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Further development of secant-type methods for solving nonlinear equations

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ABSTRACT. The objective of this paper is to define new improved Secant-type methods for finding simple root of nonlinear equations. In terms of computational cost the new iterative methods requires two evaluations of functions per iteration. It is shown and proved that the new methods have a convergence of order $1+\sqrt{3}$. Numerical comparisons are made to demonstrate exceptional convergence speed of the proposed methods. It is observed that the new iterative methods are very competitive with the similar robust methods.

1 Introduction

Finding iterative methods for solving nonlinear equations is an important area of research in numerical analysis, science and engineering [5]. In this paper, we present new iterative methods to find a simple root of the nonlinear equation. It is well established that the multipoint root-solvers is of great practical importance since it overcomes theoretical limits of one-point methods concerning the convergence order and computational efficiency. Recently, some modifications of the Newton-type methods for simple root have been proposed and analysed [5] and some work has been done on the secant-type methods. Hence, the purpose of this paper is to show further development of the secant-type methods. In view of this fact, the proposed methods are significantly better when compared with the established methods. We have found that the efficiency index of new iterative methods have a better efficiency index than the classical Newton method, the Fernandez-Torres 2.4 method [2], Tiruneh et al. 2.4 method

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[8] and the Thukral 2.4 method [7] and equivalent to Chen et al. [1], Wang et al. [10]. This paper is actually a continuation of the previous study [7] and the extension of this investigation is based on the improvement of the Thukral 2.4 method, the Fernandez-Torres 2.4 method, and Tiruneh et al. 2.4 method.

The two well-known iterative methods for finding simple root of nonlinear equations are namely, the classical Newton method and the classical secant method, given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

$$x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] f(x_n), \tag{2}$$

and their order of convergence is 2 and 1.618 respectively. However, for the purpose of this paper, we present a new secant-type methods for finding simple root of nonlinear equations.

The remaining sections of the paper are organized as follows. Some basic definitions relevant to the present work are stated in the section 2. In section 3 we improve three recently introduced secant-type methods [7] and prove their order of convergence. In section 4, two well-established methods are stated, which will demonstrate the effectiveness of the new secant-type iterative methods. Finally, in section 5, numerical comparisons are made to demonstrate the performance of the presented methods.

2 Preliminaries

In order to establish the order of convergence of an iterative method, following definitions are used [5].

Definition 1: Let $f(x)$ be a real-valued function with a root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence p is given by

$$lim_{n\to\infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \zeta \neq 0, \tag{3}$$

where $p \in \mathbb{R}^+$ and ζ is the asymptotic error constant [4,5,9].

Definition 2: Let $e_k = x_k - \alpha$ be the error in the k^{th} iteration, then the relation

$$e_{k+1} = \zeta e_k^p + O(e_k^{p+1}), \tag{4}$$

is the error equation. If the error equation exists, then p is the order of convergence of the iterative method [4,5,9].

Definition 3: Let r be the number of function evaluations of the method. The efficiency of the method is measured by the concept of efficiency index and defined as

$$EI(r,p) = \sqrt[r]{p}, \tag{5}$$

where p is the order of convergence of the method [4].

Definition 4: Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of a nonlinear equation. Then the computational order of convergence may be approximated by

$$COC \approx \frac{\ln\left|\frac{\delta_n}{\delta_{n-1}}\right|}{\ln\left|\frac{\delta_{n-1}}{\delta_{n-2}}\right|}, \tag{6}$$

where $\delta_i = \frac{f(x_i)}{f'(x_i)}$, [6].

