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Hölder regularity of the parabolic p-Laplacian

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ABSTRACT. This paper investigate quasi-linear parabolic p-Lapalace equation of the type

$$\frac{\partial\varphi}{\partial t} - \operatorname{div}(\frac{h(|\nabla\varphi|)\nabla\varphi}{|\nabla\varphi|}) = 0,$$

under suitable assumptions. We provide a single geometric setting for which a bounded weak solution φ, is locally *Hö*lder continuous in the Sobolve-Orlicz Space $W^1L^\phi(\Omega)$ using intrinsic scaling method.

1. Introduction

Natural phenomenon are mostly describe by partial differential equations, of whom, time-evolutionary problems are modelled by evolution equations. The p-Laplace equation is a second order quasi-linear partial differential equation with constant exponent. This equation is either time-dependent or time-independent. In this paper, we consider models arising in Newtonian and non-newtonian fluids evolving over time, and thus devoted to the study of regularity properties of bounded weak solutions of parabolic p-Laplace equation for which Hölder continuity of bounded weak solutions is our focus. A mathematically optimal space is provided to study the prototype

$$\frac{\partial\varphi}{\partial t} - \operatorname{div}(\frac{h(|\nabla\varphi|)\nabla\varphi}{|\nabla\varphi|}) = 0 \tag{1.1}$$

where h is nonegative monotonic increasing and continuous function define in-between two power functions $|\nabla\varphi|^{\iota_0-1}$ and $|\nabla\varphi|^{\iota_1-1}$ such that $1 < \iota_0 \leq \iota_1 < \infty$. This optimal setting is to combines both singular equations and

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degenerate equations using the method of intrinsic scaling.

The parabolic p-Laplace prototype is the equation

$$\frac{\partial\varphi}{\partial t} - \text{div}(|\nabla\varphi|^{p-2}\nabla\varphi) = 0, \text{ for } p \in (1,\infty) \tag{1.2}$$

Equation (1.1) is obtained from (1.2) and (1.2) is singular for $p \in (1,2)$ and degenerate for $p > 2$. Thus the method of intrinsic scaling has been successfully applied to (1.2) with separate treatments for the two cases as seen in [16]. To consider increasing functions trapped in between two power functions, (1.2) is generalized yielding (1.1) where h is a nonnegative and nondecreasing function $h \in C[0,\infty)$ with $h(0) = 0$. The primitive of h, defined as $\psi(s) = \int_0^s h(\sigma)d\sigma$, while the global ∇_2 and Δ_2 conditions imposed on h and ψ yields $\iota_0\psi(s) \le sh(s) \le \iota_1\psi(s)$, for any $s \in [0,\infty)$ and some constants $1 < \iota_0 \le \iota_1 < \infty$. When $\iota_0 = \iota_1 = p$, (1.2) is obtained, thus (1.2) becomes a special case of (1.1). It is shown that a bounded weak solution of the generalized p-Laplacian equation behaves in a similar way to the heat equation in an intrinsically scaled cylinder. In achieving this, two techniques are employed in our approach: the idea of "spreading positivity" from the theory of Harnack estimates and "the method of Intrinsic scaling".

Regularity and existence questions for partial differential equations dates back to Hilbert's list of 23 mathematical problems. De Giorgi [2] and De Benedetto et al [8] independently proved Hölder continuity of weak solutions thereby leading to a major turning point on regularity and a priori theories of elliptic and parabolic partial differential equations in divergence form. Improved De Giorgi's iterations was established by [14] by choosing appropriate convex test functions for nonnegative weak solutions.

The contribution of [2] assisted [12] to extend the results to quasi-linear equations of the form

$$\text{div}\mathcal{A}(x,\varphi,\nabla\varphi) = 0, \text{ in } \Omega \tag{1.3a}$$

with the assumptions

$$\begin{cases} \mathcal{A}(x,\varphi,\nabla\varphi).\nabla\varphi & \ge C_0|\nabla\varphi|^p - C \\ \\ |\mathcal{A}(x,\varphi,\nabla\varphi)| & \le C(|\nabla\varphi|^{p-1}+1) \end{cases} \tag{1.3b}$$

with $p > 1$, $C_0 > 0$ and $C \ge 0$. This generalization is in two fold: the principal part $A(x,\varphi,\nabla\varphi)$ is permitted to have a nonlinear dependence in $\nabla\varphi$, and a nonlinear growth with respect to $|\nabla\varphi|$.

Consider the parabolic equation

$$\frac{\partial\varphi}{\partial t} - (a_{ij}(x,t)\varphi_{x_j})_{x_i} = 0 \text{ in } \Omega_T \tag{1.4}$$

with bounded and measurable coefficients a_{ij} satisfying ellipticity condition. Moser [15] proved that weak solutions of (1.4) are locally Hölder continuous in Ω_T. Since linearity is immaterial, extension is possible to quasi-linear equations of the form

$$\frac{\partial\varphi}{\partial t} - \text{div}\mathcal{A}(x,t,\varphi,\nabla\varphi) = \mathcal{B}(x,t,\varphi,\nabla\varphi). \tag{1.5}$$

Ladyženkaja et al [11] established Hölder continuity of weak solutions of (1.5) assuming that the principal part has exactly a linear growth with respect to $|\nabla\varphi|$(that is $p = 2$), leaving the case $p \ne 2$ open. The problem remained open until the mid 1980's, when DiBenedetto in [3] proved Hölder continuity of weak solutions of (1.5) under the structure conditions analogues to (1.3b) for $p > 2$. This was extended in [1] to include the case when $1 < p < 2$. The generalization of (1.2) under suitable assumptions to equations of the form (1.1) is due to [13]. The technique used by DiBenedetto is called the Intrinsic scaling method, a powerful tool in the theory of regularity. This can be modified to established local Hölder continuity of weak solutions of quasi-linear porous medium-type equations. The series of papers [5], [6], [7] and the book [8] established intrinsic Harnack inequalities for nonnegative weak solutions of degenerate and singular parabolic partial differential equations with full quasi-linear structure. Hwang [9] and Hwang et al [10] established Hölder continuity of φ and $\nabla\varphi$ using a unifying framework from the theory of Orlicz Spaces and the method of intrinsic scaling.

