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Fixed point theorems for $(\alpha - \psi)$ contractions on multiplicative and partially ordered multiplicative metric spaces

K.P.R.SASTRY[1], K.K.M.SARMA[2], S.LAKSHMANA RAO[3,*]

[1]8-28-8/1, Tamil Street, Chinna Waltair Visakhapatnam-530 017, India.

[2,3] Department of Mathematics, Andhra University ,

Visakhapatnam- 530 003, India.

E-mail: laxmana.mat@gmail.com

ABSTRACT. In this paper we introduce the notion of *generaized* $(\alpha, \psi)-$ *contractive mappings* on a multiplicative metric space and partially ordered multiplicative metric space and prove some fixed point theorems for such contractions. Also we provide two supporting examples.

1 Introduction and Preliminaries

In 1922, Banach [2] proved a theorem which is now well known as "Banach's Fixed point theorem" to establish the existence and uniqueness of fixed point of a contractive mapping in a complete metric space. This principle is applicable to a variety of subjects such as integral equations, differential equations, image processing and many others. The study on the existence of fixed points of some mappings satisfying certain contractions has many applications and has been the center of various research activities. In the past years, many authors generalized Banach's fixed point theorem to various spaces such as Quasi-metric spaces, Fuzzy metric spaces, Partial metrics paces and generalized metric spaces [7,8].

On the other hand, in 2008, Bashirov et al.[1] defined a new distance so called a multiplicative distance by

* Corresponding Author.

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using the concepts of multiplicative absolute value. In 2012, Ozavsar and Cevikel [5] investigated multiplicative metric spaces by remarking its topological properties, and introduced the concept of multiplicative contraction mapping and proved some fixed point theorems for multiplicative contraction mappings of multiplicative metric spaces.

In 2012, Samet, Vetro and Vetro [3] introduced the concept of (α, ψ)- contractive maps where α is an α-admissable mapping which is a new direction in the context of generalization of contraction maps and proved the existence of fixed points of such mappings. In 2015 H.H. Alsuami, S. Chandok, M.A. Taoudi, I.M. Erhan [4] proved some fixed point theorems for (α, ψ)- rational type contractive mappings.

Recently Praveen Kumar, Shin Min Kang, Sanjay Kumar and Chahn Yong Jung [6] proved fixed point theorems for generalized (α, ψ)- contractive mappings in multiplicative metric spaces.

Motivated and inspired by the results of Praveen Kumar, Shin Min Kang, Sanjay Kumar and Chahn Yong Jung [6], in this paper, we improve the result of [6], and prove fixed point theorems for generalized (α, ψ)- contractive mappings in multiplicative metric spaces with rational inequalities. Supporting examples are provided.
The letter $\mathbb{R}^+$ denote the set of all positive real numbers.

Definition 1.1. (A.E.Bashirov, E.M.Kurplnara, A.Ozyapici [1]). Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}^+$ satisfying the following conditions:
(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$, if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iii) $d(x, y) \leq d(x, z).d(z, y)$ for all $x, y, z \in X$.(Multiplicative triangle inequality) Also (X, d) is called a multiplicative metric space.

Note: Define $d : \mathbb{R}^+ \times \mathbb{R}^+ \to [1, \infty)$ by $d(x, y) = max\{\frac{x}{y}, \frac{y}{x}\}$. Then $\mathbb{R}^+$ is a multiplicative metric space with respect to d.

Example 1.2. (M.Ozavser, A.C.Cevikel [5]). Let $(\mathbb{R}^+)^n$ be the collection of all $n-$tuples of positive real numbers. Let $d^* : (\mathbb{R}^+)^n \times (\mathbb{R}^+)^n \to \mathbb{R}^+$ be defined as follows $d^*(x, y) = \mid \frac{x_1}{y_1} \mid^* . \mid \frac{x_2}{y_2} \mid^* ...... \mid \frac{x_n}{y_n} \mid^*$.
where $x = (x_1, x_2, ...x_n), y = (y_1, y_2, ...y_n) \in \mathbb{R}^+$ and

$$\mid . \mid^* : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is } \mid a \mid^* = \begin{cases} a & if \;\; a \geq 1 \\ \frac{1}{a} & if \;\; a \leq 1 \end{cases}$$

Then $((\mathbb{R}^+)^n, d^*)$ is a multiplicative metric space.

Example 1.3. (M.Ozavser, A.C.Cevikel [5]).Let $a > 1$ be fixed real number. Then $d_a : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $d_a(w, z) = a^{\sum_{i=1}^{n} |w_i - z_i|}$
where $w = (w_1, w_2, ...w_n), z = (z_1, z_2, ...z_n) \in \mathbb{R}^n$.
Obviously, $(\mathbb{R}^n, d_a)$ is a multiplicative metric space. We can also extended multiplicative metric to $\mathbb{C}^n$ by the following definition: $d_a(w, z) = a^{\sum_{i=1}^{n} |w_i - z_i|}$ where $w = (w_1, w_2, ...w_n), z = (z_1, z_2, ...z_n) \in \mathbb{C}^n$.

Definition 1.4. (M.Ozavser, A.C.Cevikel [5]).(Multiplicative convergence). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\epsilon(x) = \{y / d(x, y) < \epsilon\}, \epsilon > 1$

there exists a natural number N such that for $n \geq N$, $x_n \in B_\epsilon(x)$, the sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n \to x \ (n \to \infty)$.

Definition 1.5. (M.Ozavsar, A.C.Civikel [5]). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if, for each $\epsilon > 1$, there exists $N \in \mathbf{N}$ such that $d(x_n, x_m) < \epsilon$, for all $m, n \geq N$

Definition 1.6. (M.Ozavsar, A.C.Civikel [5]).Let (X, d) be a multiplicative metric space. We call (x, d) is complete if every multiplicative Cauchy sequence in X is multiplicative convergent to $x \in X$.

Definition 1.7. (M.Ozavsar, A.C.Civikel [5]). Let (X, d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0, 1)$ such that $d(fx, fy) \leq d(x, y)^\lambda$ for all $x, y \in X$.

Definition 1.8. (M.Ozavsar, A.C.Civikel [5])(Multiplicative continuity). Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces and $f : X \to Y$ be a function. If for every $\epsilon > 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$, then we call f multiplicative continuous at $x \in X$.

Lemma 1.9. *(M.Ozavsar, A.C.Civikel [5]). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to x \ (n \to \infty)$ if and only if $d(x_n, x) \to 1 \ (n \to \infty)$.*

Lemma 1.10. *(M.Ozavsar, A.C.Civikel [5]). Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1 \ (m, n \to \infty)$.*

In 2012, Samet et al. [3] introduced the concept of $\alpha-$admissible mappings and established fixed point theorems for these mappings in complete metric spaces. In fact, these results extend and generalize many existing fixed point results present in the literature.