3 Development of the methods and convergence analysis

In this section, we shall define three new iterative methods, based on the Thukral 2.4 method, the Fernandez-Torres 2.4 method, and Tiruneh et al. 2.4 method. To obtain the solution of a nonlinear equation, the new secant-type methods requires two evaluations of functions and set a particular initial point, ideally close to the simple root. The recently discussed iterative methods [7], namely the Tiruneh et al. 2.4 method, the Fernandez-Torres 2.4 method and the Thukral 2.4 method are

$$x_{n+1} = x_{n-1} + \frac{x_n - x_{n-1}}{1 - (f(x_n)f(x_{n-1})^{-1})[\frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}]f'(x_{n-1})^{-1}}, \tag{7}$$

$$x_{n+1} = x_{n-1} - \frac{f(x_n)(f(x_n) - f(x_{n-1}))(x_{n-1} - x_n)}{f(x_n)(f(x_n) - f(x_{n-1}) - f(x_{n-1}))f'(x_n)(x_n - x_{n-1})} \tag{8}$$

$$x_{n+1} = x_n - \frac{2f(x_n)(f'(x_n)(x_n - x_{n-1}))}{2f'(x_n)^2(x_n - x_{n-1}) - f(x_n)(f'(x_n) - f'(x_{n-1}))} \tag{9}$$

respectively. In order to increase the order of convergence of the established methods given above, we shall insert new factors in (7)-(9). First of the new scheme to find simple root of a nonlinear equation is based on the Tiruneh et al. 2.4 method and the improved scheme is given by

$$x_{n+1} = x_{n-1} + \frac{(x_n - x_{n-1})}{1 - (f(x_n)f(x_{n-1})^{-1})[\frac{f(x_n)-f(x_{n-1})}{(x_n - x_{n-1})}](f'(x_{n-1})^{-1})} + \beta \tag{10}$$

where

$$u_n = \frac{f(x_n)}{f'(x_n)}, \tag{11}$$

$$\gamma_1 = \frac{f'(x_n) - f'(x_{n-1})}{x_n - x_{n-1}} \tag{12}$$

$$\gamma_2 = \frac{2f'(x_n) + f'(x_{n-1}) - 3\gamma_1}{x_n - x_{n-1}} \tag{13}$$

$$\beta = u_n^2[(\frac{1}{3})(\frac{\gamma_1}{f'(x_n)}) - \frac{2}{3}(\frac{\gamma_1}{f'(x_n)}) - u_{n-1}(\frac{\gamma_1}{f'(x_n)})] \tag{14}$$

where $n \in \mathbb{N}, x_0, x_1$ are the initial values and provided that the denominator of (10) is not equal to zero. It is essential to verify our finding and prove the order of convergence of the new iterative method.

Theorem: Let $f \in D \subset \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function and let for an open interval D has $\alpha in D$ be a simple zero of $f(x) = 0$ in an open interval D, with $f'(x) \neq 0$ in D. If the initial points x_0 & x_1 are sufficiently close to α then the asymptotic convergence order of the new methods defined by (10) is $1 + \sqrt{3}$.

Proof: Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the errors at $(k-1), k$ and $(k+1)$ iteration are expressed as $e_{n-1} = x_{n+1} - \alpha$, respectively.

Using Taylor expansion and taking into account that $f(\alpha) = 0$, we have

$$f(x_{n-1}) = c_1 e_{n-1} + c_2 e_{n-1}^2 + c_3 e_{n-1}^3 + ... \tag{15}$$

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + ... \tag{16}$$

where

$$c_k = \frac{f^{(k)(\alpha)}}{k!}, \quad for \;\; k = 1, 2, 3.... \tag{17}$$

Furthermore, we have

$$f'(x_{n-1}) = c_1 + 2c_2 e_{n-1} + 3c_3 e_{n-1}^2 + ... \tag{18}$$

$$f'(x_n) = c_1 + 2c_2 e_n + 3c_3 e_n^2 + ... \tag{19}$$

Using (15) and (16), we obtain

$$\omega_1 = \frac{f(x_n)}{f(x_{n-1})} = \frac{e_n}{e_{n-1}} - \frac{c_2}{c_1} e_n - (\frac{c_3}{c_1} - \frac{c_2^2}{c_1^2}) e_{n-1} e_n + ... \tag{20}$$

$$\omega_2 = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = c_1 + (e_n + e_{n-1}) c_2 + (e_n^2 + e_{n-1} e_n + e_{n-1}^2) c_3 + ... \tag{21}$$

$$\omega_3 = \frac{\omega_2}{f'(x_n)} = 1 + \frac{c_2}{c_1} e_{n-1} + \frac{c_3}{c_1} e_{n-1}^2 + ... \tag{22}$$