2. Preliminaries

2.1 Energy Estimates

Here we recall the the generalized quasi-linear parabolic partial differential equation, for $\varphi \in W^{1,G}(\Omega_T)$ is

$$\frac{\partial \varphi}{\partial t} - \text{ div } A\ (x,t,\varphi,D\varphi) \ = \ B(x,t,\varphi,D\varphi) \tag{2.1}$$

with structure conditions in the cylinder $Q_R := K_R \times [t_0, t_1]$

$$A\ (x,t,\varphi,D\varphi)\ \cdot D\varphi \ \geq \ \psi(|D\varphi|)\cdot\psi(b_0), \tag{2.2a}$$

$$|A(x,t,\varphi,D\varphi)| \ \leq \ a_1 h(|D\varphi|) + h(b_1), \tag{2.2b}$$

$$|B(x,t,\varphi,D\varphi)| \ \leq \ a_2\psi(|D\varphi|) + \psi(b_2) \tag{2.2c}$$

with b $:= \max\{b_0, b_1, b_2\}$, such that b_0, b_1, b_2 are positive parameters. In what follows, we state results on local energy estimates.

2.1.1: The Local Energy Estimates

Proposition 2.1 *Let ω be a nonnegative weak solution of (2.1) under assumption (2.2) in a cylinder $Q_R := K_R \times [t_0, t_1]$. For a nonnegative constant k and a cut off function ζ, $\exists$ constants r, s, q and $\gamma_i = \gamma_i$(data) i = 0, 1, 2, 3, 4 such that*

$$\begin{aligned}
&\int \psi^{r-1}\Big(\frac{\zeta(\omega-k)_\pm}{R}\Big)(\omega-k)_\pm^{s+2}\zeta^q dx\Big|_{t_0}^{t_1} + \\
&\gamma_0 \int\int_{Q_R} \psi(|D\omega|)\psi^{r-1}\Big(\frac{\zeta(\omega-k)_\pm}{R}\Big)(\omega-k)_\pm^{s}\zeta^q dxdt \\
&\leq \gamma_1 \int\int_{Q_R} \psi^{r-1}\ \Big(\frac{\zeta(\omega-k)_\pm}{R}\Big)(\omega-k)_\pm^{s+2}\ \zeta^{q-1}\ \zeta_t dxdt \\
&+\gamma_2 \int\int_{Q_R} \psi(\ |D\zeta|\ (\omega-k)_\pm)\psi^{r-1}\Big(\frac{\zeta(\omega-k)_\pm}{R}\Big)(\omega-k)_\pm^{s}\ \zeta^q dxdt \\
&+\gamma_3 \int\int_{Q_R} \psi^{r-1}\Big(\frac{\zeta(\omega-k)_\pm}{R}\Big)(\omega-k)_\pm^{s}\zeta^q dxdt \\
&+\gamma_4 \int\int_{Q_R} \psi^{r-1}(\frac{\zeta(w-k)I}{R})(w-k)_\pm^{s+1}\zeta^q dxdt.
\end{aligned} \tag{2.3}$$

Proof. The proof is this proposition can be seen in [9,10], hence the result. □

We assume here that the above estimates holds for (2.1) under (2.2) since the problem of differentiability of the temporal variable have been resolved using the Steklov average of a function. Introduce ordinary p-Laplacian type equation with constants lower order terms to show connection with generalized structure (2.1) with respect to local energy estimates. For $\varphi \in V'^{,p}(\Omega_q)$. The quasi-linear p-Laplacian equation is given as

$$\frac{\partial \varphi}{\partial t} - divA(x,t,\varphi,D\varphi) = B(x,t,\varphi,D\varphi) \tag{2.4}$$

satisfying the structure conditions

$$\begin{aligned} A(x,t,\varphi,D\varphi)\cdot D\varphi &\geq |D\varphi|^p - b_0 \\ |A(x,t,\varphi,D\varphi)| &\leq a_1|D\varphi|^{p-1} + b_1 \\ |A(x,t,\varphi,D\varphi)| &\leq a_2|D\varphi|^p + b_2. \end{aligned} \tag{2.5}$$

for $p \geq 1$, positive constants a_1, a_2, and nonnegative constants b_0, a_2, b_2, with the test function $(\varphi - k)_{\pm}\zeta^p$. The energy estimate are obtained as

$$\left.\begin{aligned} \sup_t \quad & \int_{K_R}(\varphi-k)^2_{\pm}\zeta^p dx + \gamma_0 \int\int_{Q_R} |D(\varphi-k)_{\pm}\zeta|^p \; dxdt \\ & \leq \int_{K_R\times\{t_0\}}(\varphi-k)^2_{\pm}\zeta^p dx + \gamma_1 \int\int_{Q_R}(\varphi-k)^p_{\pm}|D\zeta|^p \; dxdt + \\ & \gamma_2 \int\int_{Q_R}(\varphi-k)^2_{\pm}\zeta^{p-1}\zeta_t dxdt + \gamma_3 \int\int_{Q_R} b\zeta^p \; dxdt. \end{aligned}\right\} \tag{2.6}$$

By replacing lower order terms as constants $h(s) = s^{p-1}$ and $\psi(s) = \frac{1}{p}$ for any s ≥ 0, in case $h_0 = h_1 = h_1$ we have that the structure conditions (2.2) agree with(2.5).

