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A study of generalized continuous functions

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ABSTRACT. In the paper, Weak Continuity Forms, Graph Conditions and Applications, the concept of u-continuous functions are introduced and presented several applications of such functions. In the present study, by generalizing the concept of u-continuity using the notion of an α-set, introduced by O. Njastad , three classes of functions are introduced and studied.The concepts introduced here are strongly u-continuous functions, αu-continuous functions and semi-αu- continuous functions. A function $g : X \to Y$ is αu-continuous (strongly u-continuous, semi-αu-continuous) at $x \in X$, if for each α-set (α-set, open set) W which contains a closed neighborhood of $g(x)$, there exists an α-set (open set, α-set) Vwhich contains a closed neighborhood of x and satisfies condition $g(clV) \subseteq clW$. If g is αu-continuous (strongly u-continuous, semi-αu-continuous) at each $x \in X$, we say $g : X \to Y$ is αu-continuous (strongly u-continuous, semi-αu-continuous) on X.

1 Introduction

Continuous functions and their generalizations have been effective investigation of topological properties. Some of the notions used to obtain such generalized concepts of continuity are θ-closure operator[17], u-closure operator [5, 8], regularly open sets, regularly closed sets, semi-open sets of Levine [10] , and α-sets of Njastad [11]. Very often combinations of the above concepts are used towards this end. In [1, 2], some of the above mentioned concepts were used to introduce several classes of functions which were effectively used in [12]. There, some of these generalized forms of continuity were used to introduce and investigate four classes of spaces, each of which

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was characterized by characteristic functions of some of such functions [12]. In [9], using θ-closure and u-closure operators, several weak continuity forms and graph conditions were introduced. These operators were then used to give alternative proofs of some classical results and/or new characterizations of some well-known classes of spaces. They were used also to improve certain existing results. The $u - closure$ operator was utilized to isolate a "second category type" property of topological spaces. In [9] an extensive list of articles in this category is given.

In the present article we focus on the concept of u-continuity introduced in [9] and generalize this concept using α-sets. Thus three classes of functions are introduced, namely, strongly u-continuous functions, αu-continuous functions and semi-αu-continuous functions. The class of u-continuous functions contains the class of strongly u-continuous functions and is contained in the class of semi-αu-continuous functions. The class of αu-continuous functions contains the class of strongly u-continuous functions and is contained in the class of semi-αu-continuous functions. These concepts are investigated and several characterizations are established for each of them.

The article is divided into three sections: section 1 gives the preliminary concepts and results used in the sequal, section 2 gives characterizations of each of these classes of spaces and section 3 presents examples.

2 Preliminaries

Let X be a space, $A \subset X$, $x \in X$. Let Ω be a filterbase on X; $\Sigma(A)$ represents the collection of open sets containing A, $\Gamma(A)$ is the collection of closed sets containing an open set containing A and let $\Lambda(A) = \bigcup_{\Gamma(A)} \Sigma(V)$ (if $A = x$ the notation $\Sigma(x)$, $\Gamma(x)$, or $\Lambda(x)$ will be used.) The closure of A is denoted by clA and the adherence of Ω is denoted by $ad\Omega$ and $ad\Omega = \cap_{\Omega} clF$. The concept of θ-closure of a set was introduced by Veličko [17] . If $A \subseteq X$, the *θ-closure of A*, denoted as $cl_\theta(A) = \{x \in X : A \cap clU \neq \emptyset\}$ for each $U \in \Sigma(x)$. A set A is *θ-closed* if $A = cl_\theta A$. In [8] the concept of u-closure of set was introduced by Joseph and used to study, among others, compact spaces as well as Urysohn-closed spaces. Let $A \subseteq X$. The *u-closure* of A, denoted as $cl_u(A)$, = $\{x : clV \cap A \neq \emptyset, V \in \Lambda(x)\}$. This concept was further studied in [4]. A set A is u-closed if $A = cl_u A$. A function $g : X \to Y$ is *u-continuous at* $x \in X$ [7] if for each $W \in \Lambda(g(x))$ there exists a $V \in \Lambda(x)$ such that $g(clV) \subset clW$ and g is *u-continuous* On X if g is u-continuous at each $x \in X$. A filterbase Ω on a space X is said to *u-converge to* $x \in X$[9] ($\Omega \to_u x$) if for each $W \in \Lambda(x)$ some $F \in \Omega$ satisfies $F \subset clW$. A net $\{x_\eta\}$ in X *u-converges to* $x \in$ X [7] ($\{x_\eta\} \to_u x$) if $\{x_\eta\}$ is ultimately in clW for each $W \in \Lambda(x)$. [6] For a filter base Ω on X, u-adherence of Ω is denoted by $ad_u\Omega$ and is defined by $\bigcap_\Omega cl_u F$. The concept of α-set is defined by Njastad in 1965 [11]. Sets for which $A \subseteq int(cl(intA))$ are α-sets, where $intA$ represents the interior of A. One can easily see every open set is an α-set, but an α-set need not be open as is demonstrated in [11]. A is called regularly-closed (regularly-open) if $cl(int(A)) = A$ ($int(clA) = A$). Every regularly closed set is semi-open and the closure of a semi-open set is regularly closed, where a set is semi-open [8] if there exists an open set U such that $U \subseteq A \subseteq clU$. Complement of a semi-open set is semi-closed.