Definition 1.11. (Samet et al.[3]) Let X be a non empty set, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. We say that T is $\alpha-$admissible if $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$ for all $x, y \in X$.

Example 1.12. (Samet et al. [3]) Let $X = [0, \infty)$. Define mapping $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Tx = \sqrt{x}$ for all $x \in X$ and $\alpha(x, y) = \begin{cases} 2 & if \ \ x \geq y \\ 0 & if \ \ x < y \end{cases}$ respectively. Then T is an $\alpha-$admissible mapping.

Example 1.13. (Samet et al. [3]) Let $X = \mathbb{R}$. Define mapping $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by
$Tx = \begin{cases} ln \mid x \mid & if \ \ x \neq y \\ 7 \ \ if \ \ x = 0 \end{cases}$ and $\alpha(x, y) = \begin{cases} e^{x-y} & if \ \ 0 < y \leq x \\ 0 & otherwise, \end{cases}$
for all $x, y \in X$ respectively. Then T is an $\alpha-$admissible mapping.

Denote by Ψ_1 the family of non-decreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the $n-$th iterate of ψ.

Samet et al. [3] introduced the notion of $\alpha - \psi$- contractive mappings in a metric space as follows:

Definition 1.14. Let T be a mapping of a metric space (X, d) into itself. Then T is said to be an $(\alpha - \psi) -$ *contractive mappings* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_1$ such that
$\alpha(x, y).d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$.

The $(\alpha - \psi)$-contraction was further generalized by Alsulami et al. [4] in the setting of a generalized metric [4] space to rational contraction known as
$(\alpha, \psi)-$ rational type-I as follows:
Let Ψ_2 be the family of non-decreasing functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:
(i) ψ is upper semi-continuous (i.e., $x_0 \in [0, \infty) \Rightarrow \lim_{x \to x_0} sup\ \psi(x) \leq \psi(x_0)$) and non-decreasing;
(ii) $\{\psi^n(t)\}_{n \in \mathbf{N}}$ converges to 0 as $n \to \infty$ for all $t > 0$;
(iii) $\psi(t) < t$ for every $t > 0$.

Definition 1.15. (Alsulami et al. [4])Let T be a mapping of a generalized metric space (X, d) into itself. Then T is said to be an $(\alpha - \psi)$-*rational type* $-$ *I contractive mapping* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_2$ such that
$\alpha(x, y).d(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in X$, where
$M(x, y) = max\{d(Tx, x), d(Ty, y), d(x, y), \frac{d(Tx,y).d(Ty,x)}{1+d(x,y)}, \frac{d(Tx,x).d(Ty,y)}{1+d(Tx,Ty)}\}$.

Praveen Kumar et al. [6] introduced the notion of generalized $(\alpha - \psi)$-*contractive mapping* in a multiplicative metric space as follows:
Let Ψ_3 be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:
(i) ψ is upper semi-continuous and non-decreasing;
(ii) $\prod_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the $n-$th iterate of ψ;
(iii) $\psi(t) < t$ for every $t > 0$.

Definition 1.16. (Praveen Kumar et al. [6]) Let T be a mapping of a multiplicative metric space (X, d) into itself. Then T is said to be a *generalized* $(\alpha - \psi) -$ *contractive mapping* if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi_3$ such that
$\alpha(x, y).d(Tx, Ty) \leq \psi(M_1(x, y))$ for all $x, y \in X$, where
$$M_1(x, y) = max\{d(Tx, x), d(Ty, y), d(x, y), [d(Tx, y).d(Ty, x)]^{\frac{1}{2}}, \frac{d(Tx,x).d(Ty,y)}{1+d(x,y)}, \frac{d(Tx,y).d(Ty,x)}{1+d(x,y)}, \frac{d(Tx,y).d(Ty,x)}{1+d(Tx,Ty)}\}$$

Theorem 1.17. *(Praveen Kumar et al. [6])Let T be a mapping of a complete multiplicative metric space (X, d) into itself and $\alpha : X \times X \to [0, \infty)$ be a given function satisfying the following conditions:*
(C_1) *T is an $\alpha-$admissible mapping;*
(C_2) *T is a generalized (α, ψ)- contractive mapping;*
(C_3) *There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(C_4) *T is continuous.*
Then T has a fixed point.

2 Main Results

In this section we introduce the notion of $(\alpha,\psi)-$ contractive mappings in the context of a multiplicative metric space as follows:

Let Ψ_4 be the family of functions $\psi:[1,\infty)\to[1,\infty)$ satisfying the following properties:

(i) ψ is upper semi-continuous and non-decreasing;

(ii) $\prod_{n=1}^{\infty}\psi^n(t)<\infty$ for each $t>1$, where ψ^n is the $n-$th iterate of ψ.

(iii) $\psi(t)<t$ for every $t>1$.

Example 2.1. Define $\psi(t)$=$1+\frac{t-1}{2}$.

Then $\psi(t)$=$1+\frac{t-1}{2}=\frac{t+1}{2}<t$ and ψ is increasing.

and $\psi^2(t)=\psi(\psi(t))=\psi(\frac{t+1}{2})=\frac{t+3}{4}$

$\psi^3(t)=\psi(\psi^2(t))=\frac{t+7}{8}$............ $\psi^n(t)=\frac{t+2^n-1}{2^n}$

Therefore $\psi^n(t)=\frac{t+2^n-1}{2^n}=1+\frac{t-1}{2^n}$

Since $\prod x_n$ is convergent $\Leftrightarrow\sum\mid x_n-1\mid$ is convergent,

hence $\prod\psi^n(t)$ is convergent $\Leftrightarrow\sum\mid\frac{t-1}{2^n}\mid$ is convergent.

Therefore $\prod\psi^n(t)$ is convergent.

Hence $\psi\in\Psi_4$.

Now we introduce $\alpha-$admissiblity of a mapping T on a partially ordered set X and *generalized* $(\alpha,\psi)-$ *contractivity of a mapping* T in a partially ordered multiplicative metric space X.

Definition 2.2. Let$(X,\preceq)$ be a partially ordered set, T be a self mapping on X and $\alpha:X\times X\to[0,\infty)$ be a function. We say that T is $\alpha-$admissible if x and y are comparable and $\alpha(x,y)\geq1\Rightarrow\alpha(Tx,Ty)\geq1$ for all $x,y\in X$.

Definition 2.3. Let T be a mapping of a complete partially ordered multiplicative metric space $(X,\preceq,d)$ into itself. Then T is said to be a *generalized* (α,ψ)*- contractive mapping* if there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi_4$ such that

$$\alpha(x,y).d(Tx,Ty)\leq\psi(M(x,y))\quad\text{for all } x,y\in X.$$

where $M(x,y)=max\{d(Tx,x),d(Ty,y),d(x,y),[d(Tx,y).d(Ty,x)]^{\frac{1}{2}}\}$, whenever x and y are comparable.