Proposition 2.2 *Let ω be a bounded weak solution of (2.1) under assumption (2.2) in a cylinder $Q_R := K_R \times [t_0, t_1]$ and $K \in \mathbb{R}$. For a cutoff function ξ independent of t, then there are constants γ_1, γ_2, γ_3 and $b \geq 0$ depending on data such that*

$$\begin{aligned} \int_{K_R\times\{t_1\}} H(\psi^2)\xi^{h_1}dx \leq \int_{K_R\times\{t_0\}} H(\psi^2)\xi_1^h dx + \gamma_1 \int\int_{Q_R} \psi\left(\frac{|D\xi|}{\xi|\psi'|}\right)h(\psi^2)\psi|\psi'|^2\xi^{h_1}dxdt \\ + \gamma_2 \int\int_{Q_R} \psi(b)h(\psi^2)(1+\psi)|\psi'|^2\xi^{h_1}dxdt. \end{aligned} \tag{2.7}$$

Proof. The proof is this proposition can be seen in [4, 9], hence the result. ☐

The logarithmic energy estimate for (2.1) under the structure conditions (2.2) is

$$\left.\begin{aligned} \int_{K_R\times\{t_1\}} \psi^2\xi^p dx \quad & \leq \int_{K_R\times\{t_0\}} \psi^2\xi^p dx + \gamma_1 \int\int_{Q_R} \psi|\psi|^{2-p}|D\xi|^p dxdt \\ & \gamma_2 \int\int_{Q_R} b\psi|\psi'|^2\xi^p dxdt. \end{aligned}\right\} \tag{2.8}$$

which is derived from the test function $\phi(\varphi) = [\psi^2]'\xi^p$. If $h(s) = s^{p-1}$, then $\psi(s) = \frac{1}{p}^{s^{p-1}}$, and $H(s) = \frac{1}{p^2}^{s^p}$. Hence (2.8) becomes

$$\int_{K_R\times\{t_1\}} \psi^2\xi^p dx \leq \int_{K_R\times\{t_0\}} \psi^{2p}\xi^p dx + \gamma_1 \int\int_{Q_R} \psi^{2p-1}|\psi'|^{2+p}|D\xi|^p dxdt$$

$$+ \gamma_2 \int\int_{Q_R} \psi(b)\psi^{2p-2}(1+\psi)|\psi'|^2\xi^p dxdt \tag{2.9}$$

Definition 2.3. Let,

$$K^{\phi}(\Omega) = \{f : \Omega \to \bar{\mathbb{R}} : \ f \text{ is measurable and } \int_{\Omega} \phi(|f|)d\mu < \infty\}.$$

Then, $K^{\phi}(\Omega)$ is called the Orlicz class.

Definition 2.4: Let $K^{\phi}(\Omega)$ be the Orlicz class on an arbitrary measurable space (Ω, Σ, μ). Then the space $L^{\phi}(\Omega)$ of all measurable functions $f : \Omega \to \mathbb{R}$, such that $\alpha f \in K^{\phi}(\Omega)$ for some $\alpha > 0$, is called an Orlicz space. Thus $L^{\phi}(\Omega) = \{f : \Omega \to \mathbb{R}, \text{measurable} : \int_{\Omega}(\phi(\alpha f))d\mu < \infty, \text{for some } \alpha > 0\}$.

Definition 2.5 Let ϕ be a Young function, the Sobolev-Orlicz Space $W^{1,}L^{\phi}(\Omega)$ is the space of all function f such that the distributional derivatives $D^{\alpha} f \in L^{\phi}$ with $|\alpha| \leq 1$. The space $W_0^1 L^{\phi}(\Omega)$ is defined as the closure of the Schwartz space $D(\Omega)$ in $W^1 L^{\phi}(\Omega)$.

Definitions 2.6 Evolution spaces [4]

1. $V^o(\Omega_T) := L^{\infty}(0, T; L^2(\Omega)) \cap C(0, T; W^{1,2}(\Omega))$
   Where $L^{\infty}(0, T; L^2(\Omega)) = \{u : \int_0^T (\int_{\Omega} u^2 dx)^{\frac{1}{2}} dt < \infty\}$.
2. $V^{1,2}(\Omega_T) := L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))$
   Where $L^2(0, T; L^2(\Omega)) = \{u : (\int_0^T \int_{\Omega} u^2 dxdt)^{\frac{1}{2}} < \infty\}$.
3. $V^{1,p}(\Omega_T) := L^{\infty}(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,2}(\Omega))$
   Where $L^p(0, T; W^{1,2}(\Omega)) = \{u : (\int_0^T (\int_{\Omega} u^2 dx)^{\frac{p}{2}} dt)^{\frac{1}{p}} < \infty\}$.

