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On Fuzzy Sets with Fuzzy Baire Property

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ABSTRACT. In this paper, the concept of fuzzy sets with the fuzzy Baire property in fuzzy topological spaces is introduced and studied. Several characterizations of fuzzy sets with the fuzzy Baire property, are established.

1. Introduction:

In order to deal with uncertainties, the idea of fuzzy sets, fuzzy set operations was introduced by L.A.Zadeh [14] in 1965. In 1968 C.L.Chang [4], applied basic concepts of general topology to fuzzy sets and introduced fuzzy topology. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. In the recent years, there has been a growing trend among many fuzzy topologists to introduce and study different forms of fuzzy sets in fuzzy topological spaces.

The concept of Baire sets in classical topology was introduced and studied by Andrzej Szymanski [2]. The concept of Baire spaces in fuzzy setting was introduced and studied by G.Thangaraj and S.Anjalmose in [8]. The purpose of this paper is to introduce the notion of fuzzy sets with fuzzy Baire property in fuzzy topological spaces. Several characterizations of fuzzy sets with fuzzy Baire property are obtained in fuzzy topological spaces.

2. Preliminaries:

In order to make the exposition self-contained, some basic notions and results used in the sequel are given. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang (1968). Let X be a

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non-empty set and I, the unit interval [0,1]. A fuzzy set λ in X is a function from X into I. The null set 0 is the function from X into I which assumes only the value 0 and the whole fuzzy set 1 is the function from X into I which takes 1 only.

Definition 2.1. [4] Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The interior and the closure of λ are defined as follows:

(i) $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$

(ii) $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.1. [1] Let λ be any fuzzy set in a fuzzy topological space (X, T). Then $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$.

Definition 2.2. [7] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is., $intcl(\lambda) = 0$, in (X, T).

Definition 2.3. [3] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy F_σ-set in (X, T) if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$ for $i \in I$.

Definition 2.4. [3] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy G_δ-set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in I$.

Definition 2.5. [7] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy dense set if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. That is, $cl(\lambda) = 1$, in (X, T).

Definition 2.6. [7] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category.

Definition 2.7. [9] Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is called a fuzzy residual set if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where the fuzzy sets $(\lambda_i)'s$ are such that $cl[int(\lambda_i)] = 1$ in (X, T).

Definition 2.8. [10] Let (X, T) be a fuzzy topological space. A fuzzy set λ defined on X is called a fuzzy Baire set, if $\lambda = \mu \wedge \delta$ where μ is a fuzzy open set and δ is a fuzzy residual set in (X, T).

Definition 2.9. [8] Let (X,T) be a fuzzy topological space. Then (X, T) is called a fuzzy Baire space if $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T).

Definition 2.10 [12] A fuzzy topological space (X, T) is called a fuzzy globally disconnected space if each fuzzy semi-open set in (X, T) is fuzzy open.

Definition 2.11 [6] A fuzzy topological space (X, T) is called a fuzzy P - space if every non-zero fuzzy G_δ - set in (X, T), is a fuzzy open set in (X, T).

Definition 2.12 [5] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy somewhere dense if $intcl(\lambda) \neq 0$ for a non-zero fuzzy set λ in (X, T).

Definition 2.13 [11] A fuzzy topological space (X, T) is called a fuzzy perfectly disconnected space if for any two non-zero fuzzy sets λ and μ defined on X with $\lambda \leq 1 - \mu$, $cl(\lambda) \leq 1 - cl(\mu)$, in (X, T).

Theorem 2.1. [8] *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*

(1) (X, T) is a fuzzy Baire space.

(2) $int(\lambda) = 0$ for every fuzzy first category set λ in (X, T).

(3) $cl(\mu) = 1$ for every fuzzy residual set μ in (X, T).

Theorem 2.2. [10] *If λ is a fuzzy Baire set in a fuzzy submaximal space (X, T), then $\lambda = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy G_δ-set in (X, T).*

Theorem 2.3. [10] *If λ is a fuzzy Baire set in a fuzzy globally disconnected space (X, T), then $\lambda = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy G_δ-set in (X, T).*

Theorem 2.4. [10] *If λ is a fuzzy Baire set in a fuzzy Baire space (X, T), then λ is not a fuzzy dense set in (X, T).*

Theorem 2.5. [10] *If λ is a fuzzy Baire set in a fuzzy submaximal and fuzzy P-space (X, T), then λ is a fuzzy open set in (X, T).*

Theorem 2.6. [10] *If λ is a fuzzy Baire set in a fuzzy globally disconnected and fuzzy P-space (X, T), then λ is a fuzzy open set in (X, T).*

Theorem 2.7. [12] *If λ is a fuzzy somewhere dense set in a fuzzy topological space (X, T), then there exist a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\lambda)$.*

Theorem 2.8. [12] *If λ is a fuzzy somewhere dense set in a fuzzy perfectly disconnected space (X, T), then $cl(\lambda)$ is a fuzzy pre-closed set in (X, T).*

3. Fuzzy sets with fuzzy Baire property

Definition 3.1. Let (X, T) be a fuzzy topological space. A fuzzy set λ defined on X is said to have the property of fuzzy Baire, if $\lambda = (\mu \wedge \delta) \vee \eta$, where μ is a fuzzy open set, δ is a fuzzy residual set and η is a fuzzy first category set in (X, T).