Let $\{x_\lambda\}$ be a net in X with directed set D. Then the filter generated by the filter base $\mathcal{F}$ consisting of the collection of sets $\{B_{\lambda_0} = \{x_\lambda; \lambda \geq \lambda_0\}\}$ is called the filter generated by $\{x_\lambda\}$. is called the derived filter of λ. Note that a filter $\mathcal{F}$ converges to x if and only of the net associated with $\mathcal{F}$ converges to x. Furthermore; Given a topological space X and a subset A of X, a point x is in clA if and only if there is a net (equivalently, a filter) in A converging to x [16].

§2. **Main Results: Three Generalizations of u-Continuity**

In this section we introduce and characterize three generalizations of u-continuity. They are αu-continuous functions, semi-αu-continuous functions and strongly u-continuous functions. They will be introduced and characterized in sections 2.1, 2.2 and 2.3 respectively. The class of strongly u-continuous functions is contained in the class of u-continuous functions as well as in the class of αu-continuous functions; the class of semi-αu-continuous functions contains the classes of αu-continuous functions and u-continuous functions. All examples distinguishing these classes of spaces are given in section 3.

2.1 αu-Continuous Functions

Definition 2.1.1. A function $g : X \to Y$ is αu continuous at $x \in X$ if for each α-set W which contains a closed neighborhood of $g(x)$ there exists an α-set V which contains a closed neighborhood of x which satisfies $g(clV) \subset clW$ and g is αu-continuous if g is αu-continuous at each $x \in X$.

Definition 2.1.2.

(a) A filter base Ω on a space X αu-converges to $x \in X$ ($\Omega \to_{\alpha u} x$) if for each α-set W which contains a closed neighborhood of x, there exists some $F \in \Omega$ satisfies $F \subset clW$.

(b) A net $\{x_\eta\}$ in X αu converges to $x \in X$ ($\{x_\eta\} \to_{\alpha u} x$) if $\{x_\eta\}$ is ultimately in clW for each α-set W which contains a closed neighborhood of x.

In [9] it was established that the collection of α-sets of a topological space (X, τ) form a topology on X and it is the finest topology which has the same collection of semi-open sets as X. Moreover, If τ_α represents the topology generated by the α-sets of (X, τ), then $\tau_\alpha = \{U - N | U \in \tau$ and N is nowhere dense in $X\}$. Also it is clear that $\tau \subseteq \tau_\alpha$.

Definition 2.1.3. A point $x \in X$ is in α closure of A denoted as $cl_\alpha A$ if $V \cap A \neq \emptyset$ for every α-set V containing x. i.e. $cl_\alpha A = \{x \in X : V \cap A \neq \emptyset, V \in \tau_\alpha, x \in V\}$. A is called α-closed if $cl_\alpha A = A$.

Definitions 2.1.4.

(a) A point x is in the αu-closure of a set A if for every α-set U which contains a closed neighborhood of x, $A \cap cl_\alpha U \neq \emptyset$. αu-closure of a set A is denoted as $cl_{\alpha u}(A)$and defined as $cl_{\alpha u}(A) = \cap_{\Gamma(A)} cl_\alpha V$. and A is αu-closed if $A = cl_{\alpha u}(A)$. Since every open set is an α set, the αu closure of a set is contained in the u-closure of the set.

(b) Let Ω be a filter base on X. A point x is an αu- adherent point of a filter base Ω, denoted as $ad_{\alpha u}\Omega$ if x belongs to the αu-closure of every F in Ω. αu-adherence of Ω is denoted by $ad_{\alpha u}\Omega$ and is defined by $\bigcap_\Omega cl_{\alpha u} F$.

Definition 2.1.5. The $\alpha\theta$-closure of a set A is denoted by $cl_{\alpha\theta} A$ and it is defined as the intersection of all α-closed neighborhoods of A, or equivalently, α closure of all open sets containing A.

