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$\mathcal{AG}_{\mathcal{J}*}$-sets, $\mathcal{B}G_{\mathcal{J}*}$-sets and $\delta\beta_I$-open sets in ideal topological spaces

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ABSTRACT. The aim of this paper is to introduce and study the notions of $\mathcal{G}_{\mathcal{J}*}$-sets, $\mathcal{AG}_{\mathcal{J}*}$-sets and $\mathcal{B}G_{\mathcal{J}*}$-sets sets in ideal topological spaces. Properties of $\mathcal{G}_{\mathcal{J}*}$-sets, $\delta - C$-sets, $\mathcal{AG}_{\mathcal{J}*}$-sets, $\mathcal{B}G_{\mathcal{J}*}$-sets and $\mathcal{E}_{\mathcal{J}*}$-open sets are investigated. Moreover, the relationships among these sets are investigated.

1 Introduction

In this paper, $\mathcal{G}_{\mathcal{J}*}$-sets, $\mathcal{AG}_{\mathcal{J}*}$-sets, $\mathcal{B}G_{\mathcal{J}*}$-sets and $\mathcal{E}_{\mathcal{J}*}$-open sets in ideal topological spaces are introduced and studied. The relationships and properties of $\mathcal{G}_{\mathcal{J}*}$-sets, $\delta - C$-sets, $\mathcal{AG}_{\mathcal{J}*}$-sets, $\mathcal{B}G_{\mathcal{J}*}$-sets and $\mathcal{E}_{\mathcal{J}*}$-open sets in ideal topological spaces are investigated.

An ideal $\mathcal{J}$ on a nonempty set X is a nonempty collection of subsets of X which satisfies the following conditions: $A \in \mathcal{J}$ *and* $B \subset A$ implies $B \in \mathcal{J}$; $A \in \mathcal{J}$ *and* $B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$ [9]. Applications to various fields were further investigated by Jankovic and Hamlett [8]; Mukherjee et al. [10]; Arenas et al. [5]; Nasef and Mahmoud [11], etc. Given a topological space $(X, \mathcal{J})$ with an ideal $\mathcal{J}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [9] of A with respect to τ and $\mathcal{J}$ is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{J}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{J} \text{ for every } U \in \tau(x)\}$$

, where $\tau(x) = \{U \in \tau \mid x \in U\}$. Furthermore $Cl^*(A) = A \cup A^*(\mathcal{J}, \tau)$ defines a Kuratowski closure operator for the topology τ^*, called the $*$-topology, finer than τ. When there is no chance for confusion, we will simply write

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A^* for $A^*(\mathfrak{I},\tau)$. X^* is often a proper subset of X. A topological space (X,τ) with an ideal $\mathfrak{I}$ on X is called an ideal topological space and is denoted by $(X,\tau,\mathfrak{I})$. For a subset A of X, $Cl(A)$ and $Int(A)$ will denote the closure and the interior of A in (X,τ), respectively.

A subset A of an ideal topological space $(X,\tau,\mathfrak{I})$ is said to be R-I-open (resp. R-I-closed)[12] if $A = Int(Cl^*(A))$ $(resp.\ A = Cl^*(Int(A))$. A point $x \in X$ is called a δ-$\mathfrak{I}$-cluster point of A if $Int(Cl^*(U)) \cap A \neq \emptyset$ for each open set U containing x. The family of all δ-$\mathfrak{I}$-cluster points of A is called the δ-$\mathfrak{I}$-closure of A and is denoted by $\delta Cl_I(A)$. The δ-$\mathfrak{I}$-interior of A is defined by the union of all R-I-open sets of X contained in A and its denoted by $\delta Int_I(A)$. A subset A of X is said to be δ-$\mathfrak{I}$-closed (resp. δ-$\mathfrak{I}$-open) if $\delta Cl_I(A) = A$ (resp. $\delta Int_I(A) = A$) [12]. In this paper, δ-$\mathfrak{I}$-closed and δ-$\mathfrak{I}$-open are denoted by $\delta_{\mathfrak{I}}$-closed and $\delta_{\mathfrak{I}}$-open, respectively.

A subset A of an ideal topological space $(X,\tau,\mathfrak{I})$ is said to be e-$\mathfrak{I}$-open [4, 1] if $A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$. The complement of an e-$\mathfrak{I}$-open set is called an e-$\mathfrak{I}$-closed set [1, 2]. The intersection of all e-$\mathfrak{I}$-closed sets containing A is called the e-$\mathfrak{I}$-closure of A and is denoted by $Cl^*_e(A)$. The e-$\mathfrak{I}$-interior of A is defined by the union of all e-$\mathfrak{I}$-open sets contained in A and is denoted by $Int^*_e(A)$. The family of all e-$\mathfrak{I}$-open (resp. e-$\mathfrak{I}$-closed) sets of $(X,\tau,\mathfrak{I})$ containing a point $x \in X$ is denoted by $E\mathfrak{I}O(X,x)$ (resp. $E\mathfrak{I}C(X,x)$).