Example 3.1: Let $X = \{a, b, c\}$. Then the fuzzy sets α, β, γ, μ and η are defined on X as follows:

$\alpha : X \to [0, 1]$ defined as $\alpha(a) = 0.5; \alpha(b) = 0.4; \alpha(c) = 0.6.$

$\beta : X \to [0, 1]$ defined as $\beta(a) = 0.6; \beta(b) = 0.5; \beta(c) = 0.7.$

$\gamma : X \to [0, 1]$ defined as $\gamma(a) = 0.6; \gamma(b) = 0.6; \gamma(c) = 0.7.$

$\mu : X \to [0, 1]$ defined as $\mu(a) = 0.5; \mu(b) = 0.5; \mu(c) = 0.6.$

$\eta : X \to [0, 1]$ defined as $\eta(a) = 0.5; \eta(b) = 0.6; \eta(c) = 0.6.$

Then, $T = \{0, \alpha, \beta, \gamma, 1\}$ is a fuzzy topology on X. On computation, one can see that $cl(\alpha) = cl(\beta) = cl(\gamma) = cl(\mu) = 1 = cl(\eta)$; $cl(1 - \mu) = 1 - \alpha$; $cl(1 - \eta) = 1 - \alpha$; $int(1 - \alpha) = int(1 - \beta) = int(1 - \gamma) = int(1 - \mu) = int(1 - \eta) = 0$ and $int(\mu) = \alpha$, $int(\eta) = \alpha$. The fuzzy nowhere dense sets in (X, T) are $1 - \alpha, 1 - \beta, 1 - \gamma, 1 - \mu, 1 - \eta$ and $1 - \alpha = (1 - \alpha) \vee (1 - \beta) \vee (1 - \gamma) \vee (1 - \mu) \vee (1 - \eta)$ implies that $1 - \alpha$ is a fuzzy first category set in (X, T). Also, $cl[int(\alpha)] = cl(\alpha) = 1$; $cl[int(\beta)] = cl(\beta) = 1$; $cl[int(\gamma)] = cl(\gamma) = 1$; $cl[int(\mu)] = cl(\alpha) = 1$; $cl[int(\eta)] = cl(\alpha) = 1$. Then $\alpha = \alpha \wedge \beta \wedge \gamma \wedge \mu \wedge \eta$ implies that α is the fuzzy residual set in (X, T). Now, $[\alpha \wedge \alpha] \vee (1 - \alpha) = \alpha \vee (1 - \alpha) = \eta$; $[\beta \wedge \alpha] \vee (1 - \alpha) = \alpha \vee (1 - \alpha) = \eta$; $[\gamma \wedge \alpha] \vee (1 - \alpha) = \alpha \vee (1 - \alpha) = \eta$ and hence η is a fuzzy set having fuzzy Baire property.

Remark 3.1: A fuzzy set having the fuzzy Baire property need not be a fuzzy open set in a fuzzy topological space.

For, in example 3.1, η is a fuzzy set having the fuzzy Baire property in (X, T) but not a fuzzy open set in (X, T).

Proposition 3.1: Let (X, T) be a fuzzy topological space. If λ is a fuzzy set defined on X with fuzzy Baire property in (X, T), then $1 - \lambda = (\gamma \vee \beta) \wedge \alpha$, where γ is a fuzzy closed set, α is a fuzzy residual set and β is a fuzzy first category set in (X, T).

Proof: Let λ be a fuzzy set with fuzzy Baire property in (X, T). Then $\lambda = (\mu \wedge \delta) \vee \eta$, where $\mu \in T$, δ is a fuzzy residual set and η is a fuzzy first category set in (X, T). Then, $1 - \lambda = 1 - \{(\mu \wedge \delta) \vee \eta\} = [1 - (\mu \wedge \delta)] \wedge (1 - \eta) = [(1 - \mu) \vee (1 - \delta)] \wedge (1 - \eta)$. Let $1 - \mu = \gamma; 1 - \eta = \alpha$ and $1 - \delta = \beta$. Since η is a fuzzy first category set, α is a fuzzy residual set in (X, T). Also since δ is a fuzzy residual set in (X, T), β is a fuzzy first category set in (X, T). Then $1 - \lambda = (\gamma \vee \beta) \wedge \alpha$, where γ is a fuzzy closed set, α is a fuzzy residual set and β is a fuzzy first category set in (X, T).

