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Method for solving Lane-Emden type differential equations by Coupling of wavelets and Laplace transform

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ABSTRACT. The objective of this paper is to obtain the numerical solution of Lane-Emden type equations by coupling method of Laplace transform and Chebyshev wavelets. By using the properties of Laplace transform and Chebyshev wavelets, an operational matrix is derived to convert the Lane-Emden equations into a system of algebraic equations. The idea of product of operational matrix is also used to solve a numerical problem. The numerical results show that the proposed method extracts more accurate values as compared to the existing one.

1 Introduction

In this paper, we consider the Lane-Emden type equations of the forms

$$y''(t) + \frac{2}{t}y'(t) + f(t,y) = g(t),\ t \in [0,1], \tag{1}$$

subject to initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{2}$$

where a and b are constants, $f(t,y)$ is a continuous real valued function and $g(t) \in C[0,1]$. Lane-Emden type equations are arised in several phenomena of mathematical physics and astrophysics. This equation represents

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the thermionic currents, the thermal behavior of spherical cloud gas, the theory of stellar structure and isothermal gas sphere [1]-[3] etc. Many methods currently in use for handling the Lane-Emden type problems are based on either perturbation technique or series solution. A discussion of the formulation of these models and the physical system of solution can be seen in the references [4]-[6]. Many researchers are trying to find the better and efficient method for determining a solution of Lane-Emden type equations. The existing methods may be categorized in two groups: the numerical method and the analytical method. Analytical solutions are rarely possible, so often the authors are interested in developing this techniques for numerical solutions. Numerical solution of the equation (1) has been given by many authors using various numerical techniques such as Adomain decomposition [7]-[9], Homotopy perturbation [10], [11], Homotopy analysis [12], [13], Lagrangian [14], Sinc-collocation [15], rationalized Haar collocation [16], hybrid function [17], new second kind Chebyshev operational matrix [18], wavelets [19]-[30], Legendre operational matrix [31], spectral [32], power-series [33], variational iteration [34]-[36], new Galerkin operational matrix [37], an implicit series solution [38], new spectral solutions [39], pertubation technique [40], optimal homotopy method [41] etc.

The Laplace transform is a powerful tool for solving differential equations. Wavelets theory is an emerging area in mathematical research, have got considerable attention in dealing with many problems of dynamical systems. The basic idea of this method is that it reduces these problems to those of solving a system of algebraic equations, which reduces the computational efforts [4]. Recently in the article [42] Yin et al. obtained numerical solutions of Lane-Emden equations by coupling the Laplace transform and Legendre wavelets which is quite interesting. Motivated by this here we are going to discuss the numerical solution of differential equation (1) by composition of Chebyshev wavelets and Laplace transform followed by comparative study with existing reference. Rest of the discussion is summarized as follows: In section 2, we mentioned the short introduction of the Legendre wavelet and Chebyshev wavelet. In section 3, we include Laplace wavelets method. In the later section, we presented two numerical examples for comparative study with available similar technique in the literature. At the end, we give the concluding remarks.

2 Wavelets

In the current years, wavelets have found their way into many different fields of engineering and science. It constitute a family of functions constructed from dilation and translation of a single function, called mother wavelet. When the dilation parameter α and the translation parameter β vary continuously, we have the following family of continuous wavelets [43]

$$\psi_{\beta\alpha}(t) = \mid \alpha \mid^{-1/2} \psi\left(\frac{t-\beta}{\alpha}\right), \quad \alpha, \beta \in R, \quad \alpha \neq 0.$$

If we restrict the parameters α and β as discrete values as $\alpha = \alpha_0^{-l}$ and $\beta = q\beta_0\alpha_0^{-l}$, $\alpha_0 > 1$, $\beta_0 > 0$, where q and l are positive integers, we have the following family of discrete wavelets

$$\psi_{lq}(t) = \mid \alpha_0 \mid^{l/2} \psi(\alpha_0^l t - q\beta_0),$$

where $\psi_{lq}(x)$ form a wavelet basis for $L^2(R)$. In particular, when $\alpha_0 = 2$ and $\beta_0 = 1$, then $\psi_{\beta\alpha}(t)$ form an orthonormal basis [21]. Here, we discuss in briefly two types of wavelets namely Legendre and Chebyshev wavelets, which will be used in our later study.