The following Lemma, which relates αu-closure with $\alpha\theta$-closure of an open set, is used in the sequel.

Lemma 2.1.6. $cl_{\alpha u}(V) = cl_{\alpha\theta}(clV)$, for every open set $V \subset X$.

Proof. Let $x \in cl_{\alpha u}(V)$, and suppose that $x \notin cl_{\alpha\theta}(clV) = \bigcap_{\Sigma(clV)} cl_{\alpha}U$, therefore $x \notin \bigcap_{\Sigma clV} cl_{\alpha}U$, so there exists at least one open set U containing clV such that $x \notin cl_{\alpha}U$ which contradicts our assumption $x \in cl_{\alpha u}(V)$.

The converse will follow easily since $cl_{\alpha\theta}(clV) \subseteq cl_{\alpha u}(V)$, for each open set V. □

Theorem 2.1.7. A net $\{x_{\lambda}\}$, αu-converges to $x \in X$, denoted as, $\{x_{\lambda}\} \rightarrow_{\alpha u} x \in X$, if and only if the derived filter $\mathcal{F}$ does.(i.e. $\mathcal{F} \rightarrow_{\alpha u} x$)

Proof. $\{x_{\lambda}\} \rightarrow_{\alpha u} x$, So $\{x_{\lambda}\}$ is ultimately in clW for each α-set W containing a closed neighborhood of x. Thus clW belongs to the derived filter $\mathcal{F}$. Therefore $\mathcal{F} \rightarrow_{\alpha u} x$.

Conversely; If each $\mathcal{F} \rightarrow_{\alpha u} x$, then each α-set W containing a closed neighborhood of x, there exists $F \in \mathcal{F}$ such that $F \subseteq clW$, so clW belongs to F so $\{x_{\lambda}\}$ is ultimately in clW, so $\{x_{\lambda}\} \rightarrow_{\alpha u} x$. □

Theorem 2.1.8. Given a topological space X and a subset A of X, a point x is in $cl_{\alpha u}(A)$ if and only if there is a net $\{x_{\eta}\}$(equivalently, a filter Ω) in A such that $\{x_{\eta}\} \rightarrow_{\alpha u} x$. ($\Omega \rightarrow_{\alpha u} x$)

Proof. If $x \in cl_{\alpha u}(A)$; then every $\alpha-$set U which contains a closed neighborhood of x meets A; i.e. $clU \cap A \neq \emptyset$. Let $x_u \in clU \cap A$ and $\mathcal{U}_{x_u}$ be the collection of α-sets containing a closed neighborhood of x. $\mathcal{U}_{x_u}$, is a directed set with set inclusion. Let $x_u \in U \cap A$. Then $\{x_u\}$ is a net and $\{x_u\} \subseteq A$ and it αu- converges to x.

Conversely, if $\{x_{\eta}\} \rightarrow_{\alpha u} x$; then $\{x_{\eta}\}$ is ultimately in clW for each $\alpha-$set W which contains a closed neighborhood of x. Therefore, each $\alpha-$set U which contains a closed neighborhood of x meets A, hence x is in $cl_{\alpha u}A$.□

Theorem 2.1.9. The following statements are equivalent for spaces X and Y and $g : X \rightarrow Y$;

(1) The function g is αu-continuous.

(2) For each $x \in X$ each filterbase Ω on X satisfying $\Omega \rightarrow_{\alpha u} x$ also satisfies $g(\Omega) \rightarrow_{\alpha u} g(x)$.

(3) For each $x \in X$ each net $\{x_{\eta}\}$ on X satisfying $\{\eta\} \rightarrow_{\alpha u} x$ also satisfies $g \circ \{\eta\} \rightarrow_{\alpha u} g(x)$.

(4) Each filterbase Ω on X satisfies $g(ad_{\alpha u}\Omega) \subset ad_{\alpha u}g(\Omega)$.

(5) Each $A \subset X$ satisfies $g(cl_{\alpha u}A) \subset cl_{\alpha u}g(A)$.

(6) Each $B \subset Y$ satisfies $cl_{\alpha u}(g^{-1}(B)) \subset g^{-1}(cl_{\alpha u}(B))$.

(7) Each filterbase Ω on $g(X)$ satisfies $ad_{\alpha u}g^{-1}(\Omega) \subset g^{-1}(ad_{\alpha u}\Omega)$.

(8) Each $B \subset Y$ satisfies $cl_{\alpha u}[g^{-1}(int(cl_{\theta}(B)))] \subset g^{-1}(cl_{\alpha u}(B))$.