2 Preliminaries

Definition 2.1. *A subset A of an ideal topological space $(X,\tau,\mathfrak{I})$ is said to be*

1. *$\delta\alpha$-$\mathfrak{I}$-open [7] if $A \subset Int(Cl(\delta Int_I(A)))$.*
2. *$\delta\alpha^*$-$\mathfrak{I}$-set [7] if $\delta Int_I(A) = Int(Cl(\delta Int_I(A)))$.*
3. *semi*-$\mathfrak{I}$-open [7] if $A \subset Cl(\delta Int_I(A))$.*
4. *pre*-$\mathfrak{I}$-open [6] if $A \subseteq Int(\delta Cl_I(A))$.*
5. *e-$\mathfrak{I}$-open [1] if $A \subset Cl(\delta Int_I(A)) \cup Int(\delta Cl_I(A))$.*
6. *Strongly t-$\mathfrak{I}$-set [6] if $Int(A) = Int(\delta Cl_I(A))$.*
7. *$\delta\beta_I$-open [7] if $A \subset Cl(Int(\delta Cl_I(A)))$.*

The class of all *semi**-$\mathfrak{I}$-open (resp. *pre**-$\mathfrak{I}$-open, $\delta\alpha$-$\mathfrak{I}$-open, $\delta\beta_I$-open) sets of $(X,\tau,\mathfrak{I})$ is denoted by $S^*IO(X)$ (resp. $P^*IO(X)$, $\delta\alpha IO(X)$, $\delta\beta IO(X)$) [7, 6]. The complement of the above stated sets are their respective closed sets.

Lemma 2.2. *[7] Let A be a subset of an ideal topological space $(X,\tau,\mathfrak{I})$. Then*

1. *$U \cap \delta Cl_I(A) \subset \delta Cl_I(U \cap A)$ for any $\delta_{\mathfrak{I}}$-open set U in X,*
2. *$\delta Int_I(F \cup A) \subset F \cup \delta Int_I(A)$ for any $\delta_{\mathfrak{I}}$-closed set F in X,*

Lemma 2.3. *[1] Let $(X,\tau,\mathfrak{I})$ be an ideal topological space and let $A, U \subseteq X$. If A is e-$\mathfrak{I}$-open and $U \in \tau$. Then $A \cap U$ is e-$\mathfrak{I}$-open.*

Definition 2.4. *A subset A of an ideal topological space $(X,\tau,\mathfrak{I})$ is is called a $\delta - C$-set [7] if $A = U \cap V$, where U is a $\delta_{\mathfrak{I}}$-open set and V is a $\delta\alpha^*$-$\mathfrak{I}$-set.*

3 $\mathcal{AG}_{\mathcal{J}*}$-sets, $\mathcal{B}G_{\mathcal{J}*}$-sets and $\mathcal{E}_{\mathcal{J}*}$-sets

Definition 3.1. *A subset A of an ideal topological space $(X, \tau, \mathcal{J})$ is called*

1. *a $\mathcal{G}_{\mathcal{J}*}$-set if $A = U \cap V$, where U is $\delta_{\mathcal{J}}$-open and V is pre*-$\mathcal{J}$-closed.*
2. *a $\delta_{\mathcal{J}}$-locally closed set if $A = U \cap V$, where U is $\delta_{\mathcal{J}}$-open and V is $\delta_{\mathcal{J}}$-closed.*

Theorem 3.2. *For a subset A of an ideal topological space $(X, \tau, \mathcal{J})$, the following properties are equivalent:*

1. *A is a $\mathcal{G}_{\mathcal{J}*}$-set and a semi*-$\mathcal{J}$-open set in X.*
2. *$A = U \cap Cl(\delta Int_I(A))$ for a $\delta_{\mathcal{J}}$-open set U.*