Now we extend the same definition to another contraction condition:

Definition 2.4. Let T be a mapping of a complete partially ordered multiplicative metric space $(X,\preceq,d)$ into itself. Then T is said to be a *generalized rational* (α,ψ)*- contractive mapping* if there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi_4$ such that

$$\alpha(x,y).d(Tx,Ty)\leq\psi(M_2(x,y))\quad\text{for all } x,y\in X,\text{ where}$$

$M_2(x,y)$= $max\{d(Tx,x),d(Ty,y),d(x,y),[d(Tx,y).d(Ty,x)]^{\frac{1}{2}},\frac{d(Tx,x).d(Ty,y)}{d(x,y)},\frac{d(Tx,y).d(Ty,x)}{d(x,y)},\frac{d(Tx,y).d(Ty,x)}{d(Tx,Ty)}\}$, whenever x and y are comparable.

We observe that from the above contraction conditions $M_1(x,y) \leq M_2(x,y)$
Now we establish one of our main results for *generalized* $(\alpha,\psi)-$ *contractive mappings* in a multiplicative metric space and a partially ordered multiplicative metric space.

Theorem 2.5. *Let T be a mapping of a complete multiplicative metric space (X,d) into itself and $\alpha : X \times X \to [0,\infty)$ be a given function and $\psi \in \Psi_4$ satisfying the following conditions:*
(C_1) *T is an $\alpha-$admissible mapping*
(C_2) *T is a generalized (α,ψ)- contractive mapping*
i.e., $\alpha(x,y).d(Tx,Ty) \leq \psi(M(x,y))$ *for all* $x,y \in X$.
where $M(x,y) = max\{d(Tx,x), d(Ty,y), d(x,y), [d(Tx,y).d(Ty,x)]^{\frac{1}{2}}\}$ (2.5.1)
(C_3) *There exists* $x_0 \in X$ *such that* $\alpha(x_0,Tx_0) \geq 1$ *and* $\alpha(x_0,T^2x_0) \geq 1$
(C_4) *T is continuous (or)*
(C_4') *Suppose* $x_n \to x^*$ *and* $\alpha(x_n,x_{n+1}) \geq 1$ *for every n, then* $\alpha(x_n,x^*) \geq 1$.
Then T has a fixed point.
(C_5) *Further if x and y are fixed points of T then either* $\alpha(x,y) < 1$ *or* $x = y$.

Proof. From (C_3), there exists a point $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$
We construct a sequence $\{x_n\}$ in X by $x_n = T^nx_0 = Tx_{n-1}$ for all $n \in N$.
It is obvious that if $x_n = x_{n+1}$ for some $n \in N$, then x_n is a fixed point of T.
Consequently, we suppose that $x_n \neq x_{n+1}$, $\forall n \in N$
From (C_1), T is an $\alpha-$admissible, and hence $\alpha(x_0,Tx_0) = \alpha(x_0,x_1) \geq 1$
This implies $\alpha(Tx_0,Tx_1) = \alpha(x_1,x_2) \geq 1$ and $\alpha(Tx_1,Tx_2) = \alpha(x_2,x_3) \geq 1$.
By induction, we get $\alpha(x_n,x_{n+1}) \geq 1\ \forall n \geq 0$ (2.5.2)
By similar argument, we have $\alpha(x_0,x_2) = \alpha(x_0,T^2x_0) \geq 1$
and hence $\alpha(Tx_0,Tx_2) = \alpha(x_1,x_3) \geq 1$
By induction, we get $\alpha(x_n,x_{n+2}) \geq 1\ \forall n \geq 0$ (2.5.3)
From (C_2) and (2.5.2) putting $x = x_n$ and $y = x_{n-1}$ in (2.5.1)
$\alpha(x_n,x_{n-1}).d(Tx_n,Tx_{n-1}) \leq \psi(M(x_n,x_{n-1}))$
Write $a_n = d(x_{n+1},x_n)$
Now $a_n = d(x_{n+1},x_n) = 1.d(Tx_n,Tx_{n-1}) \leq \alpha(x_n,x_{n-1}).d(Tx_n,Tx_{n-1})$
$\leq \psi(M(x_n,x_{n-1}))$ (2.5.4)
where $M(x_n,x_{n-1}) =$
$max\{d(Tx_n,x_n), d(Tx_{n-1},x_{n-1}), d(x_n,x_{n-1}), [d(Tx_n,x_{n-1}).d(Tx_{n-1},x_n)]^{\frac{1}{2}}\}$
$= max\{d(x_{n+1},x_n), d(x_n,x_{n-1}), d(x_n,x_{n-1}), [d(x_{n+1},x_{n-1}).d(x_n,x_n)]^{\frac{1}{2}}\}$
$= max\{d(x_{n+1},x_n), d(x_n,x_{n-1}), [d(x_{n-1},x_{n+1})]^{\frac{1}{2}}\}$
$= max\{d(x_{n+1},x_n), d(x_n,x_{n-1})\}$
$= max\{a_n,a_{n-1}\}$ (where $a_n = d(x_{n+1},x_n)$).
From (2.5.4)
$a_n = d(x_{n+1},x_n) \leq \psi(max\{a_n,a_{n-1}\})$

Suppose $a_{n-1} < a_n$, then

$a_n \leq \psi(a_n) < a_n$ (property of ψ)

a contradiction, since $a_n > 1$

Therefore $a_n \leq a_{n-1}$.

Therefore $a_n \leq \psi(a_{n-1}) < a_{n-1}$. (2.5.5)

Therefore $a_n < a_{n-1}$ for $n = 1, 2, 3, ...$

Therefore $\{a_n\}$ is a decreasing sequence, which converges to r (≥ 1) say.

since ψ is upper semi continuous, from (2.5.5)

$r \leq \psi(r) \leq r$ (as $n \to \infty$)

Therefore $r = 1$.

Now we show that $\{x_n\}$ is a Cauchy sequence in X

Suppose $x_n = x_m$ for some $n \neq m$, Without loss of generality we may assume that $m > n + 1$.

$a_n = d(x_n, x_{n+1}) \leq d(x_n, x_m).d(x_m, x_{n+1}) = 1.a_m$

Therefore $a_n \leq a_m < a_n$, a contradiction.

Therefore $x_n \neq x_m$, for $n \neq m$.

Now $d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}).d(x_{n+1}, x_{n+2})......d(x_{n+k-1}, x_{n+k})$

$= \psi^n(d(x_0, x_1)).\psi^{n+1}(d(x_0, x_1))......\psi^{n+k-1}(d(x_0, x_1))$

$= \prod_{p=n}^{n+k-1} \psi^p(d(x_0, x_1))$

$\leq \prod_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 1$ as $n \to \infty$.

Therefore $\{x_n\}$ is a Cauchy sequence in X.