3. Hölder Continuity of Weak Bounded Solutions

3.1 The Main Lemma and Hölder Estimates

We have noted that intrinsic scaling method have been successfully applied to (1.2). The Local energy estimates of (1.2) derived with the test function $(\varphi - k)_{\pm}\zeta^p$ is

$$\begin{aligned} \sup_{t_0 \leq t \leq t_1} \int_{K_R \times \{t\}} (\varphi - k)_{\pm}^2 \zeta^p dx + \int\int_{Q_R} |D(\varphi - k)_{\pm}|^p \zeta^p dxdt \\ \leq \int_{K_R \times t_0} (\varphi - k)_{\pm}^2 \zeta^p dx \\ + C \int\int_{Q_R} (\varphi - k)_{\pm}^p |D\zeta|^p dxdt \\ + p \int\int_{Q_R} (\varphi - k)_{\pm}^2 \zeta^{p-1} \zeta_t dxdt \end{aligned} \tag{3.1}$$

For $p = 2$, (1.2) becomes the heat equation and (3.1) is homogeneous. The main results in this section concern Hölder continuity of bounded weak solution of (1.1). This results states that a nonnegative solution φ is strictly positive in a sub-cylinder if φ is near to its maximum value in more than half of a cylinder while essential oscillation of the weak solution in a sequence of shrinking and nested sub-cylinders is measured. In what follows, the statement of the Main lemma.

Lemma 3.1: Main Lemma *Suppose that φ is a nonnegative bounded weak solution of (2.1) under the structure assump-*

tions (2.2) in a cylinder $\Omega^{x_0,t_0}_{\rho,\tau}$. *Then,* $\exists$ $R > 0$ *such that* $Q^{x_0,t_0}_{\omega,4R} := K^{x_0}_{4R} \times [t_0 - \Theta M^2 \psi(\frac{M}{4R})^{-1}, t_0] \subset \Omega^{x_0,t_0}_{\rho,\tau}$, *where constants* $M > 0$. *If in addition, there exist positive constants* θ, $\mu \in (0,1)$ *and* $\lambda \in (0,1)$ *depending on data such that,* φ *satisfies*

$$meas\{(x,t) \in Q^{x_0,t_0}_{M,2R} : \varphi(x,t) > \frac{M}{2}\} > \frac{1}{2}|Q^{x_0,t_0}_{M,2R}|. \tag{3.2}$$

Then,

$$ess \inf_{K^{x_0}_R \times [t_0 - \lambda T_{M,R}, t_0]} \varphi(x,t) \geq \mu M$$

Proof. The proof of the main lemma will be presented at the end of section (3.4) □

The next result measures the essential oscillation of weak solutions in a sequence of shrinking and nested sub-cylinders use in the prove of the main lemma and the establishment of Hölder continuity.

Lemma 3.2 *Suppose that* φ *is a bounded weak solution of (2.1) under the structure assumptions (2.2) in a cylinder* $\Omega^{x_0,t_0}_{\rho,\tau}$. *Then, there exist a family of shrinking and nested cylinders* $\{Q_n\}_{n=0}^{\infty}$ *such that*

$$ess\ osc_{Q_n} \varphi(x,t) \leq \eta^n essosc_{Q^{x_0,t_0}_{\rho,\tau}} \varphi(x,t) + \frac{c}{1-\eta} r$$

where constant $\eta \in (0,1)$ *and* $c > 0$. *Proof.* The proof of lemma 3.2 can be found in [9].

□

Remarks 3.3 (i) Suppose that there are two distinct points (x_1, t_1), $(x_2, t_2) \in \Omega_T$, then the length between t_1 and t_2 is define by

$$\|t_1 - t_2\|_\psi = \frac{\|\varphi\|^2_{\infty,\Omega}}{\psi^{-1}\left(\frac{\theta\|\varphi\|^2_{\infty,\Omega_T}}{|t_1,t_2|}\right)} \tag{3.3}$$

(ii) The distance between two sets in $\mathbb{R}^N \times \mathbb{R}$ is defined as

$$\text{dist}(\kappa_1; \kappa_2) := \inf\{|x_1 - x_2| + \|t_1 - t_2\|_\psi : (x_1, t_1) \in \kappa_1, (x_2, t_2) \in \kappa_2\}$$

This remark leads to the following result

Theorem 3.4 *If* φ *is a bounded weak solution of (2.1) under the structure assumptions (2.2) in a cylinder* $\Omega^{x_0,t_0}_{\rho,\tau}$. *Then,* $\exists$ *two distinct points* (x_1, t_1) *and* (x_2, t_2) *in any set* $\Omega' \subset \Omega^{x_0,t_0}_{\rho,\tau}$ *far away from* $\partial_p \Omega_T$ *such that* $(x,t) \to \varphi(x,t)$ *has modulus of continuity. If there exists positive constants* γ, β *and* $\alpha \in (0,1)$ *depending on data. Then,*

$$|\varphi(x_1,t_1) - \varphi(x_2,t_2)| \leq \gamma(data, \|\varphi\|\infty, \Omega)\left[\frac{|x_1 - x_2| + \|t_1 - t_2\|_\psi}{dist(\Omega^{x_0,t_0}_{r,s}; \partial_p \Omega_T)}\right]^\alpha.$$

Proof. See [10] for the proof.