2.1 Legendre Wavelets (LW)

Legendre wavelets $\psi_{qp}(t) = \psi(l, g, p, t)$ have four arguments: $q = 1, 2, 3, ..., 2^{l-1}$, l is any positive integer, p is the order of Legendre polynomials, and t is the normalized time. They are defined on the interval $[0, 1)$ as follows [42]:

$$\psi_{qp}(t) = \begin{cases} \sqrt{\frac{2p+1}{2}}\ 2^{l/2} Y_p(2^l t - 2q + 1), & for\ \frac{2q-2}{2^l} \leq t < \frac{2q}{2^l}, \\ 0, & elsewhere, \end{cases} \tag{3}$$

where $p = 0, 1, 2, ..., P - 1$. The coefficient $\sqrt{p + 1/2}$ is for orthonormality, the dilation parameter $a = 2^{-l}$ and the translation parameter $b = \hat{q}2^{-l}$. Here, $Y_p(x)$ are Legendre polynomials of order p which are orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-1, 1]$, and satisfy the following recursive formula:

$$Y_0(t) = 1, \quad Y_1(t) = t,$$
$$Y_{p+1}(t) = \frac{2p+1}{p+1} t Y_p(t) - \frac{p}{p+1} Y_{p-1}(t), \quad p = 1, 2, ...$$

2.2 Chebyshev Wavelets (CW)

Chebyshev wavelets $\psi_{qp} = \psi(l, p, q, t)$ is defined as following and the four arguments, $q = 1, 2, \ldots, 2^{l-1}$, l assume any positive integer, p is the degree of Chebyshev polynomials of the first kind and t denotes the time. The wavelets are defined as [43], [44]

$$\psi_{qp}(t) = \begin{cases} 2^{l/2} \widetilde{Z_p}(2^l t - 2q + 1), & \frac{q-1}{2^{l-1}} \leq t < \frac{q}{2^{l-1}}, \\ 0, & elsewhere, \end{cases} \tag{4}$$

where

$$\widetilde{Z_p} = \begin{cases} \frac{1}{\sqrt{\pi}}, & p = 0, \\ \sqrt{\frac{2}{\pi}} Z_p, & p > 0, \end{cases} \tag{5}$$

and $p = 0, 1, \ldots, P - 1$. In (4), $Z_p(x)$ are Chebyshev polynomials of the first kind with degree p. Based on the definition, Chebyshev polynomials are orthogonal with respect to the weight function $w(t) = (1 - t^2)^{-1/2}$ within the interval $[-1, 1]$. The recursive equations as:

$$Z_0(t) = 1,\ Z_1(t) = t,\ Z_{p+1}(t) = 2tZ_p(t) - Z_{p-1}(t),\ p = 1, 2, 3, ... \tag{6}$$

It needs to be noticed that in Chebyshev wavelets, in order to obtain the orthogonal wavelets, the weight functions have to be dilated and translated as:

$$w_q(t) = w(2^l t - 2q + 1) = (1 - (2^l t - 2q + 1)^2)^{-1/2}. \tag{7}$$

A function $f(t)$ defined over $[0,1]$, may be expressed in terms of the LW or CW as follows:

$$f(t) = \sum_{q=1}^{\infty} \sum_{p=0}^{\infty} d_{qp} \psi_{qp}(t), \tag{8}$$

where

$$d_{qp} = (f(t), \psi_{qp}(t))_{w_q} = \int_0^1 w_q(t) \psi_{qp}(t) f(t) dt, \tag{9}$$

and $\langle ., . \rangle$ denoting the inner product. If the series (8) is truncated, then we obtain

$$f(t) = \sum_{q=1}^{2^{l-1}} \sum_{p=0}^{P-1} d_{qp} \psi_{qp}(t) = D^T \psi(t), \tag{10}$$

where D and $\psi(t)$ are $2^{l-1}P \times 1$ matrices given by:

$$D = [d_{10}, \ldots, d_{1(P-1)}, d_{20}, \ldots, d_{2(P-1)}, \ldots, d_{2^{l-1}0}, \ldots, d_{2^{l-1}(P-1)}]^T, \tag{11}$$

$$\psi(t) = [\psi_{10}, \ldots, \psi_{1(P-1)}, \psi_{20} \ldots, \psi_{2(P-1)}, \ldots, \psi_{2^{l-1}0}, \ldots, \psi_{2^{l-1}(P-1)}]^T. \tag{12}$$

