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Hangable graphs

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ABSTRACT. The distance $d_G(u,v)$ between vertices u and v in G is the length of a shortest $u-v$ path in G. The eccentricity of a vertex v in G is the integer $e_G(v) = \max\{d_G(v,u)\colon u \in V_G\}$. The diameter of G is the integer $d(G) = \max\{e_G(v)\colon v \in V_G\}$. The periphery of a vertex v of G is the set $P_G(v) = \{u \in V_G\colon d_G(v,u) = e_G(v)\}$, while the periphery of G is the set $P(G) = \{v \in V_G\colon e_G(v) = d(G)\}$. A graph G is said to be hangable if $P_G(v) \subseteq P(G)$ for every vertex v of G. In this paper we prove that every block graph is hangable and discuss the hangability of products of graphs.

1 Introduction

We use [1] and [3] for basic graph theory terminology and notation. Specifically, let $G = (V_G, E_G)$ be a connected graph with vertex set V_G and edge set E_G. The *distance* between vertices u and v in G, denoted by $d_G(u,v)$, is the length of a shortest $u-v$ path in G. The *eccentricity* $e_G(v)$ of a vertex v in G is the distance from v to a vertex farthest from v, that is $e_G(v) = \max\{d_G(v,u)\colon u \in V_G\}$. The *diameter* $d(G)$ of G is the maximum eccentricity of the vertices of G. It follows from these definitions that $d(G) = \max\{d_G(u,v)\colon u,v \in V_G\}$. The *periphery of vertex* v of G is the set $P_G(v)$ of the vertices farthest from v, $P_G(v) = \{u \in V_G\colon d_G(v,u) = e_G(v)\}$, whereas the *periphery of graph* G is the set $P(G)$ of vertices having the maximum eccentricity in G, that is $P(G) = \{v \in V_G\colon e_G(v) = d(G)\}$. A connected graph G is said to be *self-centered* if $P(G) = V_G$. We say that graph G is *hangable* if $P_G(v) \subseteq P(G)$ for every vertex v of G. Note that if G is a hangable graph, then the diameter of G can be found in the following way: (1) hang G at any vertex v; (2) find a vertex u farthest from v; (3) hang G at the vertex u; (4) find a vertex w farthest from u;

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(5) find the distance $d_G(u,w)$. Then $d_G(u,w)$ is the diameter $d(G)$ of G and the pair (u,w) is a *diametrical pair* of vertices in G. Thus, the class of hangable graphs is the class of graphs for which there exists a simple algorithm for finding the diameter and a diametrical pair of vertices of a graph. Note that every self-centered graph G is hangable since $P(G) = V_G$ and therefore $P_G(v) \subseteq P(G)$ for every vertex v of G. In particular, every complete graph K_n is hangable. Similarly, every cycle C_n and every n-cube Q_n are hangable. Every complete bipartite graph $K_{m,n}$ is also a hangable graph, as is easy to check. Graph G in Fig. 1 is hangable since $P_G(v) \subseteq P(G)$ for every vertex v of G (as $P_G(a) = P_G(b) = P_G(d) = \{e\} \subseteq \{a,e\} = P(G)$ and $P_G(c) = P_G(e) = \{a\} \subseteq \{a,e\} = P(G)$). Graph H in Fig. 1 is not hangable as $P_H(v) \not\subseteq\not\subseteq P(H)$ for some vertex v of H ($P_H(b) = \{a,c,d\} \not\subseteq \{a,c\} = P(H)$ and $P_H(d) = \{a,b,c\} \not\subseteq \{a,c\} = P(H)$).

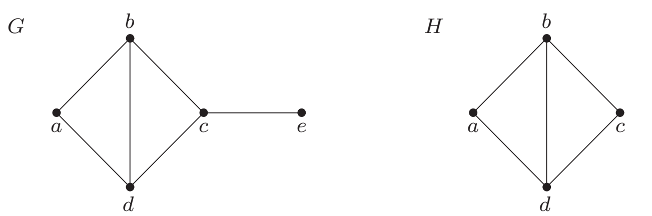

Figure 1: A hangable graph G and a non-hangable graph H

In this paper we prove that every block graph is hangable. This property of block graphs, while easy to demonstrate and prove (especially for trees), does not appear much in the literature, and seems to be overlooked so far, but would be a worthy exercise in any graph theory textbook. We also discuss hangability of three graph products, and show that any graph can be embedded as an induced subgraph in a hangable graph.

