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Sufficient condition of the Hopf bifurcation

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ABSTRACT. In this paper we propose a new sufficient condition for the Hopf bifurcation of equilibrium is derived for nonlinear autonomous differential equations with delay in three dimension, by analysis the characteristic equation.

1 Introduction

In this paper, we propose the following sufficient condition for the local Hopf bifurcation for a differential equation system with delay the form:

$$\dot{x} = f(x(t-\tau)) \tag{1}$$

The initial condition for the above system is

$$x(\theta) = \varphi(\theta), \quad -\tau \leq \theta \leq 0,$$

with $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ where $\varphi_i \in C(i = 1,2,3)$, such that $\varphi_i(\theta) \geq 0$ $(-\tau \leq \theta \leq 0, i = 1,2,3)$. Here C denotes the Banach space $C([-\tau, 0], \mathbb{R})$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}$, equipped with the supremum norm. The non-negative cone of C is defined as $C^+ = C([-\tau, 0], \mathbb{R}^+)$, where $\mathbb{R}^+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$. we here assume that $f : \mathbb{R}^3 \to \mathbb{R}$ is a locally Lipschitz continuous function on $\mathbb{R}^3$. Suppose that the equation (1) has a equilibrium point P^*, and the characteristic equation of the equation (1) around the equilibrium point P^* takes the general form (see [5, 9, 13]):

$$\Delta(\lambda, \tau) = P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0. \tag{2}$$

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with

$$P(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C$$

and

$$Q(\lambda) = D\lambda^2 + E\lambda + F,$$

The main results are as follows:

Theorem 1 (Amine Bernoussi)**.** *Assume that the equilibrium P^* of system* (1) *is locally asymptotically stable for $\tau = 0$. If $c \geq 0$, $b < 0$ and $\bar{z} \geq -\frac{4c}{b}$. Then there exists a positive τ_0 such that, when $\tau \in [0, \tau_0)$ the steady state P^* is locally asymptotically stable, and a Hopf bifurcation occurs as τ passes through τ_0, where τ_0 is given by*

$$\tau_0 = \frac{1}{\omega_0} \arccos\Big(\frac{(A\omega_0^2 - C)(F - D\omega_0^2) + (\omega_0^3 - B\omega_0)E\omega_0}{(D\omega_0^2 - F)^2 + E^2\omega_0^2}\Big),$$

and ω_0 is the least simple positive root of equation (5)*, and c, b and $\bar{z}$ are defined in lemma 2 and equation* (5)*. further we have*

$$\frac{d\Re\lambda(\tau_0)}{d\tau} > 0,$$

The organization of this paper is as follows. In section 2, we offer basic results for the proof of theorem 1. In section 3, we establish the local asymptotic stability and the Hopf bifurcation of the equilibrium P^* and prove theorem 1. Finally a discussion and conclusion is offered in section 4.

2 Basic results

We offer some basic results for the proof of theorem 1. Now we return to the study of equation (2) with $\tau > 0$. Equation (2) has a purely imaginary root $i\omega$, with $\omega > 0$:

$$\Delta(i\omega, \tau) = 0$$

if and only if

$$-A\omega^2 + C = (D\omega^2 - F)\cos(\omega\tau) - E\omega\sin(\omega\tau) \tag{3}$$

$$\omega^3 - B\omega = (D\omega^2 - F)\sin(\omega\tau) + E\omega\cos(\omega\tau) \tag{4}$$

Squaring and adding the squares together, we obtain

$$\omega^6 + a\omega^4 + b\omega^2 + c = 0, \tag{5}$$

with $a = A^2 - D^2 - 2B,\ \ b = B^2 - 2AC - E^2 + 2DF,\ \ c = C^2 - F^2$, where A, B, C, D, E *and* F are given by (2). Letting $z = \omega^2$, equation (5) becomes the following cubic equation

$$h(z) = z^3 + az^2 + bz + c = 0, \tag{6}$$

Lemma 2. *[10] Define*

$$\Delta = a^2 - 3b, \tag{7}$$

(i) If $c < 0$, then equation (6) has at least one positive root.

(ii) If $c \geq 0$ and $\Delta \leq 0$, then equation (6) has no positive roots.

(iii) If $c \geq 0$ and $\Delta > 0$, then equation (6) has positive roots if and only if $\overline{z} := \frac{1}{3}(-a + \sqrt{\Delta}) > 0$ and $h(\overline{z}) \leq 0$.

