (q^8;q^{12})_\infty}{(q^2;q^{12})_\infty^2 (q^{10};q^{12})_\infty^2}, \quad \text{By equation (45) for } i=2
\end{aligned}$$

Multiplying and dividing by $(q^4;q^4)_\infty$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} q^{12n^2+4n}\frac{1+q^{12n+2}}{1-q^{12n+2}} - \sum_{n=0}^{\infty} q^{12n^2+20n+8}\frac{1+q^{12n+10}}{1-q^{12n+10}} \\
&= \frac{(q^{12};q^{12})_\infty^3}{(q^4;q^4)_\infty}\left[\frac{(q^4;q^{12})_\infty^2 (q^8;q^{12})_\infty^2}{(q^2;q^{12})_\infty^2 (q^{10};q^{12})_\infty^2}\right]
\end{aligned}$$

which completes the proof.

Conclusion; Using method of Hickerson, we have given identities and modular form. His method is simpler and we think by his method modular equations and identities for other continued fractions can be obtained, which we propose to do later.

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