Now, we discuss the product of operational matrix [45], which will be used in the present study. Let

$$\psi(t)\psi(t)^T D = \widetilde{D}\psi(t), \tag{13}$$

where D is a known constant vector as given in equation (11) and $\widetilde{D}$ is $2^{l-1}P \times 2^{l-1}P$ product operational matrix. To get $\widetilde{D}$, let $\psi(t)$ be the function vector as defined in the relation (12), then

$$\psi(t)\psi(t)^T = \left[\psi_i(t)\psi_j(t)\right], \ 1 \le i,j \le 2^{l-1}P. \tag{14}$$

If $\psi_i(t)\psi_j(t)$ is approximated by the expression (10), then we have

$$\psi(t)\psi^T(t) \approx H_{ij}\psi(t), \tag{15}$$

with

$$H_{ij} = \left[h_{ij}^1, h_{ij}^2, ..., h_{ij}^{2^{l-1}P}\right], \tag{16}$$

and $h_{ij}^k, \quad k = 1,2,...,2^{l-1}P$ are obtained by

$$h_{ij}^k = \int_0^1 \psi_i(t)\psi_j(x)\psi_k(t)w_q(t)dt, \ 1 \le i,j \le 2^{l-1}P. \tag{17}$$

Now, if we put the above relations $(14) - (16)$ in the equation (13), the product of operation matrix $\widetilde{D} = [\widetilde{d_{ij}}]$ can be obtained by

$$\widetilde{d}_{ij} = \sum_{k=1}^{2^{l-1}P} d_k h_{i,j}^k, \quad 1 \le i,j \le 2^{l-1}P, \tag{18}$$

where $h_{i,j}^k$ are calculated by the formula (17). For example, if we take $l = 1, \ P = 5$, then in case of Legendre wavelets

$$\widetilde{D} = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 \\ d_2 & d_1 + \frac{2}{\sqrt{5}}d_3 & \frac{2}{\sqrt{5}}d_2 + 3\sqrt{\frac{3}{35}}d_4 & 3\sqrt{\frac{3}{35}}d_3 + \frac{4}{\sqrt{21}}d_5 & \frac{4}{\sqrt{21}}d_4 \\ d_3 & \frac{2}{\sqrt{5}}d_2 + 3\sqrt{\frac{3}{35}}d_4 & d_1 + \frac{2\sqrt{5}}{7}d_3 + \frac{6}{7}d_5 & 3\sqrt{\frac{3}{35}}d_2 + \frac{4}{3\sqrt{5}}d_4 & \frac{6}{7}d_3 + \frac{20\sqrt{5}}{77}d_5 \\ d_4 & 3\sqrt{\frac{3}{35}}d_3 + \frac{4}{\sqrt{21}}d_5 & 3\sqrt{\frac{3}{35}}d_2 + \frac{4}{3\sqrt{5}}d_4 & d_1 + \frac{4}{3\sqrt{5}}d_3 + \frac{6}{11}d_5 & \frac{4}{\sqrt{21}}d_2 + \frac{6}{11}d_4 \\ d_5 & \frac{4}{\sqrt{21}}d_4 & \frac{6}{7}d_3 + \frac{20\sqrt{5}}{77}d_5 & \frac{4}{\sqrt{21}}d_2 + \frac{6}{11}d_4 & d_1 + \frac{20\sqrt{5}}{77}d_3 + \frac{486}{1001}d_5 \end{bmatrix}.$$