Proof. (1) $\Rightarrow$ (2): Suppose that A is a $\mathcal{G}_{\mathcal{J}*}$-set and a *semi**-$\mathcal{J}$-open set in X. Since A is a $\mathcal{G}_{\mathcal{J}*}$-set, then we have $A = U \cap M$, where U is a $\delta_{\mathcal{J}}$-open set and M is a *pre**-$\mathcal{J}$-closed set in X. Since $A \subset M$, so $Cl(\delta Int_I(A)) \subset Cl(\delta Int_I(M))$. Since M is a *pre**-$\mathcal{J}$-closed set in X, we have $Cl(\delta Int_I(M)) \subset M$. Since A is a *semi**-$\mathcal{J}$-open set in X, we have $A \subset Cl(\delta Int_I(A))$. It follows that $A = A \cap Cl(\delta Int_I(A)) = U \cap M \cap Cl(\delta Int_I(A)) = U \cap Cl(\delta Int_I(A))$.
(2) $\Rightarrow$ (1): Let $A = U \cap Cl(\delta Int_I(A))$ for a $\delta_{\mathcal{J}}$-open set U. We have $A \subset Cl(\delta Int_I(A))$. It follows that A is a *semi**-$\mathcal{J}$-open set in X. Since $Cl(\delta Int_I(A))$ is a closed set, then $Cl(\delta Int_I(A))$ is a *pre**-$\mathcal{J}$-closed set in X. Hence, A is a $\mathcal{G}_{\mathcal{J}*}$-set in X. □

Definition 3.3. *A subset A of an ideal topological space $(X, \tau, \mathcal{J})$ is called an e-$\mathcal{J}$-regular set if A is e-$\mathcal{J}$-open and e-$\mathcal{J}$-closed.*

Definition 3.4. *A subset A of an ideal topological space $(X, \tau, \mathcal{J})$ is called*

1. *an $\mathcal{AG}_{\mathcal{J}*}$-set if $P = U \cap R$, where U is $\delta_{\mathcal{J}}$-open and R is e-$\mathcal{J}$-regular.*
2. *a $\mathcal{B}G_{\mathcal{J}*}$-set if $P = U \cap R$, where U is $\delta_{\mathcal{J}}$-open and R is e-$\mathcal{J}$-closed.*
3. *an $\mathcal{E}_{\mathcal{J}*}$-set if $P = U \cap R$, where U is $\delta_{\mathcal{J}}$-open set and R is semi*-$\mathcal{J}$-closed.*

Remark 3.5. *For a subset A of an ideal topological space $(X, \tau, \mathcal{J})$, the following diagram holds. The reverse implications in the diagram are not true in general as shown in the following examples.*

$$\begin{array}{ccccc} & & \mathcal{AG}_{\mathcal{J}*}\text{-set} & & \\ & & \downarrow & & \\ \mathcal{E}_{\mathcal{J}*}\text{-set} & \longrightarrow & \mathcal{B}G_{\mathcal{J}*}\text{-set} & \longleftarrow & \mathcal{G}_{\mathcal{J}*}\text{-set} \end{array}$$

Example 3.6. *Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{J} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Then the set $A = \{c, d\}$ is $\mathcal{B}G_{\mathcal{J}*}$-set and $\mathcal{G}_{\mathcal{J}*}$-set but it is not an $\mathcal{AG}_{\mathcal{J}*}$-set. The set $B = \{a, c, d\}$ is $\mathcal{B}G_{\mathcal{J}*}$-set and $\mathcal{AG}_{\mathcal{J}*}$-set but it is not an $\mathcal{E}_{\mathcal{J}*}$-set. The set $C = \{a, d\}$ is an $\mathcal{E}_{\mathcal{J}*}$-set but it is not an $\mathcal{AG}_{\mathcal{J}*}$-set.*

Example 3.7. *Let $X = \{a, b, c\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}\}$. Then the set $A = \{b\}$ is a $\mathcal{B}G_{\mathcal{J}*}$-set, but it is not a $\mathcal{G}_{\mathcal{J}*}$-set.*

Example 3.8. *Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{d\}, \{b\}, \{d, b\}\}$ and $\mathcal{J} = \{\emptyset\}$. Then the set $A = \{c, d\}$ is $\mathcal{AG}_{\mathcal{J}*}$-set and $\mathcal{E}_{\mathcal{J}*}$-set, but it is not a $\mathcal{G}_{\mathcal{J}*}$-set.*

Remark 3.9. *1) For an ideal topological space $(X, \tau, \mathcal{J})$, any $\delta_{\mathcal{J}}$-open set and any e-$\mathcal{J}$-regular set in X is an $\mathcal{AG}_{\mathcal{J}*}$-set but not reversible as shown in the following example.*
2) For an ideal topological space $(X, \tau, \mathcal{J})$, any $\delta_{\mathcal{J}}$-open set and any e-$\mathcal{J}$-closed set is a $\mathcal{B}G_{\mathcal{J}*}$-set but not reversible as shown in the following example.

