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On the higher order rational difference equation $x_{n+1} = \dfrac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}}$

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ABSTRACT. In this paper, we have investigated a nonlinear rational difference equation of higher order. Our concentration is on invariant intervals, periodic character, the character of semicycles and global asymptotic stability of all positive solutions of

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}}, \quad n = 0, 1, ...,$$

where the parameters α, β and A, B and the initial conditions x_k, ..., x_1, x_0 are positive real numbers $k = \{1, 2, 3, ...\}$. It is shown that the equilibrium point is globally asymptotically stable under the condition $\beta \le A$, and the unique positive solution is also globally asymptotically stable under the conditions $\beta \le A \le \beta$. It is shown that there does not exists any periodic solution for any positive parameters but it is shown computationally that there are periodic solutions with low as well as high periods.

1 Introduction

This paper presents the global stability character and the periodicity of the solutions of the following rational higher order difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}}, \quad n = 0, 1, ..., \tag{1.1}$$

where the parameters α, β and A, B and the initial conditions $x_k, ..., x_1, x_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$ is a positive integer, and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are non-negative real numbers.

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Various biological systems naturally leads to their study by means of a discrete variable. Appropriate examples include population dynamics and medicine. Some fundamental models of biological phenomena, including harvesting of fish, a single species population model, ventilation volume and blood CO2 levels, the production of red blood cells, a simple epidemics model, and a model of waves of disease that can be analyzed by difference equations are shown in [1]. Newly, there has been interest in so-called dynamical diseases, which correspond to physiological disorders for which a generally stable control system becomes unstable. One of the first papers on this subject was that of Mackey and Glass [2]. In which they investigated a first-order difference-delay equation that models the concentration of blood-level CO2. They also discussed models of a second class of diseases associated with the production of red cells, white cells, and platelets in the bone marrow. The dynamical characteristics of population system have been modeled, among others by differential equations in the case of species with overlapping generations and by difference equations in the case of species with non-overlapping generations. In process, one can developed a discrete model directly from observations and experiments. Periodically, for numerical purposes, one wants to propose a finite-difference scheme to numerically solved a given differential equation model, especially when the differential equation cannot be solved explicitly. For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points [3]. But unless we can explicitly solve both equations, it is impossible to satisfy this requirements. Most of the time, it is fascinating that a differential equation, when extracted from a difference equation, marmalade the dynamical features of the corresponding continuous-time model such as equilibria, their local and global stability characteristics, and bifurcation behaviors. If alike discrete models can be derived from continuous time models, and it will preserve the considered realities, such discrete-time models can be called 'dynamically consistent' with the continuous-time models.

The study of oscillatory and asymptotic stability properties of solution behavior of difference equations is extremely advantageous in the behavior of various biological system and other applications. This is because difference equations are relevant models for expressing situations where the variable is assumed to take only a discrete set of values and they appear frequently in the formulation and analysis of discrete time systems, in the study of biological systems, the study of deterministic chaos, the numerical integration of differential equations by finite difference schemes and so on. Difference equations are good models for describing situations where population growth is not continuous but seasonal with overlapping generations. For example, the difference equation

$$x_{n+1} = x_n e^{\left[r(1-\frac{y_n}{k})\right]}$$

has been expressed to model different animal populations.

The generalized *Beverton–Holt stock recruitment* model has been investigated in [10,11]

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1 + cx_{n-1} + dx_n}.$$

Several other researchers have studied the behavior of the solution of difference equations, for example, in [15] E.M. Elsayed investigated the solution of the following non-linear difference equation.

$$x_{n+1} = ax_n + \frac{bx_n^2}{cx_n + dx_{n-1}^2}.$$

Elabbasy et al. [16] studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation.

$$x_{n+1} = \frac{\alpha x_{n-l} + \beta x_{n-k}}{Ax_{n-l} + Bx_{n-k}}.$$

Keratas et al. [20]gave the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

Elabbasy et al. [17] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} + cx_{n-q}}.$$

Yalçınkaya et al. [18] has studied the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Saleh et. al. [19]study the solution of difference

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}.$$

Elsayed et al. [22] studied the global behavior of rational recursive sequence

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-k} + cx_{n-s}}{d + ex_{n-t}}$$

As a matter of fact, numerous papers negotiate with the problem of solving nonlinear difference equations in any way possible, see, for instance [7]—[15]. The long-term behavior and solutions of rational difference equations of order greater than one has been extensively studied during the last decade. For example, various results about periodicity, boundedness, stability, and closed form solution of the second-order rational difference equations, see [5–9, 21–29].