and for Chebyshev wavelets

$$\widetilde{D} = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \sqrt{2}d_1 & \sqrt{2}d_2 & \sqrt{2}d_3 & \sqrt{2}d_4 & \sqrt{2}d_5 \\ \sqrt{2}d_2 & \sqrt{2}d_1 + d_3 & d_2 + d_4 & d_3 + d_5 & d_4 \\ \sqrt{2}d_3 & d_2 + d_4 & \sqrt{2}d_1 + d_5 & d_2 & d_3 \\ \sqrt{2}d_4 & d_3 + d_5 & d_2 & \sqrt{2}d_1 & d_2 \\ \sqrt{2}d_5 & d_4 & d_3 & d_2 & \sqrt{2}d_1 \end{bmatrix}.$$

It is noted that here we denote $[d_{10}, d_{11}, \ldots, d_{2^{l-1}(P-1)}]$ by $[d_1, d_2, \ldots, d_{2^{l-1}P}]$.

3 Laplace Wavelets Method

In this ensuing section, we describe Laplace wavelets method [42] for solving the Lane-Emden equations given in (1).

Multiply by t on both sides and then imposing the Laplace transform to the equation (1), we obtain

$$-s^2 L^{'}\{y\} - y(0) + L\{tf(t,y) - tg(t)\} = 0, \tag{19}$$

where L is Laplace transform operator and $L^{'}\{y\} = \frac{dL\{y\}}{ds}$. The expression (19) can be also rewritten as

$$L^{'}\{y\} = -s^{-2}y(0) + s^{-2}L\{tf(t,y) - tg(t)\}. \tag{20}$$

Now integrating the equation (20) from 0 to s with respect to s and then taking the inverse Laplace transform, we have

$$y = L^{-1}\left\{-\int_0^s s^{-2}y(0)ds\right\} + L^{-1}\left\{\int_0^s s^{-2}L\{tf(t,y) - tg(t)\}\,ds\right\}. \tag{21}$$

By making use of the initial conditions given in relation (2), we can write

$$y = a + L^{-1}\left\{\int_0^s s^{-2}L\{tf(t,y) - tg(t)\}\,ds\right\}. \tag{22}$$

For convenience, we can denote $L^{-1}\left\{\int_0^s s^{-2}L\{.\}\,ds\right\}$ by Δ. Thus, the expression (22) takes the form

$$y = \alpha + \Delta\{tf(t,y) - tg(t)\}. \tag{23}$$

Below, we mention the Legendre wavelets and Chebyshev wavelets operational matrices of operator Δ. For this purpose, first of all we include three following lemmas.

Lemma 3.1. *[42] Let $\psi(t)$ be the one-dimensional wavelets vector defined in the relation (12), then*

$$t\psi(t) = RSR^{-1}\psi(t), \tag{24}$$

where S is $2^{l-1}P \times 2^{l-1}P$ *matrix*

$$S = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ & ... & ... & ... & \\ & ... & ... & ... & \\ & ... & ... & ... & \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}_{2^{l-1}P\times 2^{l-1}P}, \tag{25}$$

and R is $2^{l-1}P \times 2^{l-1}P$ *wavelet matrix.*

For example, when $l = 1$, $P = 5$, then for Legendre wavelets

$$R_{5\times5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 2\sqrt{3} & 0 & 0 & 0 \\ \sqrt{5} & -6\sqrt{5} & 6\sqrt{5} & 0 & 0 \\ -\sqrt{7} & 12\sqrt{7} & -30\sqrt{7} & 20\sqrt{7} & 0 \\ 3 & -60 & 270 & -420 & 210 \end{bmatrix}, \tag{26}$$

and for Chebyshev wavelets

$$R_{5\times5} = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 \\ 2 & -16 & 16 & 0 & 0 \\ -2 & 36 & -96 & 64 & 0 \\ 2 & -64 & 320 & -512 & 256 \end{bmatrix}. \tag{27}$$