2 Hangability of block graphs

We begin by recalling that a *block graph* is a connected graph in which every block (i.e., every maximal 2-connected subgraph) is a complete graph. Hangability of block graphs is ascertained by the following theorem.

Theorem 2.1. *Every block graph G is hangable.*

Proof. Let v be any vertex of G, and let G_v denote the graph G rooted (hanged) at v. The proof will be complete if we show that $P_G(v) \subseteq P(G)$. To prove this it suffices to show that if $u \in P_G(v)$ and x and y are vertices of G, then $d_G(u,x) \geq d_G(x,y)$ or $d_G(u,y) \geq d_G(x,y)$.

This is obvious if $x = y$ or x and y belong to the shortest $v-u$ path in G_v. Thus assume that at most one of the vertices x and y belongs to the shortest $v-u$ path in G_v, and let a_x (a_y, resp.) be the youngest common ancestor of the vertices u and x (u and y, resp.) in the rooted graph G_v.

First assume that one of the vertices x and y, say x, belongs to the shortest $v-u$ path in G_v. If $x\ (=a_x)$ is not younger than a_y in G_v (see Fig. 2 (a)), then the choice of u implies that $d_G(x,y) \le d_G(x,u)$. If x is younger than a_y (Fig. 2 (b)), then x belongs to the shortest $u-y$ path and therefore $d_G(u,y) = d_G(u,x) + d_G(x,y) \ge d_G(x,y)$.

Now assume that none of the vertices x and y belongs to the shortest $v-u$ path in G_v. If a_x is younger than a_y in G_v (Fig. 2 (c)), then $d_G(u,y) = d_G(u,a_x) + d_G(a_x,y) \ge d_G(x,a_x) + d_G(a_x,y) = d_G(x,y)$ as $d_G(a_x,u) \ge d_G(a_x,x)$. Similarly, $d_G(u,x) \ge d_G(x,y)$ if a_y is younger than a_x in G_v. Thus assume $a_x = a_y$. In this case let a_{xy} be the youngest common ancestor of the vertices x and y in G_v. We consider two possible subcases: $a_{xy} \ne a_x = a_y$, and $a_{xy} = a_x = a_y$.

If $a_{xy} \ne a_x = a_y$ and $a_{xy} \in \{x,y\}$, say $a_{xy} = x$, then x belongs to the shortest $u-y$ path and therefore $d_G(u,y) = d_G(u,x) + d_G(x,y) \ge d_G(x,y)$. Thus assume $a_{xy} \ne a_x = a_y$ and $a_{xy} \notin \{x,y\}$ (Fig. 2 (d)). Let a and b be the neighbors of a_x which belong to the shortest $u-a_x$ and $a_{xy}-a_x$ paths, respectively. Then, since $d_G(a,u) \ge d_G(b,y)$, we have $d_G(u,x) \ge d_G(u,a) + d_G(b,x) + 1 \ge d_G(y,b) + d_G(b,x) + 1 \ge d_G(y,a_{xy}) + d_G(a_{xy},x) + 1 \ge d_G(y,x) + 1 \ge d_G(y,x)$.

