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Some fixed point results in cone b-metric spaces

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ABSTRACT. In this paper, we proved some fixed point theorems of contraction mappings in cone b-metric spaces.

1 Introduction

Fixed point theory plays a basic role in application of many branches of mathematics. Finding a fixed point of contractive mapping becomes the center of strong research activity. There are many works about the fixed point of contractive maps (see, for example, [1,2]). In [2] Polish Mathematician Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle, in 1922.In [3], Bakhtin introduced b-metric spaces as a generalized of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in metric spaces. A lucid survey shows that there are many generalizations of metric spaces. One of them is b-metric space. The concept of b-metric space was introduced by Czerwik [11,12].Using this idea, he proved Banach fixed point theorem in b-metric spaces. Later on, many researchers including Aydi [8],Bota [9],Chug [10], Shi [20], Du [13], Kir [19], Huang and Zhang [4] introduced cone metric spaces as generalized of metric spaces, replacing the real number by an ordered Banach spaces and define cone metric spaces. Moreover, they proved some fixed point theorems for contraction mapping that

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expanded certain results of fixed point in metric spaces.Functional Analysis especially due to its extensive applications in random differential as well as random integral equations(see references) [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], and [30]. In [5], Hussain and Shah introduced cone b-metric spaces as a generalized of b-metric spaces and cone metric spaces. We prove some fixed point theorems in cone b-metric spaces on contraction mapping.

Before going to the main results, we define some definition, example and lemma required in sequel.

2 Preliminaries

Let E be real Banach space and P be a subset of E. Then P is called a cone if

1. P is closed , nonempty, and satisfies $P \neq \{0\}$
2. $ax + by \in P$ for all $x, y \in P$ and non-negative real number a, b
3. $x \in P$ and $-x \in P \Rightarrow x = \theta$ i.e.,$P \cap (-P) = 0$, by int$P$ the interior of P.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$. A cone p is a solid cone if $intP \neq \emptyset$. We write $\|.\|$ as the norm on E. The cone P is called normal if there is a number $k > 0$ such that $\forall x, y \in E, 0 \leq x \leq y \Rightarrow \|\|x\| \leq k\|y\|$. The least positive number k satisfying the above is called the normal constant of P.It is well known that $k \geq 1$.

In the following, we always suppose that E is a Banach space, P is cone in E with $intP \neq \emptyset$ and $\preceq$ is a partial ordering with respect to P.

Definition 2.1[4]: Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ Satisfies:

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all x, y in X,
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.2[5]: Let X be a nonempty set and $s \geq 1$ be a real number. Suppose that the mapping $d : X \times X \rightarrow E$ Satisfies:

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all x, y in X,
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a cone b-metric on X and (X, d) is called a cone b-metric space.

Remark 2.3: The class of cone b-metric spaces is larger than the class of cone metric spaces since any cone metric space must be a cone b-metric space. Therefore, it is obvious that cone b-metric space generalized b-metric spaces and cone metric spaces. Following is an example which shows that a cone b-metric spaces which are not cone metric spaces:

Example 2.4[7]: $E = R^2, P = \{(x,y) \in E : x,y \geq 0\} \subset E, X = R$ and $d : X \times X \to E$ such that $d(x,y) = (|x-y|^p, \alpha|x-y|^p)$ where $\alpha \geq 0$ and $p > 1$ are two constant. Then (X,d) is a cone b-metric space but not a cone metric space. In fact, we only need to prove (iii) in Definition2.2 as follows:

Let $x,y,z \in X$. Set $u = x-z, v = z-y$, so $x-y = u+v$ from the inequality $(a+b)^p \leq (2max\{a,b\})^p \leq 2^p(a^p+b^p)$ for all $a,b \geq 0$.

We have $|x-y|^p = |u+v|^p \leq (|u|+|v|)^p \leq 2^p(|u|^p+|v|^p) = 2^p(|x-z|^p+|z-y|^p)$,

which implies $d(x,y) \leq s[d(x,z)+d(z,y)]$ with $s = 2^p > 1$. But $|x-y|^p \leq |x-z|^p + |z-y|^p$ is impossible for all $x > y > z$, indeed, taking account of the inequality.

$$(a+b)^p > a^p + b^p \forall a,b > 0.$$

We arrive at

$$\begin{aligned} |x-y|^p &= |u+v|^p = (u+v)^p > u^p + v^p = (x-z)^p + (z-y)^p \\ &= |x-z|^p + |z-y|^p, \end{aligned}$$

for all $x > z > y$ Thus, (iii) in Definition 2.1 is not satisfied ,ie., (X,d) is not a cone metric space.