Since X is a complete metric space, so there exist x^* in X such that $x_n \to x^*$.

i.e., $\lim_{n\to\infty} d(x_n, x^*) = 1$ (2.5.6)

Suppose (C'_4) holds.

i.e., T is continuous, then from (2.5.6) we have $Tx_n \to Tx^*$

i.e., $\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 1$

By uniqueness of limit $Tx^* = x^*$.

Therefore x^* is a fixed point of T.

Now suppose (C'_4) holds, then $x_n \to x^*$ and $\alpha(x_n, x_{n+1}) \geq 1$ for every n, so that $\alpha(x_n, x^*) \geq 1$

Now from (2.5.1) $d(Tx_n, Tx^*) \leq \alpha(x_n, x^*).d(Tx_n, Tx^*)$

$\leq \psi(M(x_n, x^*))$ (2.5.7)

where $M(x_n, x^*)$=

$max\{d(Tx_n, x_n), d(Tx^*, x^*), d(x_n, x^*), [d(Tx_n, x^*).d(Tx^*, x_n)]^{\frac{1}{2}}\}$

$= max\{d(x_{n+1}, x_n), d(Tx^*, x^*), d(x_n, x^*), [d(x_{n+1}, x^*).d(Tx^*, x_n)]^{\frac{1}{2}}\}$

On letting $n \to \infty$

$= max\{1, d(Tx^*, x^*), 1, [d(Tx^*, x^*)]^{\frac{1}{2}}\}$

$= d(Tx^*, x^*)$

From (2.5.7), $d(x^*, Tx^*) \leq \psi(M(x_n, x^*)) \to \psi(d(Tx^*, x^*)) < d(Tx^*, x^*)$

a contradiction,if $d(Tx^*, x^*) \neq 1$.

Therefore $d(Tx^*, x^*) = 1$

Therefore $Tx^* = x^*$

Therefore x^* is a fixed point of T.

Suppose x and y are fixed points of T, then we show that $\alpha(x, y) < 1$ or $x = y$

If $\alpha(x, y) < 1$ then we are through.

Suppose $\alpha(x, y) \geq 1$. Then

from (2.5.1) $\alpha(x, y).d(Tx, Ty) \leq \psi(M(x, y))$ (2.5.8)

where $M(x, y) = max\{d(Tx, x), d(Ty, y), d(x, y), [d(Tx, y), d(Ty, x)]^{\frac{1}{2}}\}$

$= max\{d(x, x), d(y, y), d(x, y), [d(x, y).d(y, x)]^{\frac{1}{2}}\}$

$= max\{1, 1, d(x, y), d(x, y)\}$

$= d(x, y)$

Therefore $d(x, y) \leq \alpha(x, y).d(Tx, Ty) \leq \psi(d(x, y)) < d(x, y)$ a contradiction, if $x \neq y$

Therefore $x = y$. □

Now we establish our second main result for *generalized rational* $(\alpha, \psi) - contractivemappings$.

Theorem 2.6. *Let T be a mapping of a compete multiplicative metric space (X, d) into itself mapping and $\alpha : X \times X \to [0, \infty)$ be a given function and $\psi \in \Psi_4$ satisfying the following conditions:*

(C_1) T is an $\alpha-$admissible mapping

(C_2) T is a generalized (α, ψ)- contractive mapping

i.e., $\alpha(x, y).d(Tx, Ty) \leq \psi(M_2(x, y))$ for all $x, y \in X$, where

$$M_2(x, y) = max\{d(Tx, x), d(Ty, y), d(x, y), [d(Tx, y).d(Ty, x)]^{\frac{1}{2}}, \frac{d(Tx,x).d(Ty,y)}{d(x,y)}, \frac{d(Tx,y).d(Ty,x)}{d(x,y)}, \frac{d(Tx,y).d(Ty,x)}{d(Tx,Ty)}\} \quad (2.6.1)$$

(C_3) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$

(C_4) T is continuous (or)

(C_4'') Suppose $x_n \to x^*$ and $\alpha(x_n, x_{n+1}) \geq 1$ for every n, then $\alpha(x_n, x^*) \geq 1$.

Then T has a fixed point.

(C_5) Further if x and y are fixed points of T then either $\alpha(x, y) < 1$ or $x = y$.

Proof. From (C_3), there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$

We construct a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in N$.

It is obvious that if $x_n = x_{n+1}$ for some $n \in N$, then x_n is a fixed point of T.

Consequently, we suppose that $x_n \neq x_{n+1}, \forall n \in N$

From (C_1), T is an $\alpha-$admissible, we have $\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1$

This implies $\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1$ and $\alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1$.

By induction, we get $\alpha(x_n, x_{n+1}) \geq 1 \; \forall n \geq 0$ (2.6.2)

By similar argument, $\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1$

and hence $\alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1$

By induction, we get $\alpha(x_n, x_{n+2}) \geq 1 \; \forall n \geq 0$ (2.6.3)

From (C_2) and (2.6.2) putting $x = x_n$ and $y = x_{n+1}$ in (2.6.1)

Write $a_n = d(x_n, x_{n+1})$

$\alpha(x_{n-1}, x_n).d(Tx_{n-1}, Tx_n) \leq \psi(M(x_{n-1}, x_n))$

Now $a_n = d(x_n, x_{n+1}) = 1.d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n).d(Tx_{n-1}, Tx_n)$

$$\leq \psi(M_2(x_{n-1}, x_n)) \qquad (2.6.4)$$

where $M_2(x_{n-1}, x_n) = max\{d(Tx_{n-1}, x_{n-1}), d(Tx_n, x_n), d(x_{n-1}, x_n), [d(Tx_{n-1}, x_n).d(Tx_n, x_{n-1})]^{\frac{1}{2}},$
$\frac{d(Tx_{n-1},x_{n-1}).d(Tx_n,x_n)}{d(x_{n-1},x_n)}, \frac{d(Tx_{n-1},x_n).d(Tx_n,x_{n-1})}{d(x_{n-1},x_n)}, \frac{d(Tx_{n-1},x_n).d(Tx_n,x_{n-1})}{d(Tx_{n-1},Tx_n)}\}$

$= max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n), d(x_{n-1}, x_n), [d(x_n, x_n).d(x_{n+1}, x_{n-1})]^{\frac{1}{2}},$
$\frac{d(x_n,x_{n-1}).d(x_{n+1},x_n)}{d(x_{n-1},x_n)}, \frac{d(x_n,x_n).d(x_{n+1},x_{n-1})}{d(x_{n-1},x_n)}, \frac{d(x_n,x_n).d(x_{n+1},x_{n-1})}{d(x_n,x_{n+1})}\}$

$= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), [d(x_{n+1}, x_{n-1})]^{\frac{1}{2}}, d(x_{n+1}, x_n), \frac{d(x_{n-1},x_{n+1})}{d(x_{n-1},x_n)}, \frac{d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1})}\}$

$\leq max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1},x_n).d(x_n,x_{n+1})}{d(x_{n-1},x_n)}, \frac{d(x_{n-1},x_n).d(x_n,x_{n+1})}{d(x_n,x_{n+1})}\}$

$= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}$

$= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$

$= max\{a_{n-1}, a_n\}$ (where $a_n = d(x_{n+1}, x_n)$).