Example 3.10. *Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{d\}, \{b\}, \{d, b\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then the set $A = \{a, c, d\}$ is both $\mathcal{AG}_{\mathcal{I}*}$-set and $\mathcal{B}G_{\mathcal{I}*}$-set, but it is not $\delta_{\mathcal{I}}$-open. The set $B = \{b, d\}$ is both a $\mathcal{B}G_{\mathcal{I}*}$-set and an $\mathcal{AG}_{\mathcal{I}*}$-set but it is neither e-$\mathcal{I}$-closed nor e-$\mathcal{I}$-regular.*

Theorem 3.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. If A is an $\mathcal{AG}_{\mathcal{I}*}$-set, then A is an e-$\mathcal{I}$-open set.*

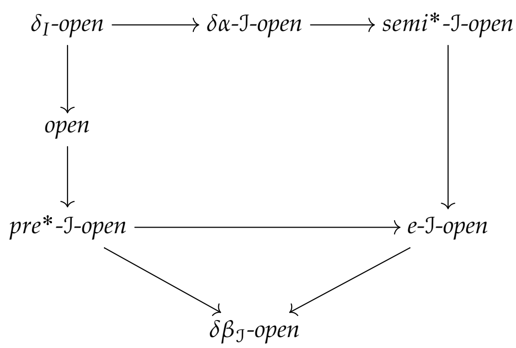
Proof. Suppose that $A \subset X$ is an $\mathcal{AG}_{\mathcal{I}*}$-set in X. It follows that $A = U \cap R$, where U is a $\delta_{\mathcal{I}}$-open set and R is an e-$\mathcal{I}$-open set in X. Since R is an e-$\mathcal{I}$-open set, then by Lemma 2.3, A is a e-$\mathcal{I}$-open set. □

Remark 3.12. *The reverse implication of Theorem 3.11 is not true in general as shown in the following example.*

Example 3.13. *Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{d\}, \{b\}, \{d, b\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then the set $A = \{a, c, d\}$ is an e-$\mathcal{I}$-open set, but it is not an $\mathcal{AG}_{\mathcal{I}*}$-set.*

Remark 3.14. *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the following diagram holds. The converses of these implications are not true as shown in [1].*

Remark 3.15. *[1]*



Theorem 3.16. *For a subset A of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*

1. *A is a $\delta\beta_I$-open set if and only if $A = A \cap Cl(Int(\delta_I Cl(A)))$.*
2. *A is a $\delta\beta_I$-closed set if and only if $A = A \cup Int(Cl(\delta Int_I(A)))$.*

Proof. (1) Suppose that A is a $\delta\beta_I$-open set in X. It follows that $A \subset Cl(Int(\delta Cl_I(A)))$ Then we have $A \subset A \cap Cl(Int(\delta Cl_I(A)))$. Also, $A \cap Cl(Int(\delta Cl_I(A))) \subset A$. Thus, $A = A \cap Cl(Int(\delta Cl_I(A)))$.
Conversely, suppose that $A = A \cap Cl(Int(\delta Cl_I(A)))$. Then we have
$A = A \cap Cl(Int(\delta Cl_I(A))) \subset Cl(Int(\delta Cl_I(A)))$. Thus, $A \subset Cl(Int(\delta Cl_I(A)))$ and hence, A is $\delta\beta_I$-open.
(2) The proof follows from (1). □

Lemma 3.17. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For any subset A of X, the following properties hold:*
(1) $Cl(A) \subset \delta Cl_I(A) \subset \delta Cl(A)$.
(2) $Cl(A) = \delta Cl_I(A) = \delta Cl(A)$ if A is an open set.
(3) $Int(A) = \delta Int_I(A) = \delta Int(A)$ if A is a closed set.

Proof. (1) Since $\tau \subset \tau^*$, $U \subset Int(Cl^*(U)) \subset Int(Cl(U))$ for every $U \in \tau$.
(2) Let $A \in \tau$ and $x \notin Cl(A)$. Then, there exists $U \in \tau$ such that $x \in U$ and $U \cap A = \emptyset$. Since $A \in \tau$, $Int(Cl(A)) \cap A = \emptyset$ and $x \notin \delta Cl(A)$. Therefore, we have $\delta Cl(A) \subset Cl(A)$ and hence (2) holds.
(3) This follows from (2). □

Theorem 3.18. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. The following properties are equivalent:*

1. *A is a $\delta\beta_I$-open set,*
2. *$\delta Cl(A) = \delta Cl_I(A) = Cl(Int(\delta Cl_I(A)))$.*

Proof. (1) $\Rightarrow$ (2) Suppose that A is a $\delta\beta_I$-open set in X. It follows that $A \subset Cl(Int(\delta Cl_I(A)))$. By Lemma 3.17 $\delta Cl_I(A) \subset \delta Cl(A) \subset \delta Cl(Cl(Int(\delta Cl_I(A))) \subset \delta Cl(Int(\delta Cl_I(A))) \subset \delta Cl_I(Int(\delta Cl_I(A))) \subset \delta Cl_I(A)$. By Lemma 3.17, we obtain

$$\delta Cl(A) = \delta Cl_I(A) = \delta Cl_I(Int(\delta Cl_I(A)))) = Cl(Int(\delta Cl_I(A))).$$