Example 2.5[7]: Let $X = l^p$ with $0 < p < 1$, where $l^p = \{\{x_n\} \subset R : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d : X \times X \to R^+$,

$$d(x,y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$$

Where $x = \{x_n\}, y = \{y_n\} \in l^p$. Then (x,d) is a b-metric space (see[5]). Put $E = l^p, P = \{\{x_n\} \in E : x_n \geq 0, \forall n \geq 1\}$. Letting the mapping $d : X \times X \to E$ be defined by $d(x,y) = \{\frac{d(x,y)}{2^n}\}_{n \geq 1}$, we conclude that (X,d) is a cone b-metric with coefficient $s = 2^{\frac{1}{p}} > 1$, but it is not a cone metric space.

Example 2.6[7]: Let $X = \{1,2,3,4\}, E = R^2, P = \{(x,y) \in E : x \geq 0, y \geq 0\}$. Define $d : X \times X \to E$ by

$$d(x,y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}) & \text{if } x \neq y \\ 0 \quad , \text{if } x = y \end{cases}$$

Then (X,d) is a cone b-metric space with the coefficient $s = \frac{6}{5}$. But it is not a cone metric space since the triangle inequality is not satisfied. Indeed, $d(1,2) > d(1,4) + d(4,2)$ and $d(3,4) > d(3,1) + d(1,4)$.

Definition 2.7[5]: Let (X,d) be a cone b-metric space, and $x \in X$ be a sequence in X. Then

1. $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$
2. $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $0 \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
3. (X,d) is a complete cone b-metric space if every Cauchy sequence is convergent.

Lemma2.8: Let (X,d) be a cone b-metric space and $\{x_n\} \in X$ such that

$$d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$$

Where $0 \leq k < 1$, then sequence $\{x_n\}$ is a Cauchy sequence.

3 Main Results

In 2013, Huaping Huang and shaoyuanXu[6] proved the following theorem: Let (X,d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $T : X \times X \to X$ satisfies the condition

$$d(Tx,Ty) \leq \alpha d(x,y), for \;\; x,y \in X.$$

Where $\alpha \in [0,1)$ is a constant .Then T has a unique fixed point in X. Furthermore, the iterative sequence $T^n x$ converges to the fixed point. Now we expended this theorem in cone b-metric space as.

Theorem3.1: Let (X,d) be a complete cone b-metric space with constant coefficient $s \geq 1$ and let $\alpha_i \geq 0 (i = 1,2,3...)$ and $T : X \times X$ be a mapping such that

$$d(Tx,Ty) \leq \alpha_1 d(x,y) + \alpha_2[d(x,Tx) + d(y,Ty)]\frac{d(x,y)+d(y,Ty)}{d(x,Ty)}$$
$$+ \alpha_3[d(x,Ty) + d(y,Tx)]\frac{[d(x,y)+d(y,Ty)+d(x,Ty)]^2}{[d(x,Ty)]^2}$$

For all $x,y \in X$ where $\alpha_1,\alpha_2,\alpha_3 > 0$, $s\alpha_1 + \frac{(1+s)}{s}\alpha_2 + \frac{(1+s)^2}{s^2}\alpha_2 \leq 1$. Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$ be a arbitrary point. Suppose there is an point $x_1 \in X$ such that $Tx_0 = x_1$. A sequence $\{x_n\}$ can be defined such that $Tx_0 = x_1$. A sequence $\{x_n\}$ can be defined such that $Tx_n = x_{n+1} (n \geq 0)$. If for some $n, x_n = x_{n+1}$, then x_n is a unique fixed point for mapping T. Therefore there is no need to go further. Suppose $x_{n+1} \neq x_n \forall n \geq 1$, Thus replacing x and y by x_{n-1} and x_n respectively in theorem (3.1), we have