□

Corollary 3.5 *If* φ *is a bounded weak solution of (2.1) under the structure assumptions (2.2) in a cylinder* $\Omega_T \subset \mathbb{R}^N \times \mathbb{R}$. *Then,* $\exists$ *two distinct point* (x_1, t_1) *and* (x_2, t_2) *in a cylinder* $\Omega^{x_0,t_0}_{r,s}$ *which is a subset of* Ω_T *far away from* $\partial_p \Omega_T$. *Then,* $(x,t) \to \varphi(x,t)$ *is locally Hölder continuous and there exist positive constants* γ, β *and* $\alpha \in (0,1)$ *depending on data such* *that*

$$|\varphi(x_1,t_1)-\varphi(x_2,t_2)| \le \gamma(data, \|\varphi\|_{\infty,\Omega_T})\Big(\frac{|x_1-x_2|+\beta\Theta^{\frac{-1}{\iota_0}}\|\varphi\|_{\infty,\Omega_T}^{(\frac{\iota_0-2}{\iota_0})}|t_1-t_2|^{\frac{1}{\iota_0}}}{H\,dist(\Omega_{r,s}^{x_0,t_0};\partial_p\Omega_T)}\Big)^{\alpha}$$

$$where\ H\,dist(\Omega_{r,s}^{x_0,t_0};\partial_p\Omega)=\inf_{(x,t)\in\Omega_{r,s}^{x_0,t_0}}\{|x-y|+\beta\Theta^{\frac{-1}{\iota_0}}\|\varphi\|_{\infty,\Omega}^{\frac{\iota_0-2}{\iota_0}}|t-s|^{\frac{1}{\iota_0}}\}$$

Proof. For the proof of this corollary we refer to [10] □

3.2 Proof of The Main Lemma

The following proposition are stated without prove because they are used in the prove of Lemma 3.1.

3.2 Proof of The Main Lemma

The following proposition are stated without prove because they are used in the prove of Lemma 3.1.

Proposition 3.2.1 [4, 10]: Let $k>0$, $\rho>0$, and a nonnegative weak solution ω of (2.1) under the structure assumptions (2.2) in a cylinder $Q_{k,2\rho}$ satisfying

$$\text{meas}\{(x,t)\in Q_{k,\rho}:\omega(x,t)>k\}\ge\frac{1}{2}|Q_{k,\rho}|. \tag{3.4}$$

Then for any $\nu_1\in(0,1)$ and $\delta_1\in(0,1)$, $\exists\, y\in K_\rho$, $\tau_1\in[\frac{T_{k,\rho}}{16},T_{k,\rho}$ and $\eta=\eta(\text{data}k)\in(0,1)$ such that $K_{2\rho}^{y}\subset K_\rho$ and

$$\text{meas}\{x\in K_{2\rho}^{y}:\omega(x-\tau_1)<\delta_1k\}<(1-\nu_1)|K_{2\rho}^{y}|$$

Proposition 3.2.2 [10]: Let $\nu\in(0,1)$, $k>0$, $\rho>0$, $\epsilon\in(0,1)$, $\exists$ a nonnegative integer $j=j(\nu N,\iota_1,\epsilon)\in(0,1)$. Then, ω is a nonnegative solution of (2.1) under the assumptions (2.2). If

$$\text{meas}\{x\in K_{\rho}^{y}:\omega(x,-\tau)<k\}<(1-\nu)|K_{\rho}^{y}| \tag{3.5}$$

for some

$$\begin{cases}\tau\le k^2\psi(\frac{k}{\rho})^{-1}, \text{for } \iota_0\ge2,\\[2ex] \tau\le\big(2^{-j}k\big)^2\psi\big(\frac{2^{-j}k}{\rho}\big)^{-1},\ \text{for } \iota_1\le2,k\end{cases} \tag{3.6}$$

Then,

$$\text{meas}\{x\in K_{\rho}^{y}:\omega(x,-t)<2^{-j}k\}<\big(1-(1-\epsilon)\nu\big)|K_{\rho}^{y}|$$

for any $t\in[0,T)$.

The following proposition says that positive information on K_ρ for all time expands to $K_{2\rho}$ for comparable times. Spreading positively is dominating when $\iota_1<2$ because the modulus of parabolicity is dominating when $|\nabla\varphi|\to0$ but when $\iota_1>2$, enough time length is required for spreading positively over spatial cube.

Proposition 3.2.3 [4]: Let $k>0$, $\rho>0$, $y\in K_{\rho}^{y}$, $\eta\in(0,1)$ and $\alpha\in(0,1)$. Then, $\exists$ a positive integer $J^*=j^*(N,\alpha,\iota_1,\eta,\nu)$ such that, a weak solution ω satisfies

$$\text{meas}\{x\in K_{2\rho}^{y}:\omega(x,t)<k\}<(1-\alpha)|K_{\eta\rho}^{y}| \tag{3.7}$$

for all $t\in(-2\tau,0]$ where

$$\begin{cases}\tau\ge k^2\psi(\frac{k}{\rho})^{-}1 \text{ for}1<\iota_1\le2\\ \tau\ge(2^{-j^*}k)^2\psi(\frac{2^{-j^*}k}{\rho})^{-}1,\ \text{for}\iota_1>2.\end{cases} \tag{3.8}$$

Then,

$$\text{meas}\{(x,t) \in K_\rho \times [-\tau, 0] : \omega(x,t) < 2^{-j^*}k\} < \nu|K_\rho \times [-\tau, 0]|$$