From lemma 2, we have the following lemma.

Lemma 3. *Assume that the equilibrium P^* of system (1) is locally asymptotically stable for $\tau = 0$.*

(i) If one of the following:

(N1) $c \geq 0$, and $\Delta \leq 0$,

(N2) $c \geq 0$, $\Delta > 0$, and $\overline{z} \leq 0$,

(N3) $c \geq 0$, $\Delta > 0$, and $h(\overline{z}) > 0$,

is true, then all roots of equation (2) have negative real parts for all $\tau \geq 0$.

(ii) If $c < 0$, or $c \geq 0$, $\Delta > 0$, $\overline{z} > 0$, and $h(\overline{z}) \leq 0$, then all roots of equation (2) have negative real parts when $\tau \in [0, \tau_0]$,
where Δ and $\overline{z} > 0$ are defined in lemma 2

3 Sufficient condition of the Hopf bifurcation

In this section, we establish the condition sufficient of the Hopf bifurcation of the equilibrium P^*.

Theorem 4. *Assume that the equilibrium P^* of system (1) is locally asymptotically stable for $\tau = 0$.*
If $c \geq 0$, $b < 0$ and $\overline{z} \geq -\frac{4c}{b}$. Then there exists a positive τ_0 such that, when $\tau \in [0, \tau_0)$ the steady state P^* is locally asymptotically stable, and a Hopf bifurcation occurs as τ passes through τ_0, where τ_0 is given by

$$\tau_0 = \frac{1}{\omega_0} \arccos\Big(\frac{(A\omega_0^2 - C)(F - D\omega_0^2) + (\omega_0^3 - B\omega_0)E\omega_0}{(D\omega_0^2 - F)^2 + E^2\omega_0^2}\Big),$$

and ω_0 is the least simple positive root of equation (5), and c, b and $\overline{z}$ are defined in lemma 2 and equation (5). further we have

$$\frac{d\Re\lambda(\tau_0)}{d\tau} > 0,$$

Proof. Consider the equation

$$f(z) = z^2 + az + b + \frac{c}{\overline{z}} = 0, \tag{8}$$

the discriminant Δ_1 of the equation (8) is given by

$$\Delta_1 = a^2 - 4(b + \frac{c}{\overline{z}})$$

or by the hypotheses $c \geq 0$ and $\overline{z} \geq \frac{-4c}{b}$ then $\Delta_1 \geq 0$, therefore equation (8) admits two solutions

$$z_1 = \frac{-a - \sqrt{\Delta_1}}{2} < 0 \text{ and } z_2 = \frac{-a + \sqrt{\Delta_1}}{2} > 0$$

then we have

$$f(z) \leq 0 \; \forall z \in [z_1, z_2]$$

the hypotheses $\overline{z} \geq \frac{-4c}{b}$ and $b < 0$ implies that $0 < \overline{z} < z_2$ then $f(\overline{z}) < 0$. Therefore

$$\overline{z} f(\overline{z}) = h(\overline{z}) < 0.$$

On the other since from the hypotheses $b < 0$ then $\Delta > 0$ then the hypotheses (iii) of the Lemma 2 is satisfied. Hence the equation (6) has at least one positive solution, more Lemma 3 implies that all roots of equation (2) have negative real parts when $\tau \in [0, \tau_0)$ (see [2, 3, 7, 8]).

Now we will determine τ_0.

Without loss of generality, we assume that the equation (6) has three positive roots, denoted by $z_1,\ z_2$ *and* z_3, respectively. Then equation (5) has three positive roots, say

$$\omega_1 = \sqrt{z_1},\ \omega_2 = \sqrt{z_2},\ \omega_3 = \sqrt{z_3}$$

According to equations (3) and (4) we have

$$\tau_l^j = \frac{1}{\omega_l}\left[\arccos\Big(\frac{(A\omega_l^2 - C)(F - D\omega_l^2) + (\omega_l^3 - B\omega_l)E\omega_l}{(D\omega_l^2 - F)^2 + E^2\omega_l^2}\Big) + 2j\Pi\right], l = 1,2,3;\ j = 0,1...$$

Then $\pm i\omega_l$ is a pair of purely imaginary roots of equation (2) with $\tau = \tau_l^j,\ l = 1,2,3;\ j = 0,1...$ clearly,

$$\lim_{j\to\infty} \tau_l^j = \infty,\ l = 1,2,3.$$

Thus, we can define

$$\tau_0 = \tau_{l_0}^{j_0} = \min_{j=0,1,..,l=1,2,3}(\tau_l^j),\ \omega_0 = \omega_{l_0}.$$

Next we need to guarantee the transversality condition of the Hopf bifurcation theorem (see [1, 4]).