Proposition 3.2: If λ is a fuzzy set with fuzzy Baire property in a fuzzy topological space (X, T), then $\lambda = [\wedge_{i=1}^{\infty}(\mu \wedge \delta_i)] \vee \eta$, where $cl[int(\delta_i)] = 1$ for $(i = 1 \; to \; \infty)$, $\mu \in T$ and η is a fuzzy first category set in (X, T).

Proof: Let λ be a fuzzy set with fuzzy Baire property in (X, T). Then $\lambda = (\mu \wedge \delta) \vee \eta$, where μ is a fuzzy open set, δ is a fuzzy residual set and η is a fuzzy first category set in (X, T). Since δ is a fuzzy residual set in (X, T), $\delta = \wedge_{i=1}^{\infty}(\delta_i)$, where $cl[int(\delta_i)] = 1$ in (X, T). Then $\mu \wedge \delta = \mu \wedge [\wedge_{i=1}^{\infty}\delta_i] = \wedge_{i=1}^{\infty}(\mu \wedge \delta_i)$ and hence $\lambda = [\wedge_{i=1}^{\infty}(\mu \wedge \delta_i)] \vee \eta$, where $cl[int(\delta_i)] = 1$ for $(i = 1 \; to \; \infty)$, $\mu \in T$ and η is a fuzzy first category set in (X, T).

Proposition 3.3: If λ is a fuzzy set with the fuzzy Baire property in a fuzzy topological space (X, T), then $int(\lambda) \neq 0$ in (X, T).

Proof: Let λ be a fuzzy set with the fuzzy Baire property in (X, T). Then $\lambda = (\mu \wedge \delta) \vee \eta$, where $\mu \in T$, δ is a fuzzy residual set and η is a fuzzy first category set in (X, T). Now $int(\lambda) = int[(\mu \wedge \delta) \vee \eta] \geq int(\mu \wedge \delta) \vee int(\eta) = [int(\mu) \wedge int(\delta)] \vee int(\eta) = [\mu \wedge int(\delta)] \vee int(\eta)$. Since $\mu, int(\delta)$ are fuzzy open sets, $\mu \wedge int(\delta)$ is also a fuzzy open set and hence $[\mu \wedge int(\delta)] \vee int(\eta)$ is a fuzzy open set in (X, T) and thus $int(\lambda) \neq 0$, in (X, T).

Proposition 3.4: If λ is a fuzzy set with the fuzzy Baire property in a fuzzy topological space (X, T), then λ is a fuzzy somewhere dense set in (X, T).

Proof: Let λ be a fuzzy set with the fuzzy Baire property in (X, T), then *by proposition 3.3* $int(\lambda) \neq 0$ in (X, T). Now $int(\lambda) \leq intcl(\lambda)$ implies that $intcl(\lambda) \neq 0$ and hence λ is a fuzzy somewhere dense set in (X, T).

Proposition 3.5: If λ is a fuzzy set with fuzzy Baire property in a fuzzy topological space (X, T), then λ is not a fuzzy nowhere dense set in (X, T).

Proof: Let λ be a fuzzy set with fuzzy Baire property in (X, T). Suppose that λ is a fuzzy nowhere dense set in (X, T). Then $intcl(\lambda) = 0$ in (X, T). But $int(\lambda) \leq intcl(\lambda)$, implies that $int(\lambda) = 0$, a contradiction *by proposition 3.3*. Hence λ is not a fuzzy nowhere dense set in (X, T).

Proposition 3.6: If λ is a fuzzy set with fuzzy Baire property in a fuzzy topological space (X, T), then there exists a fuzzy Baire set σ in (X, T) such that $\sigma \leq \lambda$.

Proof: Let λ be a fuzzy set with the fuzzy Baire property in (X, T). Then $\lambda = (\mu \wedge \delta) \vee \eta$, where $\mu \in T$, δ is a fuzzy residual set and η is a fuzzy first category set in (X, T). Let $\sigma = \mu \wedge \delta$. Since μ is a fuzzy open set and δ is a fuzzy residual set in (X, T), σ is a fuzzy Baire set in (X, T) and hence $\lambda = \sigma \vee \eta$ in (X, T). This implies that $\sigma \leq \lambda$ in (X, T).

Proposition 3.7: If λ is a fuzzy set with fuzzy Baire property in a fuzzy submaximal space (X, T), then $\mu \wedge \delta \leq \lambda$, where μ is a fuzzy open set and δ is a fuzzy G_δ-set in (X, T).