Other related work on rational difference equations see in refs. [30,31,32,36].

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

(Equilibrium Point) A point $\overline{x} \in I$ is called an equilibrium point of Eq.(1.2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(1.2), or equivalently, $\overline{x}$ is a fixed point of F.

(Periodicity) A Sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

(Fibonacci Sequence). The sequence $\{F_m\}_{m=1}^{\infty} = \{1, 2, 3, 5, 8, 13, ...\}$ *i.e.* $F_m = F_{m-1} + F_{m-2} \geq 0, F_{-2} = 0, F_{-1} = 1$ *is called Fibonacci Sequence.*

(Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.2).

(v) The equilibrium point $\overline{x}$ of Eq.(1.2) is unstable if $\overline{x}$ is not locally stable.

(vi) The linearized equation of Eq.(1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Theorem A [27] Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1, \tag{1.4}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [28]: Let $[l, m]$ be an interval of real numbers and assume that $f : [l, m]^2 \to [l, m]$,is a continuous function and consider the following equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, ..., \tag{1.5}$$

satisfying the following conditions :

(a) $f(x, y)$ is non-decreasing in $x \in [l, m]$ for each fixed $y \in [\alpha, \beta]$.and $g(x, y)$ is non-increasing in $y \in [l, m]$ for each fixed $x \in [l, m]$

(b) If $(m, M) \in [l, m] \times [l, m]$ is a solution of the system

$$M = g(M, m) \qquad and \qquad m = g(m, M),$$

then $m = M$,

then Eq. (5) has a unique equilibrium $\overline{x} \in [l, m]$ and every solution of Eq.(5) converges to $\overline{x}$.

2 Equilibrium points of Eq.(1.1)

In this section we will study the equilibrium points of Eq.(1.1). The equilibrium points of Eq.(1.1) are the positive solutions of the equation

$$\overline{x} = \frac{\alpha + \beta\overline{x}}{A + B\overline{x}}$$

Then,

$$\begin{aligned} \overline{x}(A + B\overline{x}) &= \alpha + \beta\overline{x} \\ \overline{x}(A + B\overline{x} - \beta) &= \alpha \end{aligned}$$

or

$$B\overline{x}^2 + (A - \beta)\overline{x} - \alpha = 0$$

so, the only positive equilibrium point is

$$\overline{x} = \frac{\beta - A + \sqrt{(A - \beta)^2 + 4B\alpha}}{2B}$$

Before we proceed to apprehend local stability of the fixed point, we shall see computationally what are the parameters A, B, α and β such that the fixed point is positive.

Computationally we have found the set of 5000 parameters (A, B) and (α, β) and are plotted in the Fig.1 (left, right) for which the fixed point is positive.

It is seen from the figures that for almost all the parameters (A, B) and (α, β) the fixed point is positive.

To find the linearization for our problem, consider

$$f(v, w) = \frac{\alpha + \beta v}{A + Bw}$$

now,

$$f_v = \frac{\beta}{A + Bw} \quad and \quad f_w = \frac{-B\,(\alpha + \beta v)}{(A + Bw)^2}$$

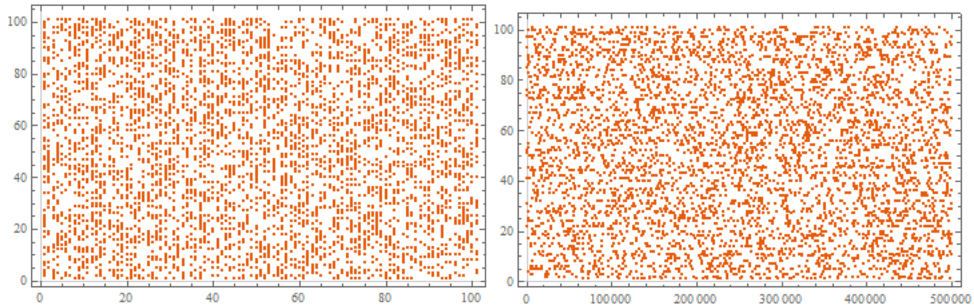

Figure 1: Set of partameters (A, B) and (α, β) (up, down)