$$\begin{aligned}
d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\
&\leq \alpha_1 d(x_{n-1},x_n) + \alpha_2[d(x_{n-1},x_n) + d(x_n,x_{n+1})]\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_{n-1},x_{n+1})} \\
&\quad + \alpha_3[d(x_{n-1},x_{n+1}) + d(x_n,x_{n+1})]\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_{n-1},x_{n+1})]^2}{[d(x_{n-1},x_{n+1})]^2} \\
&\leq \alpha_1 d(x_{n-1},x_n) + \alpha_2[d(x_{n-1},x_n) + d(x_n,x_{n+1})]\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{s[d(x_{n-1},x_n)+d(x_n,x_{n+1})]} \\
&\quad + \alpha_3[d(x_{n-1},x_{n+1}) + d(x_n,x_{n+1})]\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_{n-1},x_{n+1})]^2}{s^2[d(x_{n-1},x_n)+d(x_n,x_{n+1})]^2} \\
&\leq \alpha_1 d(x_{n-1},x_n) + \alpha_2\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{s} \\
&\quad + \alpha_3\frac{(1+s)^2}{s^2}[d(x_{n-1},x_n)+d(x_n,x_{n+1})]
\end{aligned}$$

Thus, $[1 - \frac{\alpha_2}{s} - \alpha_3\frac{(1+s)^2}{s^2}]d(x_n,x_{n+1}) \leq [\alpha_1 + \frac{\alpha_2}{s} + \alpha_3\frac{(1+s)^2}{s^2}]d(x_{n-1},x_n)$,
which implies, $d(x_n,x_{n+1}) \leq \frac{[\alpha_1+\frac{\alpha_2}{s}+\alpha_3\frac{(1+s)^2}{s^2}]}{[1-\frac{\alpha_2}{s}-\alpha_3\frac{(1+s)^2}{s^2}]}d(x_{n-1},x_n)$.
$(s\alpha_1 + \frac{(1+s)}{s}\alpha_2 + \frac{(1+s)^2}{s^2}\alpha_3) \leq 1$,
which implies $\frac{[\alpha_1+\frac{\alpha_2}{s}+\alpha_3\frac{(1+s)^2}{s^2}]}{[1-\frac{\alpha_2}{s}-\alpha_3\frac{(1+s)^2}{s^2}]} \leq \frac{1}{s} = k$.
Thus, $d(x_n,x_{n+1}) \leq kd(x_{n-1},x_n)$.
Proceeding in the same manner up to n iteration, we have
$d(x_n,x_{n+1}) \leq kd(x_{n-1},x_n) \leq k^2 d(x_{n-2},x_{n-1}) \leq .... \leq k^n d(x_0,x_1)$.

For $n \geq 1$ and letting $m > n$, we have

$$\begin{aligned}
d(x_n, x_{n+m}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m})]) \\
&\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+m})]) \\
&\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+m})]) \\
&\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + s^4 d(x_{n+3}, x_{n+4}) \\
&\quad + ... + s^{m-1} d(x_{n+m-2}, x_{n+m-1}) + s^m d(x_{n+m-1}, x_{n+m}) \\
&\leq sk^n d(x_0, x_1) + s^2 k^{n+1} d(x_0, x_1) + s^3 k^{n+2} d(x_0, x_1) + s^4 k^{n+3} d(x_0, x_1) \\
&\quad + ... + s^{m-1} k^{n+m-2} d(x_0, x_1) + s^m k^{n+m-1} d(x_0, x_1) \\
&\leq sk^n (1 + sk + (sk)^2 + (sk)^3 + ... + (sk)^{m-1}) d(x_0, x_1) \\
&\leq sk^n \frac{1 - (sk)^m}{1 - sk} d(x_0, x_1) \\
&\leq sk^n \frac{1}{1 - sk} d(x_0, x_1)
\end{aligned}$$

Since $k \in [0, \frac{1}{s}) \Rightarrow sk^n \frac{1}{1-sk} d(x_0, x_1) \to 0$ as $n \to \infty \forall m \in N^+$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Therefore sequence $\{x_n\}$ converses to a point $x^* \in X$.Now we prove that x^* is the unique fixed point of T.