From (2.6.4)

$a_n = d(x_{n+1}, x_n) \leq \psi(max\{a_n, a_{n-1}\})$

Suppose $a_{n-1} < a_n$, then

$a_n \leq \psi(a_n) < a_n$ (property of ψ)

a contradiction, since $a_n > 1$

Therefore $a_n \leq a_{n-1}$.

and hence $a_n \leq \psi(a_{n-1}) < a_{n-1}$ (2.6.5)

Therefore $a_n < a_{n-1}$ for $n = 1, 2, 3, ...$

Therefore $\{a_n\}$ is a decreasing sequence, which converges to r (≥ 1) say.

Since ψ is upper semi continuous, from (2.6.5)

$r \leq \psi(r) \leq r$ (as $n \to \infty$)

Therefore $r = 1$.

Now we show that $\{x_n\}$ is a Cauchy sequence in X

Suppose $x_n = x_m$ for some $n \neq m$, Without loss of generality we may assume that $m > n + 1$.

Therefore $x_{n+1} = x_{m+1}$

$a_n = d(x_{n+1}, x_n) = d(x_{m+1}, x_m) = a_m < a_n$ if $m > n$, a contradiction.

Therefore $x_n \neq x_m$, for $n \neq m$.

Now $d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}).d(x_{n+1}, x_{n+2})......d(x_{n+k-1}, x_{n+k})$

$= \psi^n(d(x_0, x_1)).\psi^{n+1}(d(x_0, x_1))......\psi^{n+k-1}(d(x_0, x_1))$

$= \prod_{p=n}^{n+k-1} \psi^p(d(x_0, x_1))$

$\leq \prod_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 1$ as $n \to \infty$.

Therefore $\{x_n\}$ is a Cauchy sequence in X.

Since X is a complete metric space, so there exist x^* in X such that $x_n \to x^*$.

i.e., $\lim_{n\to\infty} d(x_n, x^*) = 1$ (2.6.6)

Suppose (C_4) holds. i.e., T is continuous. Then from (2.6.6) we have $Tx_n \to Tx^*$

i.e., $\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 1$

By uniqueness of limit $Tx^* = x^*$.

Therefore x^* is a fixed point of T.

Now suppose (C_4'') holds. then $x_n \to x^*$ and $\alpha(x_n, x_{n+1}) \geq 1$ for every n, so that $\alpha(x_n, x^*) \geq 1$

Now from (2.6.1) $d(Tx_n, Tx^*) \leq \alpha(x_n, x^*).d(Tx_n, Tx^*)$

$\leq \psi(M_2(x_n, x^*))$ (2.6.7)

where $M_2(x_n, x^*)= max\{d(Tx_n, x_n), d(Tx^*, x^*), d(x_n, x^*), [d(Tx_n, x^*).d(Tx^*, x_n)]^{\frac{1}{2}}, \frac{d(Tx_n,x_n).d(Tx^*,x^*)}{d(x_n,x^*)}, \frac{d(Tx_n,x^*).d(Tx^*,x_n)}{d(x_n,x^*)}, \frac{d(Tx_n,x^*).d(Tx^*,x_n)}{d(Tx_n,Tx^*)}\}$

On letting $n \to \infty$

$= max\{1, d(Tx^*, x^*), 1, [d(Tx^*, x^*)]^{\frac{1}{2}}, d(Tx^*, x^*), d(Tx^*, x^*), \frac{d(Tx^*,x^*)}{d(x^*,Tx^*)}\}$

$= d(Tx^*, x^*)$

From (2.6.7), $d(x^*, Tx^*) \leq \psi(M_2(x_n, x^*)) \to \psi(d(Tx^*, x^*)) < d(Tx^*, x^*)$

a contradiction,if $d(Tx^*, x^*) \neq 1$.

Therefore $d(Tx^*, x^*) = 1$

Hence $Tx^* = x^*$

Therefore x^* is a fixed point of T.

Suppose x and y are fixed points of T, then we show that either $\alpha(x, y) < 1$ or $x = y$

If $\alpha(x, y) < 1$ then we are through.

Suppose $\alpha(x, y) \geq 1$. Then

from (2.6.1) $\alpha(x, y).d(Tx, Ty) \leq \psi(M_2(x, y))$ (2.6.8)

where $M_2(x, y) = max\{d(Tx, x), d(Ty, y), d(x, y), [d(Tx, y).d(Ty, x)]^{\frac{1}{2}}, \frac{d(x,x).d(y,y)}{d(x,y)}, \frac{d(x,y).d(y,x)}{d(x,y)}, \frac{d(x,y).d(y,x)}{d(x,y)}\}$

$= max\{1, 1, d(x, x), d(x, y), \frac{1}{d(x,y)}, d(x, y), d(y, x)\}$

$= d(x, y)$

Therefore $d(x, y) \leq \alpha(x, y).d(Tx, Ty) \leq \psi(d(x, y)) < d(x, y)$, a contradiction, if $x \neq y$

Therefore $x = y$. □

Now we establish our third main result concerning *generalized* $(\alpha, \psi) -$ *contractions* on partially ordered multiplicative metric spaces.

Theorem 2.7. *Let $(X, \preceq)$ be a partially ordered set and d be a multiplicative metric space on X. Let T be a self map on a complete multiplicative metric space on X and $\alpha : X \times X \to [0, \infty)$ be given function and $\psi \in \Psi_4$ satisfying the following conditions:*

(C_1) T is increasing and $\alpha-$admissible mapping

(C_2) T is a generalized (α, ψ)- contractive mapping

i.e., $\alpha(x, y).d(Tx, Ty) \leq \psi(M(x, y))$ (2.7.1)

where $M(x, y) = max\{d(Tx, x), d(Ty, y), d(x, y), [d(Tx, y).d(Ty, x)]^{\frac{1}{2}}\}$, for all $x, y \in X$, whenever x and y are compa-

rable.

(C_3) *There exists* $x_0 \in X$ *such that* $x_0 \leq Tx_0$, $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$

(C_4) *T is continuous (or)*

(C_4''') *Suppose* $\{x_n\}$ *is increasing and* $x_n \to x^* \Rightarrow \alpha(x_n, x^*) \geq 1$

Then T has a fixed point.

(C_5) *Further if x and y are fixed points of T then (i) x and y are not comparable. (ii)either x and y are comparable and* $\alpha(x, y) < 1$ *or (iii)* $x = y$.

Proof. From (C_3), there exists a point $x_0 \in X$ such that $x_0 \leq Tx_0$, $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$

We construct a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in N$.

It is obvious that if $x_n = x_{n+1}$ for some $n \in N$, then x_n is a fixed point of T.