Hence, for $\overline{x} = \frac{\beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}}{2B}$,

$$f_v(\overline{x}, \overline{x}) = \frac{2\beta}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}$$

and

$$f_w(\overline{x}, \overline{x}) = -\left(\frac{2B\alpha + \beta\left(\beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}\right)}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}\right)$$

So, the linearized equation about the $\overline{x} = \frac{\beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}}{2B}$

$$y_{n+1} = \left(\frac{2\beta}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}\right) y_n - \left(\frac{2B\alpha + \beta\left(\beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}\right)}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}\right) y_{n-k} \tag{2.1}$$

and its characteristic equation is

$$\lambda^{k+1} - \left(\frac{2\beta}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}\right) \lambda^k + \left(\frac{2B\alpha + \beta\left(\beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}\right)}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}\right) = 0. \tag{2.2}$$

3 Local Asymptotic Stability of Eq.(1.1)

3.1 Local stability about $\overline{x} = \frac{\beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}}{2B}$

Eq.(1.1) consists a positive equilibrium $\overline{x} = \frac{\beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}}{2B}$ under the condition $\beta > A$.

The characteristic equation about $\overline{x}$ is

$$\lambda^{k+1} - \left(\frac{2\beta}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}\right) \lambda^k + \left(\frac{2B\alpha + \beta\left(\beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}\right)}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}\right) = 0.$$

Lets apply Theorem1.2 under the both cases $A \geq \beta$ and $A < \beta$

(1) The case $A < \beta$

$$\left|\frac{2\beta}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}\right| + \left|\frac{2B\alpha + \beta\left(\beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}\right)}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}\right| < 1,$$

$$= \left|\frac{2\beta + 2B\alpha + \beta\left(\beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}\right)}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}}\right| < 1,$$

$$= \frac{\beta(2 + \beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}) + 2B\alpha}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}} < 1$$

Then the necessary condition

3.2

$$\frac{\beta(2+\beta-A+\sqrt{(A-\beta)^2+4B\alpha})+2B\alpha}{A+\beta+\sqrt{(A-\beta)^2+4B\alpha}} < 1,$$

$$\Longrightarrow$$

$$\beta(2 + \beta - A + \sqrt{(A-\beta)^2 + 4B\alpha}) + 2B\alpha < A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}$$

or

$$\frac{(\beta - 1)\sqrt{(A-\beta)^2 + 4B\alpha} + \beta(1+\beta) - A + 2B\alpha}{\beta} < A$$

The case $A \geq \beta$

$$\frac{\beta(2 - \beta + A + \sqrt{(A-\beta)^2 + 4B\alpha}) + 2B\alpha}{A + \beta + \sqrt{(A-\beta)^2 + 4B\alpha}} < 1$$

or

$$A < \frac{(1-\beta)\sqrt{(A-\beta)^2 + 4B\alpha} + \beta(\beta - 1) + A - 2B\alpha}{\beta}$$

Can be written after combining these two cases.

$$\frac{(\beta-1)\sqrt{(A-\beta)^2+4B\alpha}+\beta(1+\beta)-A+2B\alpha}{\beta} < A < \frac{(1-\beta)\sqrt{(A-\beta)^2+4B\alpha}+\beta(\beta-1)+A-2B\alpha}{\beta}$$

we can deduce the following theorem

Either k is even or k is odd, the unique positive equilibrium point $\overline{x} = \frac{\beta-A+\sqrt{(A-\beta)^2+4B\alpha}}{2B}$ will be locally asymptotically stable if and only if $0 < B < \frac{\beta^2}{\alpha}$ and $A > \beta$

Computationally, we wish to adumbrate the parameters (A, B) and (α, β) such that the fixed point is locally asymptotically stable. A set of 5000 parameters (A, B) and (α, β) such that the fixed point is locally asymptotically stable which are shown in the following Fig. 2 (left, right).