Proposition 3.2.4 The Modified De Giorgi iteration [4]: Let $k > 0$ and $\rho > 0$. Then, $\exists$ a nonnegative weak solution ω and $\nu_0 = \nu_0(\min\{\theta^N, \theta^{-1}\}, \text{data}) \in (0,1)$ such that, if

$$\text{meas}\{(x,t) \in Q_{k,2\rho} : \omega(x,t) < K\} < \nu_0|Q_{k,2\rho}|,$$

Then

$$\text{ess}\inf_{Q_{k,\rho}} \omega(x,t) \geq \frac{K}{2}.$$

Next we prove lemma 3.1 *Proof.* let $(x_0, t_0) := (0,0)$ depending on ι_0 and ι_1. Two cases are studied separately under the same framework. **Case I: Degenerate Equations** $(2 \leq \iota_0 \leq \iota_1 < \infty)$**.** Begin with a cylinder

$$Q_0 = K_R \times [-\tau, 0]$$

where τ is to be determined later. For a constant M, suppose that

$$\text{meas}\{(x,t) \in Q_0 : \varphi(x,t) \geq 2M\} \geq \frac{1}{2}|Q_0|.$$

By Proposition 3.2.1 with a fixed constant $\delta_1 = \frac{1}{2}$, for any constant $\nu_1 \in (0,1)$, $\exists$ a point $y \in K_R$, a time level $\tau \in [\frac{T}{16}, T]$, and a constant $\eta = \eta(M, \nu_1, \text{data})$ such that

$$K^y_{\eta R} \subset K_R$$

and

$$\text{meas}\{x \in K^y_{\eta R} : \varphi(x, -\tau) < M\} < (1-\nu_1)|K^y_{\eta R}|.$$

Then by Proposition 3.2.2 provides that for any $\epsilon \in (0,1)$, $\exists\, j = j(\nu_1, \epsilon, \text{data})$ such that , if

$$\tau \leq M^2 G(\frac{M}{\eta R})^{-1},$$

then $\forall t \in [-\tau, 0]$

$$\text{meas}\{x \in K^y_{\eta R} : \varphi(x,t) < 2^- jM\} < (1-(1-\epsilon)\nu_1)|K^y_{\eta R}| \tag{3.8}$$

Now, subdividing $K^y_{\eta R}$ into 2^{lN} congruent sub-cylinders, then for any nonnegative integer l,$\exists$ $K^{y'}_{2^{-l}}$ such that

$$\text{meas}\{x \in K^{y'}_{2^{-l}\eta R} : \varphi(x,t) < 2^{-j}M\} < (1-(1-\epsilon)\nu_1)|K^{y'}_{2^{-l}\eta R}| \tag{3.9}$$

Choose

$$l = \frac{\iota_1 - 2}{\iota_0} j,$$

that satisfies

$$2^{(\iota_1-2)j} 2^{-l\iota_0} \leq 1.$$

Therefore,

$$(2^{-j}M)^2 \psi(\frac{2^{-j}M}{2^{-l}\eta R})^{-1} \leq 2^{(\iota_1-2)j} 2^{-l\iota_0} M^2 \psi(\frac{M}{\eta R})^{-1} \leq M^2 \psi(\frac{M}{\eta R})^{-1}$$

. Hence by setting

$$Q_1 = K^{y'}_{2^{-l}}\eta R \times \left[-(2^{-j}M)^2\psi(\frac{2^{-j}M}{2^{-l}\eta R})^{-1}, 0\right],$$

The equation (3.9) implies that

$$\text{meas}\{(x,t)Q_1 : \varphi(x,t) < 2^{-j}M\} < (1-(1-\epsilon)\nu_1)|Q_1| \tag{3.10}$$

Here for a constant $\nu_0 \in (0,1)$ that is from Proposition 3.2.4 with restriction

$$b \leq \frac{2^{-j}M}{2^{-l}\eta R}, \tag{3.11}$$

fix

$$\epsilon = \frac{\nu_0}{1+\nu_0}, \nu_1 = 1-\nu_0^2 \Longrightarrow \nu_0 = (1-\epsilon)\nu_1.$$

Then with (3.10), apply proposition 3.2.4 and conclude

$$\operatorname*{ess\,inf}_{Q_2} \varphi(x,t) \geq \frac{2^{-j}M}{2} \tag{3.12}$$

where

$$Q_2 = K^{y'}_{2^{-l}\eta\frac{R}{2}} \times [-(-2^{-j}M)^2 G(\frac{2^{-j}M}{2^{-l}\eta\frac{R}{2}})^{-1}, 0]$$

Let $k = 2^{-j}M, \rho = 2^{-l}\eta\frac{R}{2}$ Then, (3.12) becomes

$$\operatorname*{ess\,inf}_{Q_2} \varphi(x,t) \geq \frac{k}{2} \tag{3.13}$$

where

$$Q_2 = K^{y'}_{\rho} \times [-K^2\psi(\frac{k}{\rho})^{-1}, 0].$$

For any $\sigma \in (0,1)$, the equation (3.8)is true if a positive constant M is replaced by σM. From the equation (3.13), the conclusion is that for a constant $\sigma \in (0,1)$, it holds such that

$$\operatorname*{ess\,inf}_{Q_3} \varphi(x,t) \geq \frac{\sigma M}{2}$$

where

$$Q_3 = K^{y'}_{\rho} \times [-(\sigma k)^2\psi(\frac{\sigma k}{\rho})^{-1}, 0]$$