Substituting the expressions (20)-(22) in (7), we have

$$e_{n+1} = (\frac{c_2^2 - c_1 c_3}{c_1^2}) e_{n-1} e_n^2 + ... \tag{23}$$

The improvement factors introduced in (10) are

$$u_{n-1} = \frac{f(x_{n-1})}{f'(x_{n-1})} = e_{n-1} - \frac{c_2}{c_1} e_{n-1}^2 + (\frac{c_2^2 - c_1 c_3}{c_1^2}) e_{n-1}^3 + ..., \tag{24}$$

$$u_n = \frac{f(x_n)}{f'(x_n)} = e_n - \frac{c_2}{c_1} e_n^2 + (\frac{c_2^2 - c_1 c_3}{c_1^2}) e_n^3 + ..., \tag{25}$$

$$\lambda_1 = \frac{f'(x_n) - f'(x_{n-1})}{x_n - x_{n-1}} = 2c_2 + 3c_3(e_{n-1} + e_n), \tag{26}$$

$$\lambda_2 = \frac{2f'(x_n) + f'(x_{n-1}) - 3\lambda_1}{x_n - x_{n-1}} = c_2 + 3c_3 e_n, \tag{27}$$

$$\beta_1 = (\frac{\lambda_1}{f'(x_n)}) u_n^2 = (\frac{2c_2 + 3c_3 e_{n-1}}{c_1}) e_n^2 + ..., \tag{28}$$

$$\beta_2 = (\frac{\lambda_2}{f'(x_n)}) u_n^2 = (\frac{c_2}{c_1}) e_n^2 + ..., \tag{29}$$

$$\beta_3 = u_{n-1} (\frac{\lambda_2}{f'(x_n)})^2 u_n^2 = (\frac{c_2}{c_1})^2 e_{n-1} e_n^2 + ..., \tag{30}$$

$$\beta = [(\frac{1}{3})\beta_1 - (\frac{2}{3})\beta_2 - \beta_3] = (\frac{c_1 c_3 - c_2^2}{c_1^2}) e_{n-1} e_n^2 + (\frac{c_2}{c_1})^3 e_{n-1}^2 e_n^2 + ... \tag{31}$$

Substituting appropriate expressions in (10), we obtain the error equation for the new iterative method

$$e_{n+1} = (\frac{c_2}{c_1})^3 e_{n-1}^2 e_n^2 + ... \tag{32}$$

In order to prove the order of convergence of (33) and we defining positive real terms of T_n and T_{n-1} as

$$T_n = \frac{|e_{n+1}|}{e_n^m}, \quad T_{n-1} = \frac{|e_n|}{e_{n-1}^m}, \tag{33}$$

$$|e_{n+1}| = (T_n T_{n-1}^m) |e_{n-1}^{m^2}|. \tag{34}$$

It is obtained from (35) that

$$\frac{|e_{n+1}|}{e_n^2 e_{n-1}^2} = |(\frac{c_2}{c_1})^3| = (T_n T_{n-1}^{m-2}) |e_{n-1}^{m^2-2m-2}|. \tag{35}$$

In order to satisfy the asymptotic equation (36), the power of the error term shall approach zero, that is

$$m^2 - 2m - 2 = 0. \tag{36}$$

It is obvious that quadratic equation (37) has two roots,

$$m = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}. \tag{37}$$

The order of convergence of the new method (10) is determined by the positive root of (38). Hence, the new method defined by (10) has a convergence order of $1 + \sqrt{3}$. This completes the proof.

Second of the new scheme to find simple root of a nonlinear equation is based on the Fernandez-Torres 2.4 method and the improved scheme is given by

$$x_{n+1} = x_{n-1} - \frac{f(x_n)(f(x_n) - f(x_{n-1}))(x_{n-1} - x_n)}{f(x_n)(f(x_n) - f(x_{n-1})) - f(x_{n-1})f'(x_n)(x_n - x_{n-1})} + \beta \tag{38}$$

where β is given by (32).