(9) Each open $W \subset Y$ satisfies $cl_{\alpha u}[g^{-1}(int(cl(W)))] \subset g^{-1}(cl_{\alpha u}(W))$.

(10) Each regular-closed $R \subset Y$ satisfies $cl_{\alpha u}[g^{-1}(int(R))] \subset g^{-1}(cl_{\alpha\theta}(R))$.

(11) Each open $W \subset Y$ satisfies $cl_{\alpha u}(g^{-1}(W)) \subset g^{-1}(cl_{\alpha u}(W))$.

Proof.

(1) $\Rightarrow$ (2) Let g be αu-continuous. Then, for every α-set W' which contains a closed neighborhood of $g(x)$, there exists an α-set V containing a closed neighborhood of x, such that $g(clV) \subset clW'$. Since V is an α-set containing closed neighborhood of x and $\Omega \rightarrow_{\alpha u} x$, there exists $F \in \Omega$ such that $F \subset clV$, so $g(F) \subset g(clV) \subset clW'$, $F \in \Omega$ and $g(F) \in g(\Omega)$,. Hence $g(\Omega) \rightarrow_{\alpha u} g(x)$.

(2) $\Rightarrow$ (3). Suppose that for each $x \in X$, each filter base Ω on X satisfying $\Omega \rightarrow_{\alpha u} x$ also satisfies $g(\Omega) \rightarrow_{\alpha u} g(x)$. Suppose that for $x \in X$ and a net $\{x_\eta\}$ on X satisfies $\{x_\eta\} \rightarrow_{\alpha u} x$. We want to show that $g \circ \{\eta\} \rightarrow_{\alpha u} g(x)$. By $\{x_\eta\} \rightarrow_{\alpha u} x, \{x_\eta\} \in clW_1$, ultimately, for each α-set W_1 which contains a closed neighborhood of x. Since g is αu-continuous, for each α-set W$'$ which contains a closed neighborhood of $g(x)$ there exists an α-set W_2 containing a closed neighborhood of x such that $g(clW_2) \subset clW'$. Now consider $V = W_1 \cap W_2$, since both W_1 and W_2 are α-sets, V is an α -set, being intersection of two α-sets, and since both of them contain a closed neighborhood of x, V is an α-set containing a closed neighborhood of x such that $g(clV) \subset clW'$. Since we have $\{x_\eta\} \subseteq clW$, ultimately, for each α-set W which contains a closed neighborhood of x, thus $\{g(\{x_\eta\})\} \in g(clW) \subset clW'$, ultimately .

(3) $\Rightarrow$ (4) Suppose that for each $x \in X$ each net $\{x_\eta\}$ on X satisfying $\{x_\eta\} \rightarrow_{\alpha u} x$ also satisfies $\{g(\{x_\eta\})\} \rightarrow_{\alpha u} g(x)$. To show that each filter base Ω on X satisfies $g(ad_{\alpha u}\Omega) \subset ad_{\alpha u}g(\Omega)$, let $x \in g(ad_{\alpha u}\Omega)$ then there exists $y \in ad_{\alpha u}\Omega$ such that $x = g(y)$. Since $y \in ad_{\alpha u}\Omega$, for each α-set U containing a closed neighborhood of y, $clU \cap F \neq \emptyset$ for every $F \in \Omega$. So $y \in cl_{\alpha u}F$, for each $F \in \Omega$. By using Theorem 2.1.8., there exists a net $\{x_\eta\}$ in $cl_{\alpha u}F$ converging to y. Since $\eta \rightarrow_{\alpha u} y$, in view of (3), $g \circ \eta \rightarrow_{\alpha u} g(y) = x$, so $g(y) \in cl_{\alpha u}g(F)$ for each $F \in \Omega$ and so $x \in ad_{\alpha u}g(\Omega)$.

(4) $\Rightarrow$ (5) Assume (4). Let $A \subseteq X$ and Ω be the filter $\mathcal{F} = \{F \subseteq X; A \subseteq F\}$. $g(ad_{\alpha u}\Omega) \subseteq ad_{\alpha u}g(\Omega)$ Therefore $g(cl_{\alpha u}A) \subseteq cl_{\alpha u}g(A)$.

(5) $\Rightarrow$ (6). Let $B \subset Y$, so $g^{-1}(B) \subset X$, In view of (5), we have $g(cl_{\alpha u}g^{-1}(B)) \subset cl_{\alpha u}g(g^{-1}(B)) \Rightarrow g(cl_{\alpha u}g^{-1}(B)) \subset cl_{\alpha u}B$. Hence $g^{-1}(g(cl_{\alpha u}g^{-1}(B))) \subset g^{-1}(cl_{\alpha u}B)$. Therefore $cl_{\alpha u}(g^{-1}(B)) \subset g^{-1}(cl_{\alpha u}(B))$.