Consequently, we suppose that $x_n \neq x_{n+1}$, $\forall n \in N$

Since $x_0 \leq Tx_0 \Rightarrow x_0 \leq x_1$

Since T is increasing, so $Tx_0 \leq Tx_1 \Rightarrow x_1 \leq x_2$

By induction $x_n \leq x_{n+1}$ $\forall n \in N$

Therefore $\{x_n\}$ is an increasing sequence.

Now $\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1$,

since T is an $\alpha-$admissible, $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$

and hence $\alpha(x_2, x_3) = \alpha(Tx_1, Tx_2) \geq 1$

By induction, we get $\alpha(x_n, x_{n+1}) \geq 1$ $\forall n \geq 0$ (2.7.2)

Now $\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1$,

since T is $\alpha-$admissible, $\alpha(x_1, x_3) = \alpha(Tx_0, Tx_2) \geq 1$

and hence $\alpha(x_3, x_4) \geq 1$.

by induction $\alpha(x_n, x_{n+1}) \geq 1$ $\forall n \geq 0$ (2.7.3)

Now $\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1$, since T is $\alpha-$admissible,

$\alpha(x_1, x_3) = \alpha(Tx_0, Tx_2) \geq 1$

and hence $\alpha(x_2, x_4) \geq 1$.

by induction $\alpha(x_n, x_{n+2}) \geq 1$ $\forall n \geq 0$ (2.7.4)

From (C_2) and (2.7.3) putting $x = x_n$ and $y = x_{n+1}$ in (2.7.1)

Write $a_n = d(x_n, x_{n+1})$

$$\text{Now } a_n = d(x_n, x_{n+1}) = 1.d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n).d(Tx_{n-1}, Tx_n)$$
$$\leq \psi(M(x_{n-1}, x_n)) \quad (2.7.5)$$

where $M(x_{n-1}, x_n) =$

$$max\{d(Tx_{n-1}, x_{n-1}), d(Tx_n, x_n), d(x_{n-1}, x_n), [d(Tx_{n-1}, x_n).d(Tx_n, x_{n-1})]^{\frac{1}{2}}\}$$
$$= max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n), d(x_{n-1}, x_n), [d(x_n, x_n).d(x_{n+1}, x_{n-1})]^{\frac{1}{2}}\}$$
$$= max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), [d(x_{n-1}, x_{n+1})]^{\frac{1}{2}}\}$$
$$= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$$
$$= max\{a_{n-1}, a_n\} \quad (\text{where } a_n = d(x_n, x_{n+1})).$$

From (2.7.5)

$a_n = d(x_n, x_{n+1}) \leq \psi(max\{a_n, a_{n-1}\})$

Suppose $a_{n-1} < a_n$, then

$a_n \leq \psi(a_n) < a_n$ (property of ψ)

a contradiction, since $a_n > 1$

Therefore $a_n \leq a_{n-1}$.

and hence $a_n \leq \psi(a_{n-1}) < a_{n-1}$ (2.7.6)

Therefore $a_n < a_{n-1}$ for $n = 1, 2, 3, ...$

Therefore $\{a_n\}$ is a decreasing sequence, which converges to r (≥ 1) say.

since ψ is upper semi continuous, from (2.7.6)

$r \leq \psi(r) \leq r$ (as $n \to \infty$)

Therefore $r = 1$.

Now we show that $\{x_n\}$ is a Cauchy sequence in X

Suppose $x_n = x_m$ for some $n \neq m$, Without loss of generality we may assume that $m > n + 1$.

$a_n = d(x_n, x_{n+1}) \leq d(x_n, x_m).d(x_m, x_{n+1}) = 1.a_m$

Therefore $a_n \leq a_m < a_n$, a contradiction.

Therefore $x_n \neq x_m$, for $n \neq m$.

Now $d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}).d(x_{n+1}, x_{n+2})......d(x_{n+k-1}, x_{n+k})$

$= \psi^n(d(x_0, x_1)).\psi^{n+1}(d(x_0, x_1))......\psi^{n+k-1}(d(x_0, x_1))$

$= \prod_{p=n}^{n+k-1} \psi^p(d(x_0, x_1))$

$\leq \prod_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 1$ as $n \to \infty$.

Therefore $\{x_n\}$ is a Cauchy sequence in X.

Since X is a complete metric space, so there exist x^* in X such that $x_n \to x^*$.

i.e., $\lim_{n\to\infty} d(x_n, x^*) = 1$ (2.7.7)

Suppose (C_4) holds. i.e., T is continuous, then from (2.7.7) we have $Tx_n \to Tx^*$

i.e., $\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 1$

By uniqueness of limit $Tx^* = x^*$.

Therefore x^* is a fixed point of T.

Now, suppose (C_4''') holds. Then $\{x_n\}$ is increasing and $x_n \to x^* \Rightarrow \alpha(x_n, x^*) \geq 1$

Now from (2.7.1) $d(Tx_n, x^*) \leq \alpha(x_n, x^*).d(Tx_n, Tx^*)$

$\leq \psi(M(x_n, x^*))$ (2.7.8)

where $M(x_n, x^*)$=

$max\{d(Tx_n, x_n), d(Tx^*, x^*), d(x_n, x^*), [d(Tx_n, x^*).d(Tx^*, x_n)]^{\frac{1}{2}}\}$

$= max\{d(x_{n+1}, x_n), d(Tx^*, x^*), d(x_n, x^*), [d(x_{n+1}, x^*).d(Tx^*, x_n)]^{\frac{1}{2}}\}$

On letting $n \to \infty$

$= max\{1, d(Tx^*, x^*), 1, [d(Tx^*, x^*)]^{\frac{1}{2}}\}$

$= d(Tx^*, x^*)$

From (2.7.8), $d(Tx^*, x^*) \leq \psi(M(x_n, x^*)) \to \psi(d(Tx^*, x^*)) < d(Tx^*, x^*)$

a contradiction,if $d(Tx^*, x^*) \neq 1$.

Therefore $d(Tx^*, x^*) = 1$

hence $Tx^* = x^*)$

Therefore x^* is a fixed point of T.

Suppose x and y are fixed points of T, then we show that either

(i) if x and y are not comparable (or)

(ii) if x and y are comparable and $\alpha(x, y) < 1$ (or)

(iii) $x = y$

If x and y are not comparable, there is nothing to prove.

If x and y are comparable and $\alpha(x, y) < 1$ then we are through.

Suppose (i) and (ii) are do not hold, then x and y are comparable and $\alpha(x, y) \geq 1$.