Remark 1: It has been shown in [7] that the Fernandez-Torres 2.4 method and Tiruneh et al. 2.4 method produce identical estimates and error equation. The improved version of these methods given (10) and (39) also produce identical estimates and error equation.

Third of the new scheme is based on the Thukral 2.4 method and the improved scheme is given by

$$x_{n+1} = x_n - \frac{2f(x_n)(f'(x_n)(x_n - x_{n-1})}{2f'(x_n)^2(x_n - x_{n-1}) - f(x_n) - f(x_{n-1})(f'(x_n) - f'(x_n))} + \phi, \tag{39}$$

$$where \;\; \phi = [\frac{1}{2}\beta_1 - \beta_2], \tag{40}$$

and β_1, β_2 are given by (29), (30) respectively. The new iterative methods requires two function evaluations and has the order of convergence $1 + \sqrt{2}$ and to determine the efficiency index of the new method, definition 3 shall be used. Hence, the efficiency index of the new iterative methods given by (10), (39), and (40) is

$$EI(1 + \sqrt{3}, 2) \approx 2.732, \tag{41}$$

and the efficiency index of the Tiruneh et al. 2.4 method, the Fernandez-Torres 2.4 method and the Thukral 2.4 method is given by

$$EI(1 + \sqrt{2}, 2) \approx 2.414. \tag{42}$$

This indicates that the efficiency index of the new methods (10), (39), (40) are equivalent to the Wang et al. method and Chen et al. method and is better than the classical secant method, the Newton method, the Fernandez-Torres 2.4 method, Tiruneh et al. 2.4 method and Thukral 2.4 method.

4 The established methods

For the purpose of comparison, two well-known iterative methods are considered namely, the Wang et al. and the Chen et al. methods. Since these methods are well established, the essential formulas are used to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new method.

In [10], Wang et al. developed a method having convergence order of 2.732 for finding simple root of nonlinear equations, the essential expression used in the method is given as

$$y_n = x_n - v_{n-1} f(x_n), \tag{43}$$

$$v_n = \frac{(y_n - x_n)}{(f(y_n) - f(x_n))}, \tag{44}$$

$$z_n = x_n - v_n f(x_n), \tag{45}$$

$$x_{n+1} = z_n - \frac{(y_n - z_n)^2}{\alpha_1 x_{n-1} + \alpha_2 y_{n-1} + (2 - \alpha_1 - \alpha_2) z_{n-1} - \beta_1 x_n \beta_2 y_n - (2 - \beta_1 - \beta_2) z_n}, \tag{46}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$.

In [1], Chen et al. also developed a method having convergence order of 2.732 for finding simple root of nonlinear equations, the essential expression used in the method is given as,

$$y_n = x_n - v_{n-1} f(x_n), \tag{47}$$

$$v_n = f'(\frac{x_n + y_n}{2})^{-1}, \tag{48}$$

$$z_n = x_n - v_n f(x_n), \tag{49}$$

$$x_{n+1} = z_n - \frac{(y_n - z_n)^2}{\alpha_1 x_{n-1} + \alpha_2 y_{n-1} + (2 - \alpha_1 - \alpha_2) z_{n-1} - \beta_1 x_n \beta_2 y_n - (2 - \beta_1 - \beta_2) z_n}, \tag{50}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$.

5 Numerical results

The proposed iterative methods given by (10), (39), and (40) are employed to solve nonlinear equation with simple root. The difference between the simple root α and the approximation x_n for test function with initial guess x_0 is displayed in tables. Furthermore, the computational order of convergence approximations are displayed in tables and we observe that this perfectly coincides with the theoretical result. The numerical computations listed in the table was performed on an algebraic system called Maple and the errors displayed are of absolute value.

Remark 2: We observe that the Tiruneh et al. 2.4 method [8] used a pair of initial points x_0, x_1. Here we use an initial point x_0 and the second initial point x_1 is given as

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \tag{51}$$

We have found that the performance of the Tiruneh et al. 2.4 method is improved. Furthermore, the second initial point (52) is also used in the new improved methods (10), (39) and (40).