(6) $\Rightarrow$ (7). Suppose (6) holds and we want to show that each filter base Ω on $g(X)$ satisfies $ad_{\alpha u}g^{-1}(\Omega) \subset g^{-1}(ad_{\alpha u}\Omega)$. Let $x \in ad_{\alpha u}g^{-1}(\Omega)$ and suppose $x \notin g^{-1}(ad_{\alpha u}\Omega)$. So $g(x) \notin ad_{\alpha u}\Omega$. Therefore, by definition, there exists at least one $F \in \Omega$ such that $g(x) \notin cl_{\alpha u}F$. In view of (6), $x \notin g^{-1}(cl_{\alpha u}F)$. Hence $x \notin cl_{\alpha u}g^{-1}(F)$. Therefore, we have $x \notin ad_{\alpha u}g^{-1}(\Omega)$ which is a contradiction.

(7) $\Rightarrow$ (8). Suppose that each filterbase Ω on $g(X)$ satisfies $ad_{\alpha u}g^{-1}(\Omega) \subset g^{-1}(ad_{\alpha u}\Omega)$. Let $B \subset Y$, satisfy $g(X) \cap int(cl_\theta(B)) \neq \emptyset$. Then in view of (6), and the fact that $cl_\theta(B)$ is closed, and in view of Lemma 2.1.6, we have, $cl_{\alpha u}[g^{-1}(int(cl_\theta(B)))] \subset g^{-1}(cl_{\alpha u}(int(cl_\theta(B)))) = g^{-1}(cl_{\alpha\theta}(clint(cl_\theta(B)))) \subset g^{-1}(cl_{\alpha\theta}(cl_\theta(B))) \subset g^{-1}(cl_{\alpha u}(B))$.

(8) $\Rightarrow$ (9). Since $clW \subseteq cl_\theta(W)$, in viewing 8, we have
$cl_{\alpha u}[g^{-1}(int(cl(W)))] \subset cl_{\alpha u}[g^{-1}(int(cl_\theta(W)))] \subset g^{-1}(cl_{\alpha u}(W))$.

(9) $\Rightarrow$ (10) For each regularly-closed $R \subset Y$, by using Lemma 2.1.6, part6 and part 9 we have; $cl_{\alpha u}[g^{-1}(int(R))] \subset$

$g^{-1}(cl_{\alpha u}(int(R))) = g^{-1}(cl_{\alpha\theta}cl(int(R))) = g^{-1}(cl_{\alpha\theta}(R))$.

(10) $\Rightarrow$ (11) Let $W \subset Y$ be open. Then clW is regular-closed in Y, consequently $cl_{\alpha u}(g^{-1}(W)) \subset cl_{\alpha u}(g^{-1}(cl(intW))) \subset g^{-1}(cl_{\alpha\theta}(\overline{clW})) = g^{-1}(cl_{\alpha u}(W))$. So (11) holds.

(11) $\Rightarrow$ (1) Let $x \in X$ and let W be an α-set containing a closed neighborhood U of g(x). Then $g(x) \notin cl_{\alpha u}(g(X - g^{-1}(cl_{\alpha u}U)))$. Therefore $g(x) \notin g(cl_{\alpha u}(X - g^{-1}(clU)))$. Hence $x \notin cl_{\alpha u}(X - g^{-1}(clW))$. So, there exists an α-set V containing a closed neighborhood of x, such that $clV \cap (X - g^{-1}(clU)) = \emptyset$. Therefore, $clV \cap (X - g^{-1}(clW)) = \emptyset$ That is, $clV \subseteq g^{-1}(clW)$, therefore $g(clV) \subseteq clW$ and hence (1) holds. □

Corollary 2.1.10. A function $f : X \to Y$ is αu-continuous if and only if inverse image of every αu-closed set is αu-closed.

Proof. In view of Theorem 2.1.9 (6) for $B \subset Y, g^{-1}(B) \subseteq cl_{\alpha u}(g^{-1}(B)) \subset g^{-1}(cl_{\alpha u}(B)) = g^{-1}(B)$, since B is αu-closed. So, $g^{-1}(B) = cl_{\alpha u}(g^{-1}(B))$.