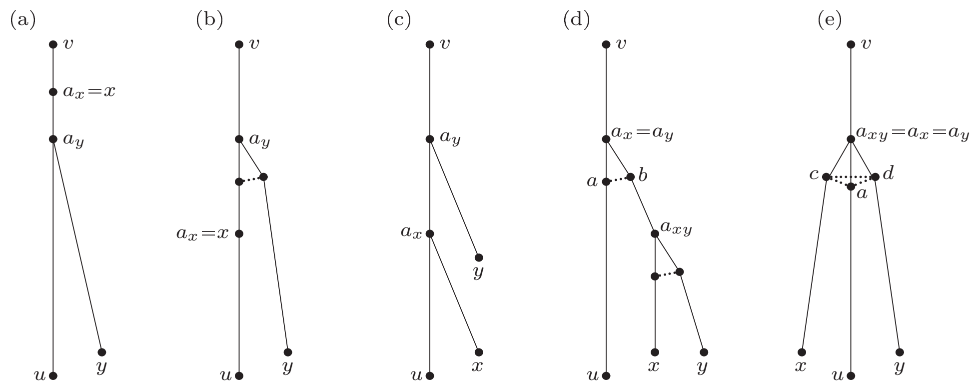

Figure 2: Block graphs rooted (hanged) at the vertex v

Finally assume that $a_{xy} = a_x = a_y$. Let a, c, and d be the neighbors of a_x which belong to the shortest $u-a_x$, $x-a_x$, and $y-a_x$ paths, respectively, Fig. 2 (e). Since G is a block graph, the subgraph of G induced by the vertices a, c, and d has either at most one or three edges. In the first case we may assume that ac is not an edge of G. Then $d_G(u,x) = d_G(u,a_x) + d_G(a_x,x) \ge d_G(y,a_x) + d_G(a_x,x) \ge d_G(y,x)$. In the second case ac, ad, and cd are edges of G, and then the choice of u implies that $d_G(a,u) \ge d_G(c,x)$ and therefore $d_G(u,x) \ge d_G(u,a) + 1 + d_G(c,x) \ge d_G(y,d) + 1 + d_G(c,x) = d_G(y,x)$.

This completes the proof of the theorem.

Since every tree is a block graph, from Theorem 2.1 we immediately have the following corollary.

Corollary 2.1. *Every tree is a hangable graph.*

3 Hangability of product graphs

We now turn our attention to hangability of coronas of graphs. Let G and H be two graphs. The *corona* of G and H, denoted by $G \circ H$, is the graph with vertex set $V_G \cup (V_G \times V_H)$ and edge set $E_G \cup \bigcup_{v \in V_G} \{v(v,x) \colon x \in V_H\} \cup \bigcup_{v \in V_G} \{(v,x)(v,y) \colon xy \in E_H\}$. In Fig. 3, the corona $G \circ H$ is shown in which $G = K_1 + (K_1 \cup K_2)$ and $H = K_2$.

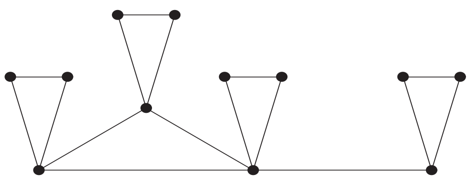

Figure 3: The corona $(K_1 + (K_1 \cup K_2)) \circ K_2$

Basic properties of distances, diameters and sets of peripheral vertices of coronas are summarized in the following lemma.

Lemma 3.1. *If $G \circ H$ is the corona of graphs G and H, where G is a connected graph of order at least two, then:*

(1) $d_{G \circ H}((u,x),(v,y)) = d_G(u,v) + 2$ if $u, v \in V_G$ and $x, y \in V_H$,

$d_{G \circ H}((u,x),v) = d_G(u,v) + 1$ if $u, v \in V_G$ and $x \in V_H$,

$d_{G \circ H}(u,v) = d_G(u,v)$ if $u, v \in V_G$;

(2) $d(G \circ H) = d(G) + 2$;

(3) $P_{G \circ H}(u) = P_{G \circ H}((u,x)) = P_G(u) \times V_H$ if $u \in V_G$ and $x \in V_H$;

(4) $P(G \circ H) = P(G) \times V_H$.

Proof. Since (1), (2), and (3) are obvious, we only prove (4). For this the inclusions $P(G \circ H) \subseteq P(G) \times V_H$ and $P(G) \times V_H \subseteq P(G \circ H)$ have to be proved.

Assume first that $a \in P(G \circ H)$. Let b be a vertex of $G \circ H$ for which $d_{G \circ H}(a,b) = d(G \circ H)$. From the definition of $G \circ H$ (and by (1)) it is obvious that $a = (v,x)$ and $b = (u,y)$ for some vertices v and u of G and some vertices x and y of H. Now, by (1) and (2) it follows that $d(G) + 2 = d(G \circ H) = d_{G \circ H}(a,b) = d_{G \circ H}((v,x),(u,y)) = d_G(v,u) + 2$. Thus $d_G(v,u) = d(G)$ and this implies that $v \in P(G)$. Therefore $a = (v,x) \in P(G) \times V_H$ and, consequently, $P(G \circ H) \subseteq P(G) \times V_H$.

Assume now that $(v,x) \in P(G) \times V_H$. Let $u \in V_G$ be such that $d_G(v,u) = d(G)$. Then, by (1), $d_{G \circ H}((v,x),(u,x)) = d_G(v,u) + 2 = d(G) + 2$. From this and from (2) it follows that $d_{G \circ H}((v,x),(u,x)) = d(G \circ H)$ and therefore $(v,x) \in P(G \circ H)$. This proves that $P(G) \times V_H \subseteq P(G \circ H)$.

Now we are ready to prove a necessary and sufficient condition for a corona of graphs to be hangable.