(2) $\Rightarrow$ (1) Let $\delta Cl_I(A) = Cl(Int(\delta Cl_I(A)))$. Then we have $A \subset \delta Cl_I(A) = Cl(Int(\delta Cl_I(A)))$. Thus, A is a $\delta\beta_I$-open set. □

Theorem 3.19. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. The following properties are equivalent:*

1. *A is a $\delta\beta_I$-open set;*
2. *there exists a pre*-$\mathcal{I}$-open set U in X such that $U \subset \delta_I Cl(A) \subset \delta_I Cl(U)$;*
3. *$\delta Cl_I(A) = Cl(Int(\delta Cl_I(A)))$;*

Proof. (1) $\Rightarrow$ (2) : Suppose that A is a $\delta\beta_I$-open set in X. We have $A \subset Cl(Int(\delta Cl_I(A)))$. We take $U = Int(\delta Cl_I(A))$. It follows that $U \subset Int(\delta Cl_I(U))$. Thus, U is a *pre**-$\mathcal{I}$-open set and also $U \subset \delta Cl_I(A) \subset \delta Cl_I(U)$.
(2) $\Rightarrow$ (3) : Suppose that there exists a *pre**-$\mathcal{I}$-open set U in X such that $U \subset \delta Cl_I(A) \subset \delta Cl_I(U)$. It follows from Lemma 3.17 that

$$\delta Cl_I(A) \subset \delta Cl_I(U) \subset Cl(Int(\delta Cl_I(U))) \subset Cl(Int(\delta Cl_I(A))) \subset \delta Cl_I(U) \subset \delta Cl_I(A).$$

Hence, $\delta Cl_I(A) = Cl(Int(\delta Cl_I(A)))$.
(3) $\Rightarrow$ (1) : The proof follows from Theorem 3.18. □

Remark 3.20. *[7] The intersection of any two $\delta\beta_I$-open sets need not be a $\delta\beta_I$-open set as shown example below.*

Example 3.21. *[7] Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}$ $, \{a, b, c\}\}$ and $\mathcal{I} = P(X)$. Then, $A = \{b, d\}$ and $B = \{a, c, d\}$ are $\delta\beta_I$-open but $A \cap B = \{d\}$ is not $\delta\beta_I$-open.*

Theorem 3.22. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. Then A is a $\delta\beta_I$-closed set if and only if A is a $\delta\alpha^*$-$\mathcal{I}$-set.*

Proof. Suppose that A is a $\delta\beta_I$-closed set in X. Then $X \backslash A$ is $\delta\beta_I$-open. It follows from Theorem 3.18 that $\delta Cl_I(X \backslash A) = Cl(Int(\delta Cl_I(X \backslash A)))$. Also,
$\delta Cl_I(X \backslash A) = X \backslash \delta Int_I(A) = Cl(Int(\delta Cl_I(X \backslash A))) = X \backslash Int(Cl(\delta Int_I(A)))$ and hence $\delta Int_I(A) = Int(Cl(\delta Int_I(A)))$. Thus, A is a $\delta\alpha^*$-$\mathcal{I}$-set. The converse is similar. □

Theorem 3.23. *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. Then A is a $\mathcal{B}G_{\mathcal{I}*}$-set if and only if $A = U \cap Cl^*_e(A)$ for a δ_I-open set U in X.*

Proof. Suppose that A is a $\mathcal{B}G_{\mathcal{I}*}$-set in X. It follows that $A = U \cap S$, where U is a δ_I-open set and S is an e-$\mathcal{I}$-closed set in X. Since $A \subset S$, then we have $Cl^*_e(A) \subset Cl^*_e(S) = S$. It implies

$$U \cap Cl^*_e(A) \subset U \cap S = A \subset U \cap Cl^*_e(A)$$

Thus, $A = U \cap Cl^*_e(A)$.
Conversely, let $A = U \cap Cl^*_e(A)$ for a δ_I-open set U. Since $Cl^*_e(A)$ is e-$\mathcal{I}$-closed, then A is a $\mathcal{B}G_{\mathcal{I}*}$-set in X. □

Theorem 3.24. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space. For a subset A of X, the following properties are equivalent:*

1. *A is a δ_I-open set,*
2. *A is a $\delta\alpha$-$\mathfrak{I}$-open set and an $\mathcal{AG}_{\mathfrak{I}*}$-set,*
3. *A is a $\delta\alpha$-$\mathfrak{I}$-open set and a $\mathcal{B}G_{\mathfrak{I}*}$-set.*