Lemma 3.2. *[42] Let* $\psi(t)$ *be the one-dimensional wavelets vector defined in the equation* (12)*, then*

$$\Delta\{\psi(t)\} \approx U\psi(t), \tag{28}$$

where $U = RTSR^{-1}$*. Also*

$$\Delta\{\phi(t)\} = TS\phi(t), \tag{29}$$

where matrix S is given in expression (25) *and matrix T is defined as*

$$T = \begin{bmatrix} -\frac{1}{2} & 0 & ... & 0 & 0 \\ 0 & -\frac{1}{6} & ... & 0 & 0 \\ . & . & ... & . & . \\ . & . & ... & . & . \\ . & . & ... & . & . \\ 0 & 0 & ... & -\frac{1}{(P-1)P} & 0 \\ 0 & 0 & ... & 0 & -\frac{1}{P(P+1)} \end{bmatrix}_{P\times P}. \tag{30}$$

Lemma 3.3. *[42] Let $\psi(t)$ be the one-dimensional wavelets vector defined in the equation (12), then*

$$\Delta\left\{t\psi(t)\right\} = \widetilde{U}\psi(t), \tag{31}$$

where $\widetilde{U} = RSTSR^{-1}$.

In order to use Laplace Chebyshev wavelets method, we assume that

$$y(t) = D^T\psi(t), \qquad f(t,y) = F^T\psi(t), \qquad tg(t) = G^T\psi(t). \tag{32}$$

Substituting (32) into (23) and using the Lemma 3.3, we get

$$\begin{aligned} D^T\psi(t) &= O^T\psi(t) + \Delta\left\{tF^T\psi(t) - G^T\psi(t)\right\} \\ &= O^T\psi(t) + F^T\widetilde{U}\psi(t) - G^T U\psi(t), \end{aligned} \tag{33}$$

and thus, we obtain

$$D^T = O^T + F^T\widetilde{U} - G^T U. \tag{34}$$

4 Numerical Examples

Example 4.1. Consider the Lane-Emden equation given in [42]:

$$y^{''}(t) + \frac{2}{t}y^{'}(t) + y(t) = 6 + 12t + t^2 + t^3, \quad 0 \le t \le 1, \tag{35}$$

with initial conditions

$$y(0) = 0, \quad y^{'}(0) = 0. \tag{36}$$

The exact solution is $y(t) = t^2 + t^3$. For $l = 1$, $P = 5$, the absolute errors for Example 4.1 by Laplace Chebyshev wavelets and Laplace Legendre wavelets are plotted in the Figures 1 and 2, respectively. Here we observe that former method shows better accuracy as compared to later one.

Example 4.2. Consider the second-order singular differential equations of Lane-Emden type [20]:

$$y^{''}(t) + \frac{2}{t}y^{'}(t) - 2(2t^2+3)y(t) = 0, \quad 0 < t \le 1, \tag{37}$$

with initial conditions

$$y(0) = 1, \quad y^{'}(0) = 0, \tag{38}$$

with the exact solution $y(t) = e^{t^2}$. In this example, we also use the idea of product operational matrix already discussed in earlier section. For $l = 1$ and $P = 5$, the absolute errors for the Example 4.2 by Laplace Chebyshev wavelets and Laplace Legendre wavelets are plotted in the Figures 3 and 4, respectively. Obviously these figures confirm that Laplace Chebyshev wavelets extracts more accurate values as compared to Laplace Legendre wavelets.

5 Conclusion

The aim of present work is to discuss an efficient and more accurate method for solving the Lane-Emden equations as singular initial value problems. The properties of the Chebyshev wavelets along with Laplace transform are used to reduce the Lane-Emden type equations into a system of algebraic equations. Two examples are included to demonstrate the validity and applicability of the considered technique. Furthermore, numerical results justify that Laplace Chebyshev wavelets gives better results as compared to Laplace Legendre wavelets. This idea may be extended for other types of wavelets also.