4 Existence of Periodic solution

In this section, we will investigate positive prime period two solution of Eq.(1.1). Let us assume the two cycle period of Eq.(1.1) will be in the form

$$...p,\ q,\ p,\ q...$$

if k is odd then,

$$x_{n+1} = x_{n-k} \tag{4.1}$$

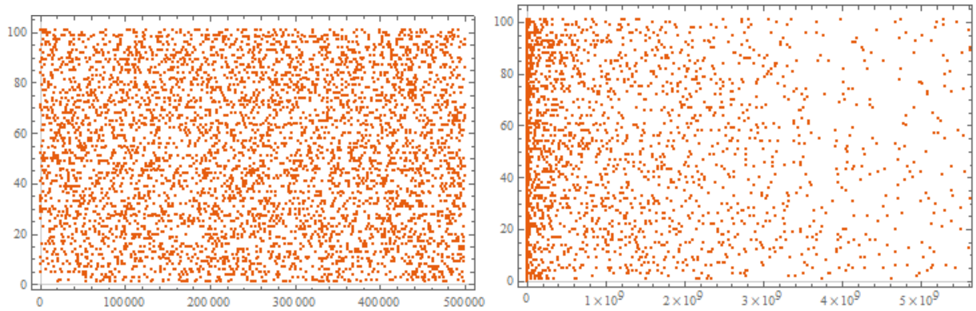

Figure 2: Set of partameters (A, B) and (α, β) (up, down)

we get,

$$\begin{aligned} p &= \frac{\alpha + \beta p}{A + Bp} \\ q &= \frac{\alpha + \beta q}{A + Bq} \end{aligned}$$

This transform to

$$\begin{aligned} p(A + Bp) &= \alpha + \beta p \\ q(A + Bq) &= \alpha + \beta q \end{aligned}$$

$$p(A + Bp - \beta) = \alpha \;\; and \;\; q(A + Bq - \beta) = \alpha$$

by subtracting we get,

$$(p - q)(A + B(p + q) - \beta) = 0$$

Then, either

$$p = q \quad or \quad (p + q) = \frac{\beta - A}{B}$$

Which is not possible for the case $\beta < A$. Since p and q both are non-negative, so we have in this case no prime period two solution of our equation.

if k is even then,

$$x_n = x_{n-k}$$

So,

$$\begin{aligned} p &= \frac{\alpha + \beta q}{A + Bq} \\ q &= \frac{\alpha + \beta p}{A + Bp} \end{aligned}$$

or

$$\begin{aligned} p(A + Bq) &= \alpha + \beta q \;\; or \\ q(A + Bp) &= \alpha + \beta p \end{aligned}$$

subtracting these two we have

$$(p-q)(A+\beta)=0$$

So, either $p = q$ *or* $A = -\beta$.(not-possible), so in this case there exists no prime period two solution for our equation. We can conclude the following thus.

There is no period two non-negative solution of Eq.(1.1)under any condition however there may exist negative prime period two solution of Eq.(1.1) as shown in our numerical discussion.

Although the equation Eq.(1.1) does not have any prime period 2 solution for any positive parameters A, B, α and β, but it possesses periodic solutions with different periods for combination of negative parameters as shown below.

$\alpha = 95$, $\beta = -72$, $A = 72$ and $B = 53$ with $k = 2$ (seems for any integer k) the trajectory is *periodic* of period 6 of which trajectory is given in the Fig. 3 (Up).

Similarly, $\alpha = 98$, $\beta = -71$, $A = 71$ and $B = 58$ with $k = 28$ (seems for any integer k) the trajectory is *periodic* of period 117 of which trajectory is given in the Fig. 3 (down).