Proof: Let λ be a fuzzy set with fuzzy Baire property in (X, T). Then, *by proposition 3.6*, there exists a fuzzy Baire set σ in (X, T) such that $\sigma \leq \lambda$, in (X, T). Since (X, T) is a fuzzy submaximal space, *by theorem 2.2*, for the fuzzy Baire set σ in (X, T), $\sigma = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy G_δ-set in (X, T). Thus, $\mu \wedge \delta \leq \lambda$, where μ is a fuzzy open set and δ is a fuzzy G_δ-set in (X, T).

Proposition 3.8: If λ is a fuzzy set with fuzzy Baire property in a fuzzy globally disconnected space (X, T), then $\mu \wedge \delta \leq \lambda$, where μ is a fuzzy open set and δ is a fuzzy G_δ - set in (X, T).

Proof: Let λ be a fuzzy set with fuzzy Baire property in (X, T). Then, *by proposition 3.6*, there exists a fuzzy Baire set σ in (X, T) such that $\sigma \leq \lambda$, in (X, T). Since (X, T) is a fuzzy globally disconnected space, *by theorem 2.3*, for the fuzzy Baire set σ in (X, T), $\sigma = \mu \wedge \delta$, where μ is a fuzzy open set and δ is a fuzzy G_δ - set in (X, T). Thus, $\mu \wedge \delta \leq \lambda$, where μ is a fuzzy open set and δ is a fuzzy G_δ - set in (X, T).

Proposition 3.9: If λ is a fuzzy set with fuzzy Baire property in a fuzzy Baire space (X, T), then there exists a fuzzy Baire set σ with $cl(\sigma) \neq 1$ such that $\sigma \leq \lambda$, in (X, T).

Proof: Let λ be a fuzzy set with fuzzy Baire property in the fuzzy topological space (X, T). Then, *by proposition 3.6*, there exists a fuzzy Baire set σ in (X, T) such that $\sigma \leq \lambda$, in (X, T). Since (X, T) is a fuzzy Baire space, *by theorem 2.4*, σ is not a fuzzy dense set in (X, T). That is, $cl(\sigma) \neq 1$, in (X, T). Thus there exists a fuzzy Baire set σ with $cl(\sigma) \neq 1$ such that $\sigma \leq \lambda$ in (X, T).

Proposition 3.10: If λ is a fuzzy set with fuzzy Baire property in a fuzzy submaximal and fuzzy P-space (X, T), then there exists a fuzzy Baire and fuzzy open set σ in (X, T) such that $\sigma \leq \lambda$.

Proof: Let λ be a fuzzy set with fuzzy Baire property in (X, T). Then, *by proposition 3.6*, there exists a fuzzy Baire set σ in (X, T) such that $\sigma \leq \lambda$. Since (X, T) is a fuzzy submaximal and fuzzy P-space, *by theorem 2.5*, σ is a fuzzy open set in (X, T). Thus, if λ is a fuzzy set with fuzzy Baire property in (X, T), then there exists a fuzzy Baire and fuzzy open set σ in (X, T) such that $\sigma \leq \lambda$.

Proposition 3.11: If λ is a fuzzy set with fuzzy Baire property in a fuzzy globally disconnected space and fuzzy P-space (X, T), then there exists a fuzzy Baire and fuzzy open set σ in (X, T) such that $\sigma \leq \lambda$.

Proof: Let λ be a fuzzy set with fuzzy Baire property in (X, T). Then, *by proposition 3.6*, there exists a fuzzy Baire set σ in (X, T) such that $\sigma \leq \lambda$. Since (X, T) is a fuzzy globally and fuzzy P-space, *by theorem 2.6*, σ is a fuzzy open set in (X, T). Thus, if λ is a fuzzy set with Baire property in (X, T), then there exists a fuzzy Baire and fuzzy open set σ in (X, T) such that $\sigma \leq \lambda$.

Proposition 3.12: If λ is a fuzzy set with fuzzy Baire property in a fuzzy topological space (X, T), then there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\lambda)$.

Proof: Let λ be a fuzzy set with fuzzy Baire property in (X, T). Then, *by proposition 3.4*, λ is a fuzzy somewhere dense set in (X, T). *By theorem 2.7*, there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\lambda)$.

Proposition 3.13: If λ is a fuzzy set with the fuzzy Baire property in a fuzzy perfectly disconnected space (X, T), then $cl(\lambda)$ is a fuzzy pre-closed set in (X, T).

Proof: Let λ be a fuzzy set with fuzzy Baire property in (X, T). Then, *by proposition 3.4*, λ is a fuzzy somewhere dense set in (X, T). Since (X, T) is a fuzzy Perfectly disconnected space, *by theorem 2.8*, $cl(\lambda)$ is a fuzzy pre-closed set in (X, T).

4. Conclusion

In this paper, the concepts of fuzzy sets with the fuzzy Baire property in fuzzy topological space, are introduced and studied. Several characterizations of fuzzy sets with the fuzzy Baire property are established.

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