Theorem 3.1. *If G is a connected graph of order at least two and H is any graph, then the corona $G \circ H$ is a hangable graph if and only if G is hangable.*

Proof. Let G be a hangable graph. Let u and x be any vertex of G and H, respectively. Then $P_G(u) \subseteq P(G)$ and consequently $P_G(u) \times V_H \subseteq P(G) \times V_H$. Thus, by statements (3) and (4) of Lemma 3.1, $P_{G\circ H}(u) \subseteq P(G \circ H)$ and $P_{G\circ H}((u,x)) \subseteq P(G \circ H)$. This proves that graph $G \circ H$ is hangable.

Assume now that graph $G \circ H$ is hangable. Then $P_{G\circ H}(u) \subseteq P(G \circ H)$ and $P_{G\circ H}((u,x)) \subseteq P(G \circ H)$, where $u \in V_G$ and $(u,x) \in V_G \times V_H$ are two possible types of vertices of $G \circ H$. Now, since $P_{G\circ H}(u) = P_{G\circ H}((u,x)) = P_G(u) \times V_H$ and $P(G \circ H) = P(G) \times V_H$ (by statements (3) and (4) of Lemma 3.1, it follows that $P_G(u) \times V_H \subseteq P(G) \times V_H$. Consequently, $P_G(u) \subseteq P(G)$ and this proves that G is hangable.

Let G and H be graphs. The *Cartesian product* of G and H, denoted by $G\Box H$, is the graph with vertex set $V_G \times V_H$ and where two vertices (a,b) and (c,d) are adjacent if and only if $ac \in E_G$ and $b = d$ or $a = c$ and $bd \in E_H$. Distances, eccentricities, diameters and sets of peripheral vertices of Cartesian products of graphs are discussed in the next lemma.

Lemma 3.2. *If $G\Box H$ is the Cartesian product of connected graphs G and H, then:*

(1) $d_{G\Box H}((a,b),(c,d)) = d_G(a,c) + d_H(b,d)$, if (a,b), $(c,d) \in V_{G\Box H}$;

(2) $e_{G\Box H}(a,b) = e_G(a) + e_H(b)$, if $(a,b) \in V_{G\Box H}$;

(3) $d(G\Box H) = d(G) + d(H)$;

(4) $P_{G\Box H}((a,b)) = P_G(a) \times P_H(b)$, if $(a,b) \in V_{G\Box H}$;

(5) $P(G\Box H) = P(G) \times P(H)$.

Proof. The statements (1)–(3) without any proof were presented in [2]. A formal proof of (1) was given in [4]. We prove the statements (4) and (5).

(4) The definition of periphery of a vertex and properties (1) and (2) validate the following chain equalities

$$\begin{aligned} P_{G\Box H}(a,b) &= \{(x,y) \in V_{G\Box H} \colon d_{G\Box H}((a,b),(x,y)) = e_{G\Box H}(a,b)\} \\ &= \{(x,y) \in V_{G\Box H} \colon d_G(a,x) + d_H(b,y) = e_G(a) + e_H(b)\} \\ &= \{(x,y) \in V_{G\Box H} \colon d_G(a,x) = e_G(a) \text{ and } d_H(b,y) = e_H(b)\} \\ &= \{(x,y) \in V_{G\Box H} \colon x \in P_G(a) \text{ and } y \in P_H(b)\} \\ &= P_G(a) \times P_H(b). \end{aligned}$$

(5) From the definition of periphery of a graph and from (2) and (3) it follows that

$$\begin{aligned} P(G\Box H) &= \{(a,b) \in V_{G\Box H} \colon e_{G\Box H}((a,b)) = d(G\Box H)\} \\ &= \{(a,b) \in V_{G\Box H} \colon e_G(a) + e_H(b) = d(G) + d(H)\} \\ &= \{(a,b) \in V_{G\Box H} \colon e_G(a) = d(G) \text{ and } e_H(b) = d(H)\} \\ &= \{(a,b) \in V_{G\Box H} \colon a \in P(G) \text{ and } b \in P(H)\} \\ &= P(G) \times P(H). \end{aligned}$$

The following theorem specifies when the Cartesian product of graphs is a hangable graph and shows how to construct one hangable graph from other graphs.