6 Conclusion

New secant-type methods for solving nonlinear equations with simple root have been presented. The effectiveness of the new iterative methods are examined by showing the accuracy of the simple root of several nonlinear equations. We have shown numerically and verified that the new iterative methods have convergence of order

2.732. The major advantages of the new methods are that they are very effective and produces high precision of approximation of the simple root. Finally, we conclude that the new iterative methods may be considered a very good alternative to the established methods.

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Figure 1: Table 1 Test functions and their simple roots.

Functions	*Simple Root*	*Initial point*
$f_1(x)=(x-2)(x^{10}+x+1)\exp(-x-1)$	$\alpha=2$	$x_0=2.001$
$f_2(x)=\ln(1+x^2)+\exp(x)\sin(x)$	$\alpha=0$	$x_0=0.1$
$f_3(x)=x^6-x^4-x^3-1$	$\alpha=1.4036$	$x_0=1.35$
$f_4(x)=\exp(x^2+7x-30)-1$	$\alpha=3$	$x_0=3.1$
$f_5(x)=\cos(x)^2-x^2+1$	$\alpha=1.0985$	$x_0=1$
$f_6(x)=\exp(-x^2)+\sin(x)+\ln(x+2)$	$\alpha=-0.7967$	$x_0=-1$
$f_7(x)=\exp(x)\sin(x)\cos(x)-x^5+7$	$\alpha=1.4904$	$x_0=1.5$
$f_8(x)=\exp(2-x)-\cos(3^{-1}x)$	$\alpha=2.3417$	$x_0=2$
$f_9(x)=\cos(x)^2-2^{-1}x$	$\alpha=0.8570...$	$x_0=1.4$
$f_{10}(x)=x^2-\exp(x)-3x+2$	$\alpha=0.2575$	$x_0=0.5$

Figure 2: Table 2 Errors occurring in the estimates of the simple root of the methods described.

f_i	**(9)**	**(7)=(8)**	**(47)**	**(51)**	**(10)=(39)**	**(40)**
f_1	0.256e-172	0.392e-152	0.855e-164	0.433e-160	0.535e-257	0.231e-300
f_2	0.654e-76	0.330e-38	0.180e-83	0.340e-55	0.116e-81	0.767e-106
f_3	0.143e-59	0.154e-61	0.120e-13	0.472e-14	0.560e-106	0.873e-107
f_4	0.199e-19	0.350e-28	-	0.897e-28	0.186e-32	0.422e-41
f_5	0.e731-103	0.182e-105	0.110e-131	0.105e-178	0.426e-193	0.290e-180
f_6	0.111e-88	0.281e-91	0.348e-179	0.403e-145	0.532e-141	0.465e-142
f_7	0.131e-136	0.623e-138	0.269e-125	0.474e-125	0.186e-232	0.421e-241
f_8	0.364e-49	0.555e-39	0.871e-57	0.764e-57	0.962e-76	0.116e-87
f_9	0.456e-59	0.373e-46	0.178e-58	0.123e-48	0.192e-98	0.542e-118
f_{10}	0.462e-98	0.696e-65	0.328e-189	0.245e-170	0.372e-181	0.774e-170

Figure 3: Table 3 Performance of COC

f_i	(9)	(7)=(8)	(47)	(51)	(10)=(39)	(40)
f_1	2.4089	2.4114	2.7616	2.7599	2.7106	2.7205
f_2	2.4051	2.4118	2.6724	2.7425	2.7916	2.7474
f_3	2.4118	2.3990	2.5627	2.6480	2.6955	2.7326
f_4	2.1906	2.4097	-	2.7670	2.6861	2.6834
f_5	2.4099	2.4098	2.7497	2.7405	2.7229	2.7246
f_6	2.4080	2.4085	2.7341	2.7752	2.7332	2.7330
f_7	2.4073	2.4113	2.7415	2.7440	2.7102	2.7223
f_8	2.3766	2.4108	2.7262	2.7258	2.7126	2.7211
f_9	2.3879	2.4111	2.7283	2.6865	2.7068	2.7367
f_{10}	2.4153	2.4147	2.7128	2.7474	2.7291	2.7294