From (2.7.1) $\alpha(x, y) = 1.d(x, y) \leq \alpha(x, y).d(Tx, Ty) \leq \psi(M(x, y))$ (2.7.9)

where $M(x, y) = max\{d(Tx, x), d(Ty, y), d(x, y), [d(Tx, y), d(Ty, x)]^{\frac{1}{2}}\}$

$= max\{d(x, x), d(y, y), d(x, y), [d(x, y).d(y, x)]^{\frac{1}{2}}\}$

$= max\{1, 1, d(x, y), d(x, y)\}$

$= d(x, y)$

Therefore $d(x, y) \leq \alpha(x, y).d(Tx, Ty) \leq \psi(d(x, y)) < d(x, y)$ a contradiction, if $x \neq y$

Therefore $x = y$. □

Now we establish our fourth main result concerning *generalized rational* $(\alpha, \psi) - $ *contraction* on partially ordered multiplicative metric spaces.

Theorem 2.8. *Let $(X, \preceq)$ be a partially ordered set and d be a multiplicative metric space on X. Let T be a self map on a complete multiplicative metric space on X and $\alpha : X \times X \to [0, \infty)$ be given function and $\psi \in \Psi_4$ satisfying the following conditions:*

(C_1) *T is increasing and $\alpha-$admissible mapping*

(C_2) *T is a generalized (α, ψ)- contractive mapping*

i.e., $\alpha(x, y).d(Tx, Ty) \leq \psi(M_2(x, y))$ (2.8.1)

where $M_2(x, y)= max\{d(Tx, x), d(Ty, y), d(x, y), [d(Tx, y).d(Ty, x)]^{\frac{1}{2}},$

$\frac{d(Tx,x).d(Ty,y)}{d(x,y)}, \frac{d(Tx,y).d(Ty,x)}{d(x,y)}, \frac{d(Tx,y).d(Ty,x)}{d(Tx,Ty)}\}$ *for all $x, y \in X$, whenever x and y are comparable.*

(C_3) *There exists $x_0 \in X$ such that $x_0 \leq Tx_0$, $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$*

(C_4) *T is continuous (or)*

(C_4''') *Suppose $\{x_n\}$ is increasing and $x_n \to x^* \Rightarrow \alpha(x_n, x^*) \geq 1$*

Then T has a fixed point.

(C_5) *Further if x and y are fixed points of T then (i) x and y are not comparable. (ii)either x and y are comparable and $\alpha(x, y) < 1$ or (iii) $x = y$.*

Proof. From (C_3), there exists a point $x_0 \in X$ such that $x_0 \leq Tx_0$, $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$

We construct a sequence $\{x_n\}$ in X by $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in N$.

It is obvious that if $x_n = x_{n+1}$ for some $n \in N$, then x_n is a fixed point of T.

Consequently, we suppose that $x_n \neq x_{n+1}, \forall n \in N$

Since $x_0 \leq Tx_0 \Rightarrow x_0 \leq x_1$

Since T is increasing, so $Tx_0 \leq Tx_1 \Rightarrow x_1 \leq x_2$

By induction $x_n \leq x_{n+1}\ \forall n \in N$

Therefore $\{x_n\}$ is an increasing sequence.

Now $\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1$,

since T is an $\alpha-$admissible, $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$

and hence $\alpha(x_2, x_3) = \alpha(Tx_1, Tx_2) \geq 1$

By induction, we get $\alpha(x_n, x_{n+1}) \geq 1\ \forall n \geq 0$ (2.8.2)

Now $\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1$,

since T is $\alpha-$admissible, $\alpha(x_1, x_3) = \alpha(Tx_0, Tx_2) \geq 1$

and hence $\alpha(x_3, x_4) \geq 1$.

by induction $\alpha(x_n, x_{n+1}) \geq 1\ \forall n \geq 0$ (2.8.3)

Now $\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1$, since T is $\alpha-$admissible,

$\alpha(x_1, x_3) = \alpha(Tx_0, Tx_2) \geq 1$

and hence $\alpha(x_2, x_4) \geq 1$.

by induction $\alpha(x_n, x_{n+2}) \geq 1\ \forall n \geq 0$ (2.8.4)

From (C_2) and (2.8.3) putting $x = x_n$ and $y = x_{n+1}$ in (2.8.1)

Write $a_n = d(x_n, x_{n+1})$

$\alpha(x_{n-1}, x_n).d(Tx_{n-1}, Tx_n) \leq \psi(M_2(x_{n-1}, x_n))$

Now $a_n = d(x_n, x_{n+1}) = 1.d(Tx_{n-1}, Tx_n) \leq \alpha(x_{n-1}, x_n).d(Tx_{n-1}, Tx_n)$

$$\leq \psi(M_2(x_{n-1}, x_n)) \qquad (2.8.5)$$

where $M_2(x_{n-1}, x_n) = max\{d(Tx_{n-1}, x_{n-1}), d(Tx_n, x_n), d(x_{n-1}, x_n), [d(Tx_{n-1}, x_n).d(Tx_n, x_{n-1})]^{\frac{1}{2}},$
$\frac{d(Tx_{n-1},x_{n-1}).d(Tx_n,x_n)}{d(x_{n-1},x_n)}, \frac{d(Tx_{n-1},x_n).d(Tx_n,x_{n-1})}{d(x_{n-1},x_n)}, \frac{d(Tx_{n-1},x_n).d(Tx_n,x_{n-1})}{d(Tx_{n-1},Tx_n)}\}$

$$= max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n), d(x_{n-1}, x_n), [d(x_n, x_n).d(x_{n+1}, x_{n-1})]^{\frac{1}{2}}, \frac{d(x_n,x_{n-1}).d(x_{n+1},x_n)}{d(x_{n-1},x_n)}, \frac{d(x_n,x_n).d(x_{n+1},x_{n-1})}{d(x_{n-1},x_n)}, \frac{d(x_n,x_n).d(x_{n+1},x_{n-1})}{d(x_n,x_{n+1})}\}$$

$$= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), [d(x_{n+1}, x_{n-1})]^{\frac{1}{2}}, d(x_{n+1}, x_n), \frac{d(x_{n-1},x_{n+1})}{d(x_{n-1},x_n)}, \frac{d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1})}\}$$

$$\leq max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1},x_n).d(x_n,x_{n+1})}{d(x_{n-1},x_n)}, \frac{d(x_{n-1},x_n).d(x_n,x_{n+1})}{d(x_n,x_{n+1})}\}$$

$$= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}$$

$$= max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$$

$$= max\{a_{n-1}, a_n\} \quad (\text{where } a_n = d(x_{n+1}, x_n)).$$

From (2.8.5)

$a_n = d(x_n, x_{n+1}) \leq \psi(max\{a_n, a_{n-1}\})$

Suppose $a_{n-1} < a_n$, then

$a_n \leq \psi(a_n) < a_n$ (property of ψ)

a contradiction, since $a_n > 1$

Therefore $a_n \leq a_{n-1}$.

and hence $a_n \leq \psi(a_{n-1}) < a_{n-1}$ (2.8.6)

Therefore $a_n < a_{n-1}$ for $n = 1, 2, 3, ...$

Therefore $\{a_n\}$ is a decreasing sequence, which converges to r (≥ 1) say.

since ψ is upper semi continuous, from (2.8.6)

$r \leq \psi(r) \leq r$ (as $n \to \infty$)

Therefore $r = 1$.