Now, let $t = (2^{-l}\eta)^{\iota}_0(\sigma k)^2 G(\frac{\sigma k}{R})^{-1}$ then σ is a function of t. Also, let $\varphi(x,t) = \nu(x_0,t)\sigma(t)$. Applying Proposition 3.2.3 with a linear cutoff function ξ in the cylinder $Q_4 = K_R \times [-t, 0]$ satisfying $|D\xi| \leq \frac{2}{R}, 0 \leq \xi_t \leq \frac{2}{t}$ and the level sets for ν that $(\nu - K_i)_-$ for $K_i = 2^{-i}K$. The local energy estimate by ignoring the first term on the left hand side yields

$$\begin{aligned}
&\int\int_{Q_4} \psi(\sigma|D(\nu-K_i)_-|)\psi^{r-1}(\frac{\xi\sigma(\nu-K_i)_-}{R})\sigma^s(\nu-K_i)^s_-\xi^q dxdt \\
&\leq \gamma_1 \int\int_{Q_4} \psi^{r-1}(\frac{\xi\sigma(\nu-K_i)_-}{R})\sigma^{s+2}(\nu-K_i)^{s+2}_-\xi^{q-1} dxdt \\
&+ \gamma_2 \int\int_{Q_4} \psi^{r}(\frac{\xi\sigma(\nu-K_i)_-}{R})\sigma^{s+2}(\nu-K_i)^s_-\xi^q dxdt \\
&+ \gamma_3 \int\int_{Q_4} \psi(b)\psi^{r-1}(\frac{\xi\sigma(\nu-K_i)_-}{R})\sigma^s(\nu-K_i)^s_-\xi^{q-1-2\iota_1} dxdt
\end{aligned} \tag{3.14}$$

with the restriction $b \leq \frac{\sigma K}{R}$ and $(v-K)_- \leq K_i$ and an increasing function $\omega \to \psi^r(\omega)\omega^2, \omega \to \psi^{r-1}(\omega)\omega^{s+2}$, a deceasing function $\omega \to \psi^{r-1}(\omega)\omega^s$,the inequality (3.13) is simplified to

$$\int\int_{Q_4} \psi(\sigma|D(v-K_i)_-|)\chi_{\{K_{i+1}\leq v\leq K_i\}}\xi^q dxdt$$

$$\leq \gamma\psi(\frac{\sigma K_i}{R})|K_R \times [-t,0]|.$$

Then by following lines from the proof of Proposition 3.2.3, we conclude that for any $\nu \in (0,1)$, $\exists\, j^* = j^*(N,\eta,\nu,\text{data})$ such that

$$\text{meas}\{(x,s) \in K_{\frac{R}{2}} \times [\frac{-t}{2},0] : v(x,s) < 2^{-j^*}K\} < \nu|K_{\frac{R}{2}} \times [-\frac{t}{2},0]|$$

where for any $\sigma \in (0,1)$

$$\tau^* = (2^{-l}\eta)^{\iota_0}(\sigma k)^2\psi(\frac{\sigma k}{R})^{-1} \tag{3.15}$$

By fixing σ in (3.15) to be 2^{-j^*}, we carry Proposition 3.2.4 for $v(x,t)$ using that $u(x,t) = v(x,t)\sigma(t)$. Therefore in the cylinder

$$Q = K_{\frac{R}{4}} \times [-(2^{-l}\eta)^{\iota_0}(2^{-j^*}k)^2\psi(\frac{2^{-j^*}k}{\frac{R}{2}})^{-1},0].$$

It holds that

$$\operatorname*{ess\,inf}_{Q} v(x,t) \geq \frac{2^{j^*}k}{2}$$

Hence,

$$\text{ess inf}\, u(x,t) \geq 2^{-2j^*-j-1}M$$

leading to our conclusion and choosing

$$T = 16(2^{-}l\eta)^{\iota_0}(2^{-j^*-j}M)^2\psi(\frac{2^{-j^*-j}M}{R})^{-1}.$$

The equation (3.11) and (3.14) imply that we pick

$$M \geq \max\{2^{\frac{\iota_0+\iota_1-2}{\iota_0}}b\eta R, 2^{j+j^*}bR\}$$

otherwise,

$$\text{essosc}\omega \leq cbR$$

for some constant $c > 0$ that leads to Hölder continuity.

Case II: Singular Equation $(1 < \iota_0 \leq \iota_1 \leq 2)$. We begin with the cylinder $Q_0 = K_R \times [-T,0]$ where T is to be determined later. For a constant M, suppose that

$$\text{meas}\{(x,t) \in Q_0 : \varphi(x,t) \geq 2M\} \geq \frac{1}{2}|Q_0|.$$

Then Proposition 3.2.1 with $\delta_1 = \frac{1}{2}$ provides that for any $\nu_1 \in (0,1)$, $\exists$ a point $y \in K_R$, a time level $\tau \in [\frac{\tau}{16},T]$, and $\eta = \eta(M,\delta_1,\nu_1,\text{data}) \in (0,1)$ such that $K^y_{\eta R} \subset K_R$ and

$$\text{meas}\{x \in K^y_{\eta R} : \varphi(x,-\tau) < M\} < (1-\nu_1)|K^y_{\eta R}|.$$

By Proposition 3.2.2, for any $\epsilon \in (0,1)$, $\exists$ a positive integer $j = j(\nu_1,\epsilon,\text{data})$ if

$$\tau \leq (2^{-j}m)^2\psi(\frac{2^{-j}M}{\eta R})^{-1}$$

. Then $\forall t \in [-\tau, 0]$

$$\text{meas}\{x \in K^y_{\eta R} : \varphi(x,t) < 2^{-j}M\} < (1-(1-\epsilon)\nu_1)|K^y_{\eta R}|$$