$$\begin{aligned}
d(x^*, Tx^*) &\leq s[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \\
&\leq s[d(x^*, x_{n+1}) + d(Tx_n, Tx^*)] \\
&\leq s\{d(x^*, Tx_n) + \alpha_1 d(x^*, x_n) + \alpha_2 [d(x_n, Tx_n) + d(x^*, Tx^*)] \frac{d(x_n, Tx_n) + d(x^*, Tx^*)}{s[d(x_n, Tx_n) + d(x^*, Tx^*)]}\} \\
&\quad + \alpha_3 [d(x_n, Tx^*) + d(x^*, Tx^*)] \frac{[d(x_n, Tx_n) + d(x^*, Tx^*) + s[d(x_n, Tx_n) + d(x^*, Tx^*)]]^2}{s^2 [d(x_n, Tx_n) + d(x^*, Tx^*)]^2} \\
&\leq s[d(x^*, Tx_n) + \alpha_1 d(x_n, x^*) + \alpha_2 \frac{d(x_n, x^*) + d(x^*, Tx^*)}{s} \\
&\quad + \alpha_3 (1+s)^2 \frac{d(x_n, x^*) + d(x^*, Tx^*)}{s^2}][1 - \frac{\alpha_2}{s} - \frac{\alpha_3 (1+s)^2}{s^2}] \\
&\leq s[d(x^*, x_{n+1}) + (\alpha_1 + \frac{\alpha_2}{s} - \frac{\alpha_3 (1+s)^2}{s^2})[d(x^*, x_{n+1}) + d(x_n, x_{n+1})]]
\end{aligned}$$

Which implies,

$$\begin{aligned}
(1 - \alpha_2 - \frac{\alpha_3 (1+s)^2}{s^2}) d(x^*, Tx^*) &\leq s(1 + \alpha_1 + \frac{\alpha_2}{s} - \frac{\alpha_3 (1+s)^2}{s^2}) d(x^*, x_{n+1}) \\
&\quad + s(\alpha_1 + \frac{\alpha_2}{s} - \frac{\alpha_3 (1+s)^2}{s^2}) d(x_n, x_{n+1})
\end{aligned}$$

Thus,

$$\begin{aligned}
(s - s\alpha_2 - \alpha_3 (1+s)^2) d(x^*, Tx^*) &\leq (s^2 + s^2 \alpha_1 + s\alpha_2 + \alpha_3 (1+s)^2) d(x^*, x_{n+1}) \\
&\quad + (s^2 \alpha_1 + s\alpha_1 + \alpha_3 (1+s)^2) d(x_n, x_{n+1})
\end{aligned}$$

Hence,

$$\begin{aligned}
d(x^*, Tx^*) \leq & \frac{(s^2 + s^2 \alpha_1 + s\alpha_2 + \alpha_3 (1+s)^2)}{(s - s\alpha_2 - \alpha_3 (1+s)^2)} d(x^*, x_{n+1}) \\
& + \frac{(s^2 \alpha_1 + s\alpha_1 + \alpha_3 (1+s)^2)}{(s - s\alpha_2 - \alpha_3 (1+s)^2)} d(x_n, x_{n+1})
\end{aligned}$$

Since $\{x_n\}$ is a Cauchy sequence, therefore for any $c \in intP$, we select an $n_i \in N$ for all $n \geq n_i$ such that

$$d(x^*, Tx_n) \ll \frac{(s^2\alpha_1 + s\alpha_1 + \alpha_3(1+s)^2)c}{2(s - s\alpha_2 - \alpha_3(1+s)^2)}$$

and

$$d(x_n, x^*) \ll \frac{(s^2\alpha_1 + s\alpha_2 + \alpha_3(1+s)^2)c}{2(s - s\alpha_2 - \alpha_3(1+s)^2)}$$

Thus, for any $c \in intP, d(x^*, Tx^*) \ll c$, for all $n \geq n_i$. Therefore x^* is fixed point of T.
i.e.,$x^* = Tx^*$.
Now we will prove that x^* is unique. For that let x' be another fixed point such that $x' = Tx'$. Now from theorem (3.1), we have

$$\begin{aligned}
d(x^*, x') &= d(Tx^*, Tx^{'})\\
&\leq \alpha_1 d(x^*, x^{'}) + \alpha_2[d(x^*, Tx^{*'}) + d(x', Tx'')]\frac{d(x^*, x^{''}) + d(x', Tx'')}{d(x^*, Tx'')}\\
&\quad + \alpha_3[d(x^*, Tx^{''}) + d(x', Tx^{*'})]\frac{d(x^*, x^{''}) + d(x', Tx'') + d(x^*, Tx'')}{d(x^*, Tx'')}\\
&\leq \alpha_1 d(x^*, x^{''}) + \alpha_2[d(x^*, x^{*'}) + d(x', x'')]\frac{d(x^*, x^{''}) + d(x', x'')}{d(x^*, x'')}\\
&\quad + \alpha_3[d(x^*, x^{''}) + d(x', x^{*'})]\frac{d(x^*, x^{''}) + d(x', x'') + d(x^*, x'')}{d(x^*, x'')}\\
&\leq \alpha_1 d(x^*, x') + 2\alpha_3 d(x^*, x')\\
&\leq (\alpha_1 + 2\alpha_3) d(x^*, x')
\end{aligned}$$

This is a contradiction. Hence $x^* = x'$, i.e. x^* is a unique fixed point of the mapping T. This completes the proof of the theorem.