Now we show that $\{x_n\}$ is a Cauchy sequence in X

Suppose $x_n = x_m$ for some $n \neq m$, Without loss of generality we may assume that $m > n+1$.

Therefore $x_{n+1} = x_{m+1}$

$a_n = d(x_{n+1}, x_n) = d(x_{m+1}, x_m) = a_m < a_n$ if $m > n$, a contradiction.

Therefore $x_n \neq x_m$, for $n \neq m$.

Now $d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}).d(x_{n+1}, x_{n+2})......d(x_{n+k-1}, x_{n+k})$

$= \psi^n(d(x_0, x_1)).\psi^{n+1}(d(x_0, x_1))......\psi^{n+k-1}(d(x_0, x_1))$

$= \prod_{p=n}^{n+k-1} \psi^p(d(x_0, x_1))$

$\leq \prod_{p=n}^{\infty} \psi^p(d(x_0, x_1)) \to 1$ as $n \to \infty$.

Therefore $\{x_n\}$ is a Cauchy sequence in X.

Since X is a complete metric space, so there exist x^* in X such that $x_n \to x^*$.

i.e., $\lim_{n\to\infty} d(x_n, x^*) = 1$ (2.8.7)

Suppose (C_4) holds. i.e., T is continuous, then from (2.8.7) we have $Tx_n \to Tx^*$

i.e., $\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 1$

By uniqueness of limit $Tx^* = x^*$.

Therefore x^* is a fixed point of T.

Now suppose (C_4'''') holds. Then $\{x_n\}$ is increasing and $x_n \to x^* \Rightarrow \alpha(x_n, x^*) \geq 1$

Now from (2.8.1) $d(Tx_n, x^*) \leq \alpha(x_n, x^*).d(Tx_n, Tx^*)$

$\leq \psi(M_2(x_n, x^*))$ (2.8.8)

where $M_2(x_n, x^*) = max\{d(Tx_n, x_n), d(Tx^*, x^*), d(x_n, x^*), [d(Tx_n, x^*).d(Tx^*, x_n)]^{\frac{1}{2}}, \frac{d(Tx_n,x_n).d(Tx^*,x^*)}{d(x_n,x^*)}, \frac{d(Tx_n,x^*).d(Tx^*,x_n)}{d(x_n,x^*)}, \frac{d(Tx_n,x^*).d(Tx^*,x_n)}{d(Tx_n,Tx^*)}\}$

On letting $n \to \infty$

$= max\{1, d(Tx^*, x^*), 1, [d(Tx^*, x^*)]^{\frac{1}{2}}, d(Tx^*, x^*), d(Tx^*, x^*), \frac{d(Tx^*,x^*)}{d(x^*,Tx^*)}\}$

$= d(Tx^*, x^*)$

From (2.8.8), $d(x^*, Tx^*) \leq \psi(M_2(x_n, x^*)) \to \psi(d(Tx^*, x^*)) < d(Tx^*, x^*)$

a contradiction,if $d(Tx^*, x^*) \neq 1$.

Therefore $d(Tx^*, x^*) = 1$

Hence $Tx^* = x^*$.

Therefore x^* is a fixed point of T.

Suppose x and y are fixed points of T, then we show that either

(i) if x and y are not comparable (or)

(ii) if x and y are comparable and $\alpha(x, y) < 1$ (or)

(iii) $x = y$

If x and y are not comparable, there is nothing to prove.

If x and y are comparable and $\alpha(x,y) < 1$ then we are through.

Suppose (i) and (ii) are do not hold, then x and y are comparable and $\alpha(x,y) \geq 1$

From (2.8.1) $\alpha(x,y) = 1.d(x,y) \leq \alpha(x,y).d(Tx,Ty) \leq \psi(M_2(x,y))$ (2.8.9)

where $M_2(x,y) = max\{d(Tx,x), d(Ty,y), d(x,y), [d(Tx,y), d(Ty,x)]^{\frac{1}{2}}\}$
$\frac{d(Tx,x).d(Ty,y)}{d(x,y)}, \frac{d(Tx,y).d(Ty,x)}{d(x,y)}, \frac{d(Tx,y).d(Ty,x)}{d(Tx,Ty)}\}$

$= max\{d(x,x), d(y,y), d(x,y), [d(x,y).d(y,x)]^{\frac{1}{2}}\}$
$\frac{d(x,x).d(y,y)}{d(x,y)}, \frac{d(x,y).d(y,x)}{d(x,y)}, \frac{d(x,y).d(y,x)}{d(x,y)}\}$

$= max\{1, 1, d(x,y), d(x,y), \frac{1}{d(x,y)}, d(x,y), d(x,y)\}$

$= d(x,y)$

Therefore $d(x,y) \leq \alpha(x,y).d(Tx,Ty) \leq \psi(d(x,y)) < d(x,y)$ a contradiction, if $x \neq y$

Therefore $x = y$. □

Corollary 2.9. *Theorem 1.17 (Praveen Kumar et al.[6], theorem 2.2)*

Proof. We have that $M_1(x,y) \leq M_2(x,y)$

From theorem 2.6 i.e., $\alpha(x,y).d(Tx,Ty) \leq \psi(M_1(x,y)) \leq \psi(M_2(x,y))$

Therefore T has a fixed point. □

Now we give two examples supporting our results, showing the significance of (C_4') and (C_4'').

Example 2.10. Let $X = \{a, b\}$ and a, b are not comparable. Let d be the multiplicative metric on X defining $d(a,b) = 2$, $d(a,a) = d(b,b) = 1$.

Define $T : X \to X$ by $T(a) = a$, $T(b) = b$ and

$\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \begin{cases} 1 & if \;\; x = y \\ \frac{1}{2} & otherwise \end{cases}$

Define ψ by $\psi(t) = \frac{t+1}{2}$, if $t > 1$

Now $\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$, Since $\alpha(Ta,Ta) = \alpha(a,a) \geq 1$.

Therefore T satisfies conditions (C_1)-(C_4)

Therefore a and b are fixed points of T.

This example shows that T may have two fixed points x and y with $\alpha(x,y) < 1$

Example 2.11. Let $X = [1,\infty)$. Let d be the multiplicative metric on X defined by

$d(x,y) = \begin{cases} 1 & if \;\; x = y \\ 2 & otherwise \end{cases}$ and $\alpha(x,y) = \begin{cases} 1 & if \;\; x = y \\ \frac{1}{2} & otherwise \end{cases}$

$T : X \to X$ by $T(x) = x, \forall\, x$, and $\psi(t) = \frac{t+1}{2}$, if $t > 1$

Then $\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) = \alpha(x,y) \geq 1$

Therefore T satisfies all conditions of theorem 2.7

Therefore Every point of X is a fixed points of T.

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