Let $\epsilon = \frac{\nu_0}{1+\nu_0}, \nu_1 = 1-\nu_0^2$, where ν_0 is a constant from Proposition 3.2.4, applying De Giorgi iteration with the restriction

$$b \leq \frac{2^{j-1}M}{\eta R} \tag{3.16}$$

to conclude that

$$\text{ess}\inf_{Q_1} \varphi(x,t) \geq \frac{2^{-j}M}{2} \tag{3.17}$$

where

$$Q_1 = K^y_{\eta\frac{R}{2}} \times [-(2^{-j}M)^2\psi(\frac{2^{-j}M}{\eta\frac{R}{2}})^{-1}, 0]$$

setting $k = 2^{-j}M, \rho = \eta\frac{R}{2}$. Then from (3.17) we have that

$$\text{ess}\inf_{Q_1} \varphi(x,t) \geq \frac{1}{2}$$

where

$$Q_1 = K^y_\rho \times [-k^2\psi(\frac{k}{\rho}), 0]$$

since $k^2\psi(\frac{k}{\rho}) \geq \eta_1^\iota k^1\varphi(\frac{k}{R})^{-1}$ by Proposition 3.2.3, for any $\nu \in (0,1)$ $\exists$ a positive integer $j^* = j^*(\text{data})$ such that

$$\text{meas}\{(x,t) \in K_R \times [-\tau, 0] : \varphi(x,t) < 2^{-J^*}k\} < \nu|K_R \times [-\tau, 0]| \tag{3.18}$$

Now set

$$\varphi(x,t) = 2^{-j^*}v(x,t).$$

Then, let

$$R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, K_n = \frac{K}{2} + \frac{K}{2^{n+1}}, \text{for}, n = 0,1,2,...$$

and set

$$Q_n = K_{R_n} \times [-K_n^2\psi(\frac{K_n}{R_n})^{-1}, 0]$$

$$\xi_n = \begin{cases} 1 & \text{Inside of } Q_{n+1} \\ 0 & \text{Outside of } Q_n \end{cases}$$

satisfying

$$|D\xi_n| = \frac{2^{n+2}}{R}, 0 \leq (\xi_n)_t \leq 2^nK^{-2}G(\frac{K_n}{R_n})$$

Then the local energy estimates with a level set of v provides that

$$\begin{aligned}
&\sup_t \int_{K_{R_n}} \psi^{r-1}[\frac{2^{-j^*}\xi_n(v-k_n)_-}{R_n}](2^{-j^*}(v-k_n)_-)^{s+2}\xi_n^q dx \\
&+ \int\int_{Q_n} \psi(|2^{-j^*}D(v-k_n)_-|)\psi^{r-1}(\frac{2^{-j^*}\xi_n(v-k_n)_-}{R_n})(2^{-j^*}(v-k_n)_-)\xi_n^q dxdt \\
&\leq \gamma_0 \int\int_{Q_n} \psi^{r-1}(\frac{2^{-j^*}\xi_n(v-k_n)_-}{R_n})(2^{-j^*}(v-k_n)_-)^{s+2}\xi_n^{q-1}(\xi_n)_t dxdt \\
&+ \gamma_1 \int\int_{Q_n} \psi^r(\frac{2^{-j^*}\xi_n(v-k_n)_-}{R_n})(2^{-j^*}(v-k_n)_-)^s\xi_n^{q-1-2\iota_1} dxdt \\
&+ \gamma_2 \int\int_{Q_n} \psi(b)\psi^{r-1}(\frac{2^{-j^*}\xi_n(v-k_n)_-}{R_n})(2^{-j^*}(v-k_n)_-)^s\xi_n^q dxdt
\end{aligned} \tag{3.19}$$

Now from the properties that $\omega \to \psi^r(\omega)\omega^2, \omega \to \psi^{r-1}(\omega)\omega^{s+2}$, are increasing functions and $\omega \to \psi^{r-1}(\omega)\omega^s$ is a decreasing function and that

$$(v-k_n)_- \le k_n \le k$$

and implying

$$b \le \frac{2^{-j^*k}}{R} \tag{3.20}$$

Since

$$\psi^{r-1}\big(\frac{2^{-j^*}\xi_n(v-k_n)_-}{R_n}\big)\big(2^{-j^*}(v-k_n)_-\big)^{s+2}\xi_n^{q-1}(\xi_n)_t$$
$$\le \psi^{r-1}\big(\frac{2^{-j^*}K_n}{R_n}\big)\big(2^{-j^*}K_n\big)^{s+2}K^{-2}\psi(\frac{K}{R_n})$$
$$\le \psi^r\big(\frac{2^{-j^*}K_n}{R_n}\big)\big(2^{-j^*}K_n\big)^s$$

Since

$$2^{-2j^*}\psi(\frac{k}{R_n}) \le \psi(\frac{2^{-j^*}}{R_n})$$

due to $1 < \iota_0 \le \iota_1 \le 2$. Hence the right hand side of (3.19) is simplified to

$$RHS = \gamma\psi^r(\frac{2^{-j^*}}{R_n})(2^{-j^*})^s \int\int_{Q_n} \chi_{\{v<k_n\}}dxdt.$$

By Proposition 3.2.4,

$$Q_2 = K_R \times [-K^2\psi(\frac{K}{R})^{-1}, 0]$$

If

$$\text{meas}\{(x,t) \in Q_2 : v(x,t) < k\} < \nu_0|Q_2|.$$

Then,

$$\operatorname*{ess\,inf}_{Q_3} v(x,t) \ge \frac{k}{2}$$

where $Q_3 = K_{\frac{K}{2}} \times [-K^2\psi(\frac{k}{\frac{R}{2}})^{-1}, 0]$.Hence $\text{ess}\inf_{Q_3}\varphi(x,t) \ge 2^{-j-j^*-1}M$. The time length T is now chosen to be $T = 16\eta_1^g(2^{-j}M)\psi(\frac{2^{-j}M}{R})^{-1}$ from (3.16) and (3.20), yields $M \ge \max\{2^j\eta bR, 2^{j+j^*}bR\}$. Hence the result.

□

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