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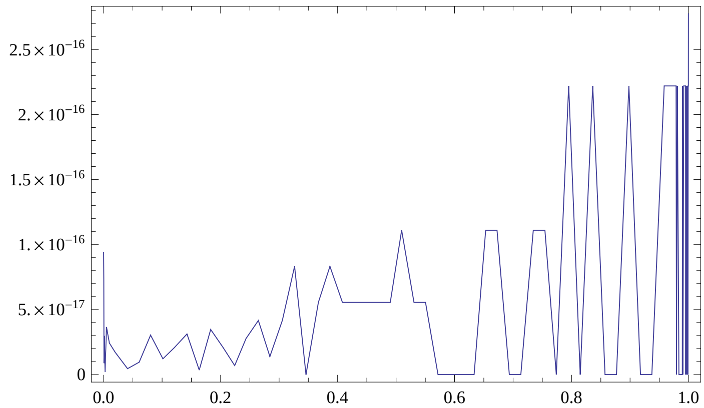

Figure 1: Absolute error by Laplace Chebyshev wavelets for Example 5.1.

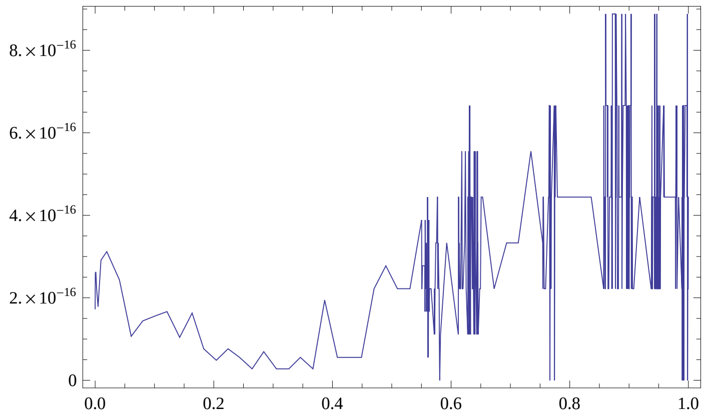

Figure 2: Absolute error by Laplace Legendre wavelets for Example 5.1.

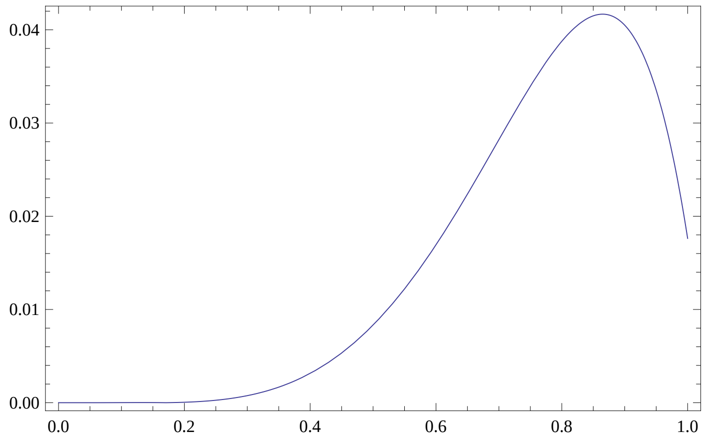

Figure 3: Absolute error by Laplace Chebyshev wavelets for Example 5.2.

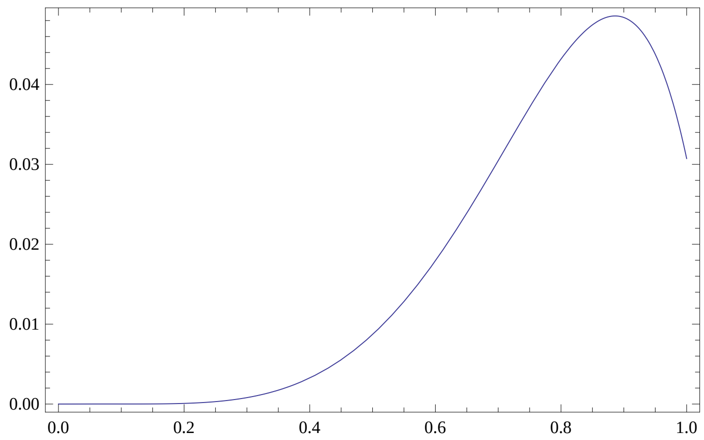

Figure 4: Absolute error by Laplace Legendre wavelets for Example 5.2.