Theorem 3.2: Let (X, d) be a complete cone b-metric space with constant coefficient $s \geq 1$ and let $(\alpha_i \geq 0(i = 1, 2, 3))$ and $T : X \times X \to X$ be a mapping such that

$$d(Tx, Ty) \leq \alpha_1 d(x, y) + \alpha_2 \frac{(d(x, Ty))^2}{d(x, y) + d(y, Ty)} + \alpha_3 \frac{(d(y, Ty))^2}{d(x, y) + d(x, Ty)}$$

for all $x, y \in X$ where $\alpha_1, \alpha_2, \alpha_3 > 0, s\alpha_1 + s^2(1+s)\alpha_2 + s^3(2+s)\alpha_2 \leq 1$. Then T has a unique fixed point in X.

Proof: Let $x_0 inX$ be a arbitrary point. Suppose there is an point $x_1 inX$ such that $Tx_0 = x_1$. A sequence $\{x_n\}$ can be defined such that $Tx_n = x_{n+1}$,, then x_n is a unique fixed point for mapping T. Therefore there is no need to go further. Suppose $x_{n+1} \neq x_n \forall n \geq 1$, Thus replacing x and y by x_{n-1} and $x_n$respectively in theorem (3.2), we have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \frac{(d(x_{n-1}, Tx_n))^2}{d(x_{n-1}, x_n) + d(x_n, Tx_n)} + \alpha_3 \frac{(d(x_n, Tx_n))^2}{d(x_{n-1}, x_n) + d(x_{n-1}, Tx_n)} \\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \frac{(d(x_{n-1}, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})} + \alpha_3 \frac{(d(x_n, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})} \\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 \frac{s^2(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_n, x_{n+1})} + \alpha_3 \frac{(d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1}))^2}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})} \\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 [s^2(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))^2] + \alpha_3 [s^2[d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})] \\
&\leq \alpha_1 d(x_{n-1}, x_n) + \alpha_2 [s^2(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))^2] \\
&\quad + \alpha_3 [s^2[d(x_{n-1}, x_n) + s[d(x_{n-1}, x_n + d(x_n, x_{n+1})]]
\end{aligned}$$

$$(1 - \alpha_2 s^2 - \alpha_3 s^3) d(x_n, x_{n+1}) \leq (\alpha_1 + \alpha_2 s^2 + \alpha_3 s^2 (1+s)) d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq \frac{(\alpha_1 + \alpha_2 s^2 + \alpha_3 s^2(1+s))}{(1 - \alpha_2 s^2 - \alpha_3 s^3)} d(x_{n-1}, x_n)$$

Since $s\alpha_1 + s^2(1+s)\alpha_2 + s^3(2+s)\alpha_2 \leq 1$, which implies,

$$\frac{(\alpha_1 + \alpha_2 s^2 + \alpha_3 s^2(1+s))}{(1 - \alpha_2 s^2 - \alpha_3 s^3)} \leq \frac{1}{s} = k; implies$$

$$d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n)$$

Proceeding in the same manner up to n iteration, we have

$d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq ... \leq k^n d(x_0, x_1)$ for $n \geq 1$ and letting $m > n$, we have

$$\begin{aligned}
d(x_n, x_{n+m}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m})] \\
&\leq s d(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+m})] \\
&\leq s d(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + s^3[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+m})] \\
&\leq s d(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + s^4 d(x_{n+3}, x_4) \\
&\quad + s^{m-1} d(x_{n+m-2}, x_{n+m-1}) + s^m d(x_{n+m-1}, x_{n+m}) \\
&\leq s k^n d(x_0, x_1) + s^2 k^{n+1} d(x_0, x_1) + s^3 k^{n+2} d(x_0, x_1) + s^4 k^{n+3} d(x_0, x_1) \\
&\quad s^{m-1} k^{n+m-2} d(x_0, x_1) + s^m k^{n+m-1} d(x_0, x_1) \\
&\leq s k^n (1 + sk + (sk)^2 + (sk)^3 + ... + (sk)^{m-1}) d(x_0, x_1) \\
&\leq s k^n \frac{1 - (sk)^m}{1 - sk} d(x_0, x_1) \\
&\leq s k^n \frac{1}{1 - sk} d(x_0, x_1)
\end{aligned}$$