2.2. Semi-αu-Continuous Functions

Definition 2.2.1. A function $g : X \to Y$ is *semi-αu-continuous at $x \in X$* if for each open set W which contains a closed neighborhood of $g(x)$ there exists an α-set V which contains a closed neighborhood of x which satisfies $g(clV) \subset clW$ and g is semi αu-continuous if g is semi αu-continuous at each $x \in X$.

Clearly every u-continuous and αu-continuous is semi-αu-continuous.

Theorem 2.2.2. The following statements are equivalent for spaces X and Y and $g : X \to Y$;

(1) The function g is semi-αu-continuous.

(2) For each $x \in X$ each filterbase Ω on X satisfying $\Omega \to_{\alpha u} x$ also satisfies $g(\Omega) \to_u g(x)$.

(3) For each $x \in X$ each net $\{x_\eta\}$ on X satisfying $\{x_\eta\} \to_{\alpha u} x$ also satisfies $\{g(x_\eta)\} \to_u g(x)$.

(4) Each filterbase Ω on X satisfies $g(ad_{\alpha u}\Omega) \subset ad_u g(\Omega)$.

(5) Each $A \subset X$ satisfies $g(cl_{\alpha u}A) \subset cl_u g(A)$.

(6) Each $B \subset Y$ satisfies $cl_{\alpha u}(g^{-1}(B)) \subset g^{-1}(cl_u(B))$.

(7) Each filterbase Ω on $g(X)$ satisfies $ad_{\alpha u}g^{-1}(\Omega) \subset g^{-1}(ad_u\Omega)$.

(8) Each $B \subset Y$ satisfies $cl_{\alpha u}[g^{-1}(int(cl_{\alpha\theta}(B)))] \subset g^{-1}(cl_u(B))$.

(9) Each open $W \subset Y$ satisfies $cl_{\alpha u}[g^{-1}(int(cl(W)))] \subset g^{-1}(cl_u(W))$.

(10) Each regular-closed $R \subset Y$ satisfies $cl_{\alpha u}[g^{-1}(int(R))] \subset g^{-1}(cl_\theta(R))$.

(11) Each open $W \subset Y$ satisfies $cl_{\alpha u}(g^{-1}(W)) \subset g^{-1}(cl_u(W))$.

Proof. Can be done similar to the proof of the Theorem 2.1.9.□

Corollary 2.2.3. A function $f : X \to Y$ is semi-αu-continuous if and only if inverse image of every u-closed set is αu-closed.

Proof. In view of Theorem 3.2.3 (6) for $B \subset Y, g^{-1}(B) \subseteq cl_{\alpha u}(g^{-1}(B)) \subset g^{-1}(cl_u(B)) = g^{-1}(B)$. Since $cl_u B = B, g^{-1}(B) = cl_{\alpha u}(g^{-1}(B))$.

2.3 Strongly$u$-Continuous Functions

Definition 2.3.1 A function $g : X \to Y$ is *strongly u-continuous at* $x \in X$ if for each α set Wwhich contains a closed neighborhood of $g(x)$ there exists an open set V which contains a closed neighborhood of x such tht $g(clV) \subset clW$. We say g is strongly u-continuous if g is strongly u-continuous at each $x \in X$.

The following theorem gives sense of characterization and can be proved similar to the Theorem 2.1.9.

Theorem 2.3.2. The following statements are equivalent for spaces X and Y and $g : X \to Y$;

(1) The function g is strongly u-continuous.

(2) For each $x \in X$ each filterbase Ω on X satisfying $\Omega \to_{\alpha u} x$ also satisfies $g(\Omega) \to_u g(x)$.

(3) For each $x \in X$ each net $\{x_\eta\}$ on X satisfying $\{x_\eta\} \to_{\alpha u} x$ also satisfies $\{g(x\eta)\} \to_u g(x)$.

(4) Each filterbase Ω on X satisfies $g(ad_u\Omega) \subset ad_{\alpha u} g(\Omega)$.

(5) Each $A \subset X$ satisfies $g(cl_u A) \subset cl_{\alpha u} g(A)$.

(6) Each $B \subset Y$ satisfies $cl_u(g^{-1}(B)) \subset g^{-1}(cl_{\alpha u}(B))$.

(7) Each filterbase Ω on $g(X)$ satisfies $ad_{\alpha u} g^{-1}(\Omega) \subset g^{-1}(ad_u\Omega)$.

(8) Each $B \subset Y$ satisfies $cl_{\alpha u}[g^{-1}(int(cl_{\alpha\theta}(B)))] \subset g^{-1}(cl_u(B))$.

(9) Each open $W \subset Y$ satisfies $cl_{\alpha u}[g^{-1}(int(cl(W)))] \subset g^{-1}(cl_u(W))$.