Clearly $\lambda(\tau) = u(\tau) + i\omega(\tau)$ is a root of equation (2) if and only if

$$u^3 - 3u\omega^2 + Au^2 - A\omega^2 + Bu + C = -e^{-u\tau}\Big(Du^2\cos(\omega\tau) - D\omega^2\cos(\omega\tau) \tag{9}$$

$$+Eu\cos(\omega\tau) + F\cos(\omega\tau) + 2Du\omega\sin(\omega\tau) + E\omega\sin(\omega\tau)\Big)$$

and

$$3u^2\omega - \omega^3 + 2Au\omega + B\omega = -e^{-u\tau}\Big(-Du^2\sin(\omega\tau) + D\omega^2\sin(\omega\tau) \tag{10}$$

$$-Eu\sin(\omega\tau) - F\sin(\omega\tau) + 2Du\omega\cos(\omega\tau) + E\omega\cos(\omega\tau)\Big)$$

Let $u(\tau)$ and $\omega(\tau)$ satisfying $u(\tau_0) = 0$, and $\omega(\tau_0) = \omega_0$. By differentiating the equations (9) and (10) with respect to τ and then set $\tau = \tau_0$. Doing this, we get

$$G_1\frac{du(\tau_0)}{d\tau} + G_2\frac{d\omega(\tau_0)}{d\tau} = h_1,$$

$$-G_2\frac{du(\tau_0)}{d\tau} + G_1\frac{d\omega(\tau_0)}{d\tau} = h_2,$$

where

$$G_1 = -3\omega_0^2 + B + (E + D\omega_0^2\tau_0 - F\tau_0)\cos(\omega_0\tau_0) + (2D\omega_0 - E\omega_0\tau_0)\sin(\omega_0\tau_0),$$

$$G_2 = -2A\omega_0 + (E + D\omega_0^2\tau_0 - F\tau_0)\sin(\omega_0\tau_0) + (-2D\omega_0 + E\omega_0\tau_0)\cos(\omega_0\tau_0),$$
$$h_1 = (-D\omega_0^3 + F\omega_0)\sin(\omega_0\tau_0) - E\omega_0^2\cos(\omega_0\tau_0),$$

and

$$h_2 = (-D\omega_0^3 + F\omega_0)\cos(\omega_0\tau_0) + E\omega_0^2\sin(\omega_0\tau_0).$$

Solving for $\frac{du(\tau_0)}{d\tau}$ we get

$$\frac{du(\tau_0)}{d\tau} = \frac{G_1h_1 - G_2h_2}{G_1^2 + G_2^2},$$

Therefore, we have

$$\frac{du(\tau_0)}{d\tau} = \frac{\omega_0^2 h'(\omega_0^2)}{G_1^2 + G_2^2}$$

Note that if $h(\overline{z}) < 0$, then $h'(\omega_0^2) \neq 0$, because $h(\pm\infty) = \pm\infty$ and $h(0) = c \geq 0$ (see [6]).
Thus, if $h'(\omega_0^2) \neq 0$ we have the transversality condition:

$$\frac{du(\tau_0)}{d\tau} \neq 0.$$

If $\frac{du(\tau_0)}{d\tau} < 0$ for $\tau < \tau_0$ and close to τ_0, then equation (2) has a root $\lambda(\tau) = u(\tau) + i\omega(\tau)$ satisfying $u(\tau) > 0$, which contradicts (ii) of Lemma 3. This completes the proof. □

Remark 5. *Our main result also valid for any differential equation of higher dimension three provided that the study of the characteristic equation of the equation reduces to study the equation* (2) *(see [11, 12]) and more generally all the problems in* $\mathbb{R}^n$ *has equation of the form:*

$$(x^3)^n + a(x^2)^n + bx^n + c = 0$$

4 Concluding

In this work, we presented a mathematical analysis for a nonlinear autonomous differential equation with delays in three dimension. The originality of this work is to have a new sufficient condition for the Hopf bifurcation of equilibria. On the other hand, the main results can be generalized to differential equation of dimension greater than three (see Remark 5).
The interesting results obtained in this paper can be applied to study the dynamical behaviors of systems existing in many fields such as virology, epidemiology, economics and ecology.

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