Proof. (1) ⇒ (2): It follows from the fact that any δ_I-open set is a $\delta\alpha$-$\mathfrak{I}$-open set and an $\mathcal{AG}_{\mathfrak{I}*}$-set.
(2) ⇒ (3): It follows Remark 3.5.
(3) ⇒ (1): Suppose that A is a $\delta\alpha$-$\mathfrak{I}$-open set and a $\mathcal{B}G_{\mathfrak{I}*}$-set. Then $A = U \cap S$, where U is a δ_I-open set and S is an e-$\mathfrak{I}$-closed set. By Remark 3.14 and Theorem 3.22, S is a $\delta\alpha^*$-$\mathfrak{I}$-set. Thus, A is a $\delta - C$-set in X. Since A is a $\delta - C$-set and a $\delta\alpha$-$\mathfrak{I}$-open set, then by Theorem 4 [7], A is a δ_I-open set. □

4 I_δ-submaximality

Definition 4.1. *An ideal topological space $(X, \tau, \mathfrak{I})$ is said to be $\delta\mathfrak{I}$-extremally disconnected if $\delta Cl_I(A) \in \tau$ for each $A \in \tau$.*

Lemma 4.2. *An ideal topological space $(X, \tau, \mathfrak{I})$ is $\delta\mathfrak{I}$-extremally disconnected if and only if it is extremally disconnected.*

Proof. This follows from Lemma 3.17. □

Theorem 4.3. *For an ideal topological space $(X, \tau, \mathfrak{I})$, the following properties are equivalent:*

1. *X is $\delta\mathfrak{I}$-extremally disconnected,*
2. *$\delta Int_I(A)$ is closed for every closed subset A of X,*
3. *$\delta Cl_I(Int(A)) \subset Int(\delta Cl_I(A))$ for every subset A of X,*
4. *Every semi*-$\mathfrak{I}$-open set is pre*-$\mathfrak{I}$-open.*
5. *The $\delta_{\mathfrak{I}}$-closure of every $\delta\beta_{\mathfrak{I}}$-open subset of X is open.*
6. *Every $\delta\beta_{\mathfrak{I}}$-open is pre*-$\mathfrak{I}$-open.*
7. *For every subset A of X, A is $\delta\alpha$-$\mathfrak{I}$-open if and only if it is semi*-$\mathfrak{I}$-open.*

Proof. (1) ⇒ (2) : Let $A \subset X$ be a closed set. Then $X \backslash A$ is open. By (1) $\delta Cl_I(X \backslash A) = X \backslash \delta Int_I(A)$ is open. Thus $\delta Int_I(A)$ is closed.
(2) ⇒ (3) : Let A be any set in X. Then $X \backslash Int(A)$ is closed in X and by (2) $\delta Int_I(X \backslash Int(A))$ is closed in X. Therefore $\delta Cl_I(Int(A))$ is open in X and hence, $\delta Cl_I(Int(A)) \subset Int(\delta Cl_I(A))$.
(3) ⇒ (4) : Let A be any *semi**-$\mathfrak{I}$-open set. By (3), we have $A \subset Cl(\delta Int_I(A)) \subset \delta Cl_I(Int(A)) \subset Int(\delta Cl_I(A))$. Thus, A is *pre**-$\mathfrak{I}$-open.
(4) ⇒ (5) : Let $A \subset X$ be a $\delta\beta_{\mathfrak{I}}$-open set. By Theorem 3.18 and Lemma 3.17, $\delta Cl_I(A) = Cl(Int(\delta_I(A))) = Cl(\delta Int_I(\delta Cl_I(A)))$. Then $\delta Cl_I(A)$ is *semi**-$\mathfrak{I}$-open. By (4) $\delta Cl_I(A)$ is *pre**-$\mathfrak{I}$-open. Thus $\delta Cl_I(A) \subset Int(\delta Cl_I(A))$ and hence $\delta Cl_I(A)$ is open.
(5) ⇒ (6) : Let A be $\delta\beta_{\mathfrak{I}}$-open. By (5) $\delta Cl_I(A) = Int(\delta Cl_I(A))$. Thus $A \subset \delta Cl_I(A) = Int(\delta Cl_I(A))$ and hence A is *pre**-$\mathfrak{I}$-open.
(6) ⇒ (7) : Let A be a *semi**-$\mathfrak{I}$-open set. Since a *semi**-$\mathfrak{I}$-open set is $\delta\beta_{\mathfrak{I}}$-open, then by (6) A is *pre**-$\mathfrak{I}$-open. Since A is *semi**-$\mathfrak{I}$-open and *pre**-$\mathfrak{I}$-open, A is $\delta\alpha$-$\mathfrak{I}$-open.
(7) ⇒ (1) : A be an open set of X. Then A is $\delta\beta_I$-open and $\delta Cl_I(A)$ is *semi**-$\mathfrak{I}$-open and by (7) $\delta Cl_I(A)$ is $\delta\alpha$-$\mathfrak{I}$-open. Therefore $\delta Cl_I(A) \subset Int(Cl(\delta Int_I(\delta Cl_I(A)))) \subset Int(\delta Cl_I(A))$ and hence $\delta Cl_I(A) = Int(\delta Cl_I(A))$. Hence $\delta Cl_I(A)$ is open and X is $\delta\mathfrak{I}$-extremally disconnected. □