(10) Each regular-closed $R \subset Y$ satisfies $cl_{\alpha u}[g^{-1}(int(R))] \subset g^{-1}(cl_\theta(R))$.

(11) Each open $W \subset Y$ satisfies $cl_{\alpha u}(g^{-1}(W)) \subset g^{-1}(cl_u(W))$.

Corollary 2.3.2. A function $f : X \to Y$ is strongly u-continuous if and only if inverse image of every αu-closed set is u-closed.

Proof. In view of Theorem 2.3.2 (6) for $B \subset Y, g^{-1}(B) \subseteq cl_u(g^{-1}(B)) \subset g^{-1}(cl_{\alpha u}(B)) = g^{-1}(B)$. Thus $g^{-1}(B) = cl_u(g^{-1}(B))$.

We close this section with the following implication diagram. Examples are provided in the next section.

$$\begin{array}{ccc} \alpha u - continuous & \Leftarrow & Strongly\ \ u - continuous \\ \Downarrow & & \Downarrow \\ Semi - \alpha u - continuous & \Leftarrow & u - continuous \end{array}$$

3 Counterpart for Examples

In [9], it was established that the collection of α-sets of a topological space (X, τ) form a topology on X. Moreover, if τ_α represents the topology generated by the α-sets of X, then $\tau_\alpha = \{U - N | U$ is open and N is nowhere dense in $X\}$. In [3], this topology on X is represented by $F(\tau)$, as the largest topology on X which has the same collection of open sets as (X, τ). There exist topologies on a set X such that $\tau \neq \tau_\alpha$ and clearly $\tau \subset \tau_\alpha$. In [15] it was established that a topological property is semi-topological [3] if and only if $\tau = F(\tau)$. For a topology τ on X, if $\tau \neq F(\tau)$, then for such spaces, there exists nowhere dense subsets in (X, τ) which are not closed and α-sets which are not open.

From the definitions of u-continuity and the generalizations of this concept presented here, an open set or an α-set containing a closed neighborhood of the point is considered. So, for these difnitions an added layer of boundary of a point is the focus. It is well-known that for a regular space, a non-empty open set contains a closed neighborhood of each of its points. In a regular space, continuity agrees with u-continuity. In [3], it is established that regularity is not semi-topological and hence there exist regular spaces (X, τ) so that $\tau \neq F(\tau) = \tau_\alpha$.

When $\tau \neq F(\tau) = \tau_\alpha$, The identity function $f : (X, \tau) \to (X, F(\tau))$ has the property that inverse image of every open set is an α-set, which is not open and hence f is α-continuous [1], but not continuous. On the other hand if we consider the identity function $g : (X, F(\tau)) \to (X, \tau)$, inverse image of every α-set is open. Also if there is an α-set which contains a closed neighborhood of a point, it is an α-set which contains a regularly closed neighborhood of the point. Ganster [7] characterized a strongly s-regular space as the space which has the property that every non-empty open set contains a regularly closed set around each of its points. There exist strongly s-regular spaces which are not regular [7]. So, if (X, τ) is a strongly s-regular space which is not regular and if $\tau \neq F(\tau)$, then the identity function $f : (X, \tau) \to (X, F(\tau))$ is significant in classification of these classes of functions.

Examples 4.1 and 4.2 give strongly u-continuous functions which are not continuous. So, these functions represent αu-continuous functions which are not continuous.

4 Examples

Example 4.1 . Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Let f $f : (X, \tau_1) \to (X, \tau_2)$ be defined by $f(a) = b, f(b) = a, f(c) = c$. This function is strongly u- continuous which is not continuous.

Remark. Note that the topology τ_2 in Example 4.1 is infact the topology $F(\tau) = \tau_\alpha$. The identity function $f : (X, \tau) \to (X, \tau_2)$ is strongly u-continuous which is not continuous.

Example 4.2. Let $X = [-1, 1]$ and let $\tau_1 = \{U \subset X :$ either $0 \notin U$ or $(-1, 1) \subseteq U\}$.Let τ_2 be generated by $\{[-1, b), b > 0; (a, 1], a < 0; (a, b), a < 0 < b\}$. Define function $f : (X, \tau_1) \to (X, \tau_2)$ as $f(1) = -1, f(-1) = 1$; for $x \in (-1, 1), f(x) = x$. Since closure of every open set or α-set equals to the set X, f is strongly u-continuious. However, f is not continuous since $f^{-1}((a, b))$ with $a < 0 < b$ is not open in (X, τ_1).