Theorem 3.2. *If G and H are connected graphs, then the Cartesian product $G\Box H$ is a hangable graph if and only if G and H are hangable graphs.*

Proof. Let (a,b) be a vertex of $G\Box H$. Since $P_{G\Box H}((a,b)) = P_G(a)\times P_H(b)$ (by Lemma 3.2 (4)) and $P(G\Box H) = P(G)\times P(H)$ (by Lemma 3.2 (5)), the result follows from the equivalences

$$\begin{aligned} P_{G\Box H}((a,b)) \subseteq P(G\Box H) \quad &\Leftrightarrow \quad P_G(a)\times P_H(b) \subseteq P(G)\times P(H)\\ &\Leftrightarrow \quad P_G(a)\subseteq P(G) \text{ and } P_H(b)\subseteq P(H). \end{aligned}$$

Corollary 3.1. *The Cartesian product $P_m\Box P_n$ is a hangable graph for every paths P_m and P_n.*

The *join of graphs* G and H, denoted by $G+H$, is the graph obtained from the disjoint union of G and H by adding an edge between each vertex of G and each vertex of H. The following theorem relates to hangability of join graphs.

Theorem 3.3. *For graphs G and H, the join $G+H$ is a hangable graph if and only if either*

(1) *$G+H$ is a complete graph, or*

(2) *at most one vertex of $G+H$ is adjacent to every other vertex of $G+H$.*

Proof. Assume first that $G+H$ is a non-complete hangable graph. Let U be the set of all vertices, each of which is adjacent to every other vertex of $G+H$. It remains to show that U has at most one element. Suppose that U is non-empty. Then, since $e_{G+H}(x) = 1$ for every $x\in U$ and $e_{G+H}(y) = 2$ for every $y\in V_{G+H}-U$, it follows that $P(G+H) = V_{G+H}-U$. Now, since $P_{G+H}(x) = V_{G+H}-\{x\}$ for every $x\in U$, hangability of $G+H$ implies that $V_{G+H}-\{x\}$ is a subset of $V_{G+H}-U$ $(= P(G+H))$ and this clearly forces that U has at most one element.

If $G+H$ is a complete graph, then $G+H$ is a hangable graph as we have already observed. If no vertex of $G+H$ is adjacent to every other vertex of $G+H$, then $e_{G+H}(x) = 2$ for every $x\in V_{G+H}$. Therefore $P(G+H) = V_{G+H}$ and this implies that $G+H$ is hangable. Finally assume that $G+H$ has exactly one vertex, say v_0, which is adjacent to every other vertex of $G+H$. Then $e_{G+H}(v_0) = 1$ and $e_{G+H}(x) = 2$ for every $x\in V_{G+H}-\{v_0\}$. Consequently $P(G+H) = V_{G+H}-\{v_0\}$ and now it is obvious that $G+H$ is a hangable graph since $P_{G+H}(v_0) = V_{G+H}-\{v_0\}\subseteq P(G+H)$ and $P_{G+H}(x)\subset P(G+H)$ for every $x\in V_{G+H}-\{v_0\}$.

The next theorem allows us to obtain hangable graphs from other graphs. Using the join operation, it is also easy to describe how to embed (as an induced subgraph) a graph in a hangable supergraph.

Corollary 3.2. *Every graph is an induced subgraph of a hangable graph.*

Proof. Let H be a graph. We shall prove that H is an induced subgraph of some hangable graph. We may assume that H is not a hangable graph. Then H is not a complete graph. If no vertex of H is adjacent to every other vertex of H, then the join K_1+H has exactly one vertex adjacent to every other vertex of K_1+H and it follows from Theorem 3.3 (2) that K_1+H is a hangable supergraph of H. Now assume that H has a vertex adjacent to every other vertex of H. Let U be the set of all such vertices in H. Since H is not a complete graph, the set $V_H - U$ is nonempty and from the choice of U it follows that $H = H[U] + H[V_H - U]$. Now no vertex of the graph $G = (K_1 \cup H[U]) + H[V_H - U]$ (obtained from H by adding a new vertex and joining it by an edge to every vertex belonging to $V_H - U$) is adjacent to every other vertex of G and it follows from Theorem 3.3 (2) that G is a hangable supergraph of H.

4 Open problems

We conclude this paper with the list of open problems.

Problem 1. *Determine all hangable subgraphs of the Cartesian product $P_m \Box P_n$.*

Problem 2. *Determine all hangable subgraphs of the n-cube Q_n.*

Problem 3. *Determine all graphs G such that G and $\overline{G}$ are hangable graphs.*

Problem 4. *Which self-complementary graph G (such that $G = \overline{G}$) is hangable?*

Problem 5. *For a connected graph G determine the smallest positive integer k such that the power G^k is a hangable graph.*

Problem 6. *For a graph G determine the algorithm that checks with complexity $O(|V_G|^2)$ whether G is hangable.*

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