Theorem 4.4. *Let $(X, \tau, \mathcal{I})$ be $\delta\mathcal{I}$-extremally disconnected. For a subset A of X, the following properties hold:*

1. *A is a $\mathcal{B}G_{\mathcal{I}*}$-set if and only if A is a $\mathcal{G}_{\mathcal{I}*}$-set,*
2. *Any $\mathcal{E}_{\mathcal{I}*}$-set is a $\mathcal{G}_{\mathcal{I}*}$-set.*

Proof. (1) Suppose that A is a $\mathcal{B}G_{\mathcal{I}*}$-set in X. It follows that $A = U \cap S$, where U is a $\delta\mathcal{I}$-open set and S is an e-$\mathcal{I}$-closed set in X. Then $X \backslash S$ is an e-$\mathcal{I}$-open set in X. We have $X\backslash S \subset Cl(\delta Int_I(X\backslash S)) \cup Int(\delta Cl_I(X\backslash S))$. Since $(X, \tau, \mathcal{I})$ is $\delta\mathcal{I}$-extremally disconnected, then by Theorem 4.3, $Cl(\delta Int_I(X\backslash S)) \subset Cl(Int(X\backslash S)) \subset \delta Cl_I(Int(X\backslash S)) \subset Int(\delta Cl_I(X\backslash S))$ and hence

$$X\backslash S \subset Cl(\delta Int_I(X\backslash S)) \cup Int(\delta Cl_I(X\backslash S)) = Int(\delta Cl_I(X\backslash S)).$$

Thus, $X\backslash S$ is a *pre**-$\mathcal{I}$-open set and hence S is a *pre**-$\mathcal{I}$-closed set. Consequently, A is a $\mathcal{G}_{\mathcal{I}*}$-set in X. The converse follows from Remark 3.5.

(2) Let A be an $\mathcal{E}_{\mathcal{I}*}$-set. Then $A = U \cap M$, where U is $\delta\mathcal{I}$-open and M is *semi**-$\mathcal{I}$-closed. Since X is $\delta\mathcal{I}$-extremally disconnected, then by Theorem 4.3, M is *pre**-$\mathcal{I}$-closed. Hence, A is a $\mathcal{G}_{\mathcal{I}*}$-set. □

Definition 4.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A be a subset of X. (1) A is said to be $\mathcal{I}_\delta$-dense if $\delta Cl_{\mathcal{I}}(A) = X$, (2) $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_\delta$-submaximal if every $\mathcal{I}_\delta$-dense set of X is δ_I-open.*

Lemma 4.6. *If $(X, \tau, \mathcal{I})$ is $\mathcal{I}_\delta$-submaximal, then $P^*IO(X) = \tau^{\delta_{\mathcal{I}}}$, where $\tau^{\delta_{\mathcal{I}}} = \{$all $\delta_{\mathcal{I}}$-open sets in $(X, \tau, I)\}$.*

Proof. It is obvious that $\tau^{\delta_{\mathcal{I}}} \subset P^*IO(X)$. Conversely, suppose that $A \in P^*IO(X)$. Then $A \subset Int(\delta Cl_{\mathcal{I}}(A))$ and let $D = Int(\delta Cl_{\mathcal{I}}(A))\backslash A$. Then

$$\begin{aligned}\delta Cl_{\mathcal{I}}(X\backslash D) =&\delta Cl_{\mathcal{I}}[(X\backslash Int(\delta Cl_{\mathcal{I}}(A))) \cup A]\\ =&\delta Cl_{\mathcal{I}}(X\backslash Int(\delta Cl_{\mathcal{I}}(A))) \cup \delta Cl_{\mathcal{I}}(A)\\ =&(X\backslash \delta Int_{\mathcal{I}}(Int(\delta Cl_{\mathcal{I}}(A)))) \cup \delta Cl_{\mathcal{I}}(A)\\ \supset&(X\backslash \delta Cl_{\mathcal{I}}(A)) \cup \delta Cl_{\mathcal{I}}(A)\\ =&X.\end{aligned}$$

Hence $(X\backslash D)$ is $\mathcal{I}_\delta$-dense and it is δ_I-open. Therefore, $A = Int(\delta Cl_{\mathcal{I}}(A)) \cap (X\backslash D)$ is δ_I-open. Consequently, we obtain $P^*IO(X) = \tau^{\delta_{\mathcal{I}}}$. □