Example 4.3. Example of an αu-continuous function which is not u-continuous. This is the Example 1.3 of [9], given to illustrate that continuity at a point need not imply u-continuity.

Let $X = \{0\} \cup (1, \infty)$ and let $\{D(k); k = 1, 2, 3, 4, 5\}$ be a partition of $(1, \infty)$ into subsets dense in $(1, \infty)$ in the usual topology and suppose $N - \{1\} \subset D(1)$ where N is the set of positive integers. Let $\mathcal{B}$ be the base of open intervals for the usual topology on $(1, \infty)$ with the end point in $D(1)$. Declare W to be open in X if W satisfies the following properties: If $k = 1, 3$ or 5 and $x \in W \cap D(k)$ some $B \in \mathcal{B}$ satisfies $x \in B \cap D(k) \subset W$; if k=2 or k=4 and $x \in W \cap D(k)$ some $B \in \mathcal{B}$ satisfies $x \in B \cap (D(k-1) \cup D(k) \cup D(k+1)) \subset W$; if $0 \in W$ then some $m \in N$ satisfies $\bigcup_{n \geq m}(2n, 2n+1) \cap D(1) \subset W$. Let Y have the same description as X except that $N - \{1\} \subset D(5)$. Let $g : X \to Y$ be identity function. Then g is continuous at 0 and g is not u-continuous at 0, let $W = \{0\} \cup \bigcup_N ((2n, 2n+1) \cap D(1))$ and $V = W \cup \bigcup_N ((2n, 2n+1) \cap D(2))$. Then $clW \subset V$ and $V, W \in \Sigma(0)$ in Y, so $V \in \Lambda(0)$. If $E \subset N$ is the set of multiples of 2 then $E \cap clA \neq \emptyset$ for each $A \in \Sigma(0)$ in X and $E \cap clV = \emptyset$. However, this function is αu-continuous. Let $V' = \{0\} \cup \bigcup_N ([2n, 2n+1] \cap (D(1) \cup D(5)))$. Then V' is an α-set containing clW, and $E \cap clV \neq \emptyset$. Consider $W = \{0\} \cup \bigcup_N ([2n, 2n+1] \cap D(1))$, and $V' = \{0\} \cup \bigcup_N ([2n, 2n+1] \cap (D(1) \cup D(5)))$. Then $clW' \subset V'$.

Example 4.4. Example of a u-continuous function which is not strongly u-continuous. This is stated as Example 1.6 in [9].

Let $X = Y = \{0\} \cup (1, \infty)$. A set V is open in X if V satisfies the following properties; if $0 \in V$ there is a sequence $\setminus$ in $(0, 1)$ and a $k \in N$ such that, $\bigcup_{n \geq k}(2n - \setminus, 2n + \setminus) \subset V$; if $x \in V - \{0\}$ there is an open interval I such that $x \in I \subset V$. A set V is open in Y if V satisfies the following properties: If $0 \in V$ there is a $k \in N$ such that $\bigcup_{n \geq k}(2n, 2n+1) \subset V$; if $x \in V - \{0\}$ there is an open interval I such that $x \in I \subset V$. Let $g : X \to Y$ be identity function except for 0, define f(0)= 2.5. It is not difficult to see g is u-continuous. $W = \{0\} \cup \bigcup_{n \geq 1}[2n, 2n+1]$ is an α-set containing a closed neighborhood of 2.5 but there is no open set V in X containing closed neighborhood of 0 which satisfies $g(clV) \subset clW$.

5 Conclusion

We conclude this article with the following observations. In [9], where the concept of u-continuity was introduced, several applications of this concept were presented. For example, graph conditions using the u-closure operator was used to give significant characterizations of quasi Urysohn-closed spaces, functionally compact spaces and C-compact spaces. Further, this concept was used to give generalizations of Uniform Boundedness Principle from classical analysis and also provided a "second category type" property for the class of quasi Urysohn-closed spaces.

In [14], the class of C-compact spaces were characterized by functions satisfying certain graph conditons using θ-closure operator and in [13], the class of compact and extremally disconnected spaces was characterized, among others, using a generalization of θ-closure operator. The class of functions introduced here can effectively be used to provide similar results for the above mentioned class of spaces and properties.

For any set A, the u-closure of A contains the θ-closure of A which contains the closure of A. So, as we consider boundaries of a set based on these different closure operators, the boundaries get larger or smaller, depending on the closure operator in question. These types of properties and generalizations of continuity based on these properties could be effectively used to create models using fuzzy set concept, distinguishing the degree of closeness of a collection of points in a set with another collection of points.

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