Since $k \in [0, \frac{1}{s}) \Rightarrow s k^n \frac{1}{1-sk} d(x_0, x_1) \to 0$ as $n \to \infty \forall m \in N^+$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Therefore sequence $\{x_n\}$ converses to a point $x^* \in X$. Now we prove that x^* is the unique fixed point of T.

$$
\begin{aligned}
d(x^*, Tx^*) &\leq s[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)] \\
&\leq s[d(x^*, Tx_n) + d(Tx_n, Tx^*)] \\
&\leq s[d(x^*, Tx_n)\alpha_1 d(x^*, x_n) + \alpha_2 \frac{(d(x_n, Tx^*))^2}{d(x_n, x^*) + d(x^*, Tx^*)} + \alpha_3 \frac{(d(x^*, Tx^*))^2}{d(x_n, x^*), d(x_n, Tx^*)}] \\
&\leq s[d(x^*, Tx_n)\alpha_1 d(x^*, x_n) + \alpha_2 \frac{s^2(d(x_n, x^*) + d(x^*, Tx^*))^2}{d(x_n, x^*) + d(x^*, Tx^*)} + \alpha_3 \frac{s^2(d(x^*, Tx^*))^2}{d(x^*, Tx^*)}] \\
&\leq s[d(x^*, Tx_n)\alpha_1 d(x^*, x_n) + \alpha_2 s^2(d(x_n, x^*) + d(x^*, Tx^*)) + \alpha_3 s^2 d(x^*, Tx^*)
\end{aligned}
$$

which implies,

$$(1 - \alpha_2 s^2 - \alpha_3 s^2)d(x^*, Tx^*) \leq s^2 d(x^*, x_{n+1}) + s(\alpha_1 + \alpha_2 s^2 d(x^*, x_n)).$$

Thus,

$$d(x^*, Tx^*) \leq \frac{s^2}{(1 - \alpha_2 s^2 - \alpha_3 s^2)} d(x^*, x_{n+1}) + \frac{s(\alpha_1 + \alpha_2 s^2}{(1 - \alpha_2 s^2 - \alpha_3 s^2)} d(x^*, x_n)$$

Since $\{x_n\}$ is a Cauchy sequence, therefore for any $c \in intP$, we select an $n_i \in N$ for all $n \geq n_i$ such that

$$d(x^*, Tx_n) \ll \frac{s^2 c}{2(1 - \alpha_2 s^2 - \alpha_3 s^2)}$$

and

$$d(x_n, x^*) \ll \frac{s(\alpha_1 + \alpha_2 s^2)}{2(1 - \alpha_2 s^2 - \alpha_3 s^2)}$$

Thus, for any $c \in intP, d(x^*, Tx^*) \ll c \forall n \geq n_i$. Therefore x^* is fixed point of T. i.e.,$x^* = Tx^*$.

Now we will prove that x^* is unique. For that let x' be another fixed point such that $x' = Tx'$.

Now from theorem (3.2), we have

$$
\begin{aligned}
d(x^*, x') &\leq d(Tx^*, Tx^{''}) \\
&\leq \alpha_1 d(x^*, x^{''}) + \alpha_2 \frac{(d(x^*, Tx^{''}))^2}{d(x^*, x'') + d(x', Tx'')} + \alpha_3 \frac{(d(x', Tx^{''}))^2}{d(x^*, x'') + d(x^*, Tx'')} \\
&\leq \alpha_1 d(x^*, x^{''}) + \alpha_2 \frac{(d(x^*, x^{''}))^2}{d(x^*, x'') + d(x', x'')} + \alpha_3 \frac{(d(x', x^{''}))^2}{d(x^*, x'') + d(x^*, x'')} \\
&\leq \alpha_1 d(x^*, x^{''}) + \alpha_2 d(x^*, x^{''}) \\
&\leq (\alpha_1 + \alpha_2) d(x^*, x^{''})
\end{aligned}
$$

This is a contradiction. Hence $x^* = x'$ i.e.,x^* is a unique fixed point of the mapping T. This completes the proof.

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