Theorem 4.7. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

1. *X is $\mathcal{I}_\delta$-submaximal,*
2. *Every pre*-$\mathcal{I}$-open set is $\delta_{\mathcal{I}}$-open,*
3. *Every pre*-$\mathcal{I}$-open set is semi*-$\mathcal{I}$-open and every $\delta\alpha$-$\mathcal{I}$-open set is $\delta_{\mathcal{I}}$-open.*

Proof. (1) $\Rightarrow$ (2): It follows from Lemma 4.6.
(2) $\Rightarrow$ (3): Suppose that every *pre**-$\mathcal{I}$-open set is δ_I-open. Since every δ_I-open set is *semi**-$\mathcal{I}$-open, every *pre**-$\mathcal{I}$-open set is *semi**-$\mathcal{I}$-open. Let a subset A of X be a $\delta\alpha$-$\mathcal{I}$-open set. By Lemma 3.17, $A \subset Int(Cl(\delta Int_I(A))) = Int(\delta Cl_I(\delta Int_I(A))) \subset Int(\delta Cl_I(A))$. Therefore, every $\delta\alpha$-$\mathcal{I}$-open set is *pre**-$\mathcal{I}$-open, then by (2), A is δ_I-open.
(3) $\Rightarrow$ (1): Let A be a $\mathcal{I}_\delta$-dense subset of X. Then $\delta Cl_I(A) = X$ and A is *pre**-$\mathcal{I}$-open. By (3), A is *semi**-$\mathcal{I}$-open. Since A is *pre**-$\mathcal{I}$-open and *semi**-$\mathcal{I}$-open, by Lemma 3.17, $A \subset Int(\delta Cl_I(A)) \subset Int(\delta Cl_I(Cl(\delta Int_I(A)))) \subset$

$Int(\delta Cl_I(\delta Int_I(A))) = Int(Cl(\delta Int_I(A)))$. Therefore, A is $\delta\alpha$-$\mathcal{I}$-open. Thus, by (3), A is δ_I-open and hence X is $\mathcal{I}_\delta$-submaximal. □

Theorem 4.8. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*
(1) X is $\mathcal{I}_\delta$-submaximal and $\delta\mathcal{I}$-extremally disconnected,
(2) A subset of X is $\delta\beta_\mathcal{I}$-open if and only if it is δ_I-open.

Proof. (1) ⇒ (2): Let X be $\mathcal{I}_\delta$-submaximal and $\delta\mathcal{I}$-extremally disconnected. By Theorem 4.3, every $\delta\beta_\mathcal{I}$-open set is *pre**-$\mathcal{I}$-open. By Theorem 4.7, every *pre**-$\mathcal{I}$ open set is δ_I-open. Thus, every $\delta\beta_\mathcal{I}$-open set is δ_I-open. The converse follows from the fact that every δ_I-open set is $\delta\beta_\mathcal{I}$-open.
(2) ⇒ (1): Suppose that a subset of X is $\delta\beta_\mathcal{I}$-open if and only if it is δ_I-open. Since every $\delta\beta_\mathcal{I}$-open set is δ_I-open and so *pre**-$\mathcal{I}$-open, by Theorem 4.3, X is $\delta\mathcal{I}$-extremally disconnected. Since every *pre**-$\mathcal{I}$-open set is $\delta_\mathcal{I}$-open, by Theorem 4.7, X is $\mathcal{I}_\delta$-submaximal. □

Theorem 4.9. *Let an ideal topological space $(X, \tau, \mathcal{I})$ be $\mathcal{I}_\delta$-submaximal and $\delta\mathcal{I}$-extremally disconnected. Then, for a subset A of X, the following properties are equivalent:*

1. *A is $\delta\beta_\mathcal{I}$-open,*
2. *A is semi*-$\mathcal{I}$-open,*
3. *A is pre*-$\mathcal{I}$-open,*
4. *A is δ_I-open.*

Proof. This follows from Theorem 4.8. □

Theorem 4.10. *Let an ideal topological space $(X, \tau, \mathcal{I})$ be $\mathcal{I}_\delta$-submaximal and $\delta\mathcal{I}$-extremally disconnected. Then, for a subset A of X, the following properties are equivalent:*

1. *A is a $\mathcal{B}G_{\mathcal{I}*}$-set;*
2. *A is an $\mathcal{E}_{\mathcal{I}*}$-set;*
3. *A is a $\mathcal{G}_{\mathcal{I}*}$-set;*
4. *A is a $\delta_\mathcal{I}$-locally closed set.*

Proof. The proof follows from Theorem 4.9. □

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