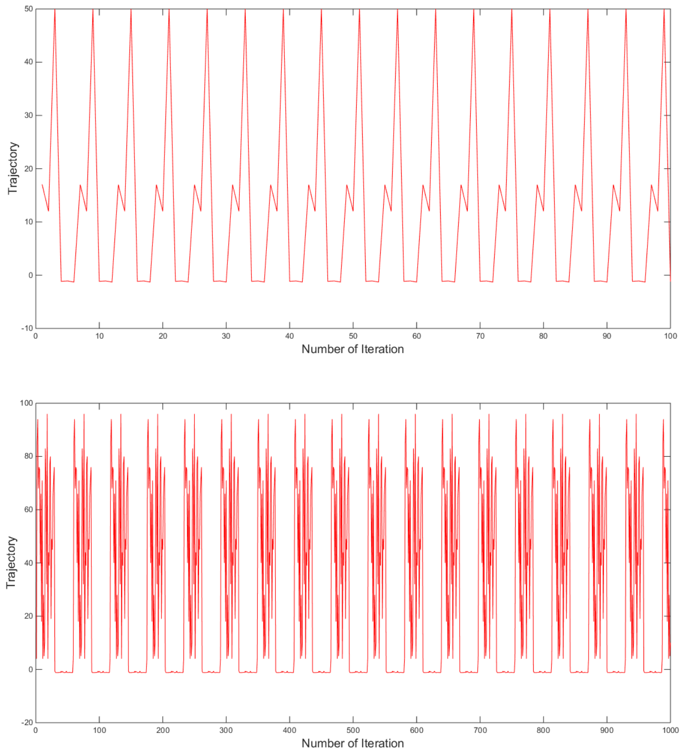

Figure 3: Periodic trajectories of period 6 and 117 respectively (up, down)

5 Global Stability of Eq.(1.1)

In this section we will study the global stability character of the solutions of Eq.(1.1).

The equilibrium point $\bar{x}$ of Eq.(1.1) is a global attractor.

Proof. Let $p,\ q$ are real numbers and assume that $f : [p,q]^2 \to [p,q]$ be a function defined by

$$f(v,w) = \frac{\alpha + \beta v}{A + Bw}$$

we can easily see that the function is increasing in v and decreasing in w. Suppose that $(m,,M)$ is a solution of the system

$$m = f(m,M) \quad \text{and} \quad M = f(M,m)$$

Then from Eq.(1.1) we see that

$$m = \frac{\alpha + \beta m}{A + BM} \ \textit{and} \ M = \frac{\alpha + \beta M}{A + Bm}$$

by subtracting we get

$$A(m - M) = \beta(m - M)$$

and

$$(m - M)(A - \beta) = 0, \ \ A \neq \beta.$$

Thus

$$m = M.$$

It follows by Theorem that $\bar{x}$ is a global attractor of Eq.(1.1) and then the proof is complete. □

6 Boundedness of Solution of Eq.(1.1)

In this section we will study the boundedness of solutions of Eq.(1.1).

Every solution of Eq.(1.1) is boumded and persist.

Proof. Every solution of Eq. (1.1) is bounded and persists. Let $\{x_n\}_{n=-p}^{\infty}$ be a solution of Eq. (1.1). It follows from Eq. (1.1) that

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}} = \frac{\alpha}{A + Bx_{n-k}} + \frac{\beta x_{n-k}}{A + Bx_{n-k}}.$$

Hence

$$x_n \leq \frac{\alpha}{A} + \frac{\beta}{B} = M \quad \forall \ n \geq 1. \tag{6.1}$$

Now we wish to show that there exists $m > 0$ such that

$$x_n \geq m \quad \forall \ n \geq 1$$

The transformation

$$x_n = \frac{1}{y_n}$$

will lead Eq.(1.1) to the equivalent form

$$\begin{aligned} y_{n+1} &= \frac{B + Ay_{n-k}}{\beta + \alpha y_{n-k}} \\ &= \frac{B}{\beta + \alpha y_{n-k}} + \frac{Ay_{n-k}}{\beta + \alpha y_{n-k}}. \end{aligned}$$

It follows that

$$y_{n+1} \leq \frac{B}{\beta} + \frac{A}{\alpha} = \frac{B\alpha + A\beta}{\alpha\beta} = H \qquad \forall\, n \geq 1.$$

Thus we obtain

$$x_n = \frac{1}{y_n} \geq \frac{1}{H} = \frac{\alpha\beta}{B\alpha + A\beta} = m \quad for \ all \ \ n \geq 1. \tag{6.2}$$

From (6.1) and (6.2) we conclude that

$$m \leq x_n \leq M \qquad for \ all \ \ n \geq 1.$$

Therefore, every solution of Eq. (1.1) is bounded and persists.

□

7 Conclusion

The dynamics of a rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}}, \quad n = 0, 1, ...,$$

is studied. The global stability of the unique positive fixed point and boundedness are adumbrated. In near future, we wish to extend the parameters to any real number in order to gather other kind of dynamics (viz. higher order periodic solutions, chaotic behavior(if any)) of the rational difference equation. Also, we would like to explore an addaptive rational difference equation

$$x_{n+1} = ax_n + \frac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}}, \quad n = 0, 1, ...,$$

Body Math

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