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Total resolving number of subdivision and total graphs

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ABSTRACT. Let $G = (V, E)$ be a simple connected graph. An ordered subset W of V is said to be a *resolving set* of G if every vertex is uniquely determined by its vector of distances to the vertices in W. The minimum cardinality of a resolving set is called the *resolving number* of G and is denoted by $r(G)$. As an extension, the *total resolving number* was introduced in [6] as the minimum cardinality taken over all resolving sets in which $\langle W \rangle$ has no isolates and it is denoted by $tr(G)$. In this paper, we obtain the bounds on the total resolving number of subdivision graphs and total graphs. Also, we characterize the extremal graphs.

1 Introduction

Let $G = (V, E)$ be a finite, simple, connected and undirected graph. The *degree* of a vertex v in a graph G is the number of edges incident to v and it is denoted by $d(v)$. The maximum degree in a graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest u-v path in G. The maximum value of distance between vertices of G is called its *diameter*. P_n denote the *path* on n vertices. C_n denote the *cycle* on n vertices. K_n denote the *complete graph* on n vertices. A graph is *acyclic* if it has no cycles. A *tree* is a connected acyclic graph. A *spider* is a tree with one vertex of degree at least 3 and all others with degree at most 2. A complete bipartite graph is denoted by $K_{s,t}$. A *star* is denoted by $K_{1,n-1}$. A *tree* obtained by joining the centres of two stars $K_{1,s}$ and $K_{1,t}$ by an edge is called a *bistar* and it is denoted by $B_{s,t}$.

For a cut vertex v of a connected graph G, suppose that the disconnected graph $G \setminus \{v\}$ has k components $G_1, G_2, \ldots, G_k$ $(k \geq 2)$. The induced subgraphs $B_i = G[V(G_i) \cup \{v\}]$ are connected and referred to as the *brances* of G at v. A cut vertex v is a *path support* if there is a nontrivial path as a branch at v; a *simple path support* if there is

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exactly one path support at v; a *multi path support* if there are more than one path support at v. A graph contains exactly one cycle is called a *unicyclic graph.* To *identify* non adjacent vertices x and y of a graph G is to replace these vertices by a single vertex which is incident to all the edges which were incident in G to either x or y. A *(k, l)-kite* is a graph obtained by identifying any vertex of a cycle C_k with an end vertex of a path P_l. A *partition* of a set A is a list $A_1, A_2, \ldots, A_k$ of subsets of A such that each element of A appears in exactly one subset in the list.

A vertex of degree at least 3 in a graph G is called a *major vertex* of G. Any end vertex u of G is said to be a *terminal vertex* of a major vertex v of G if $d(u,v) < d(u,w)$ for every other major vertex w of G. The *terminal degree* $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\sigma(G)$ be the sum of the terminal degrees of the major vertices of G and $ex(G)$ be the number of exterior major vertices of G. Let $\theta(G)$ be the number of exterior major vertices of G with terminal degree at least two.

If $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ is an ordered set, then the ordered k-tuple $(d(v,w_1), d(v,w_2), ..., d(v,w_k))$ is called the representation of v with respect to W and it is denoted by $r(v|W)$. Since the representation for each $w_i \in W$ contains exactly one 0 in the i^{th} position, all the vertices of W have distinct representations. W is called a *resolving set* for G if all the vertices of $V \setminus W$ also have distinct representations. The minimum cardinality of a resolving set is called the *resolving number* of G and it is denoted by $r(G)$. In [6] we introduced and studied total resolving number. If W is a resolving set and the induced subgraph $\langle W \rangle$ has no isolates, then W is called a *total resolving set* of G. The minimum cardinality taken over all total resolving sets of G is called the *total resolving number* of G and is denoted by $tr(G)$.

In this paper, we obtain the bounds on the total resolving number of subdivision graphs and total graphs. Also, we characterize the extremal graphs.

2 Total Resolving Number of Graphs

The following results are used in subsequent sections.

Observation 2.1. *[6] Let $\{w_1, w_2\} \subset V(G)$ be a total resolving set in G. Then the degrees of w_1 and w_2 are at most 3.*

Theorem 2.2. *[6] For $n \geq 3$, $tr(P_n) = 2$ and $tr(C_n) = 2$.*

Observation 2.3. *[6] Let G a graph of order $n \geq 3$. Then $2 \leq tr(G) \leq n-1$.*

Observation 2.4. *[6] Let G be a unicyclic graph with even cycle C_k. Then $tr(G) = 2$ if and only if at most two adjacent vertices of C_k are simple path supports and no vertex of G is a multi path support.*

Theorem 2.5. *[6] If T is a tree that is not a path, then $tr(T) = \sigma(T) - ex(T) + \theta(T)$.*

Notation 2.6. *[7] Let $\mathscr{G}$ be the collection of graphs G such that G is the union of two distinct paths $P_1 : x_1x_2 \ldots x_r$, $P_2 : y_1y_2 \ldots y_s$, $r \leq s$ and $x_1y_1 \in E(G)$, $x_iy_i \in E(G)$ for at least one i, $2 \leq i \leq r$.*

Theorem 2.7. *[7] If G is a bipartite graph that is not a path, then $tr(G) = 2$ if and only if $G \in \mathscr{G}$.*

3 Subdivision Graphs

In this section, we obtain the bounds for the total resolving number of subdivision graph of a general graph and characterize the extremal graphs.

Definition 3.1. *The subdivision graph $S(G)$ of a graph G is the graph obtained from G by deleting every edge uv of G and replacing it by a vertex w of degree 2 that is joined to u and v.*

By Theorems 2.2 and 2.5, we have the following Observation.

Observation 3.2. *For $n \geq 3$,*
(i)$tr(S(P_n)) = tr(P_n) = 2$
(ii)$tr(S(C_n)) = tr(C_n) = 2$
(iii)$tr(S(T)) = tr(T) = \sigma(T) - ex(T) + \theta(T)$.

Theorem 3.3. *Let G be a graph of order $n \geq 3$. Then $2 \leq tr(S(G)) \leq n-1$.*

Proof. By Observation 2.3, $tr(S(G)) \geq 2$. Next, we claim that $tr(S(G)) \leq n-1$. We consider the following two cases.

Case $1 : \Delta(G) = n-1$.

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $d(v_1) = n-1$. Let v_{ij} be the new vertex of the edge v_iv_j in $S(G)$. Let $W = \{v_1\} \cup \{v_{12}, v_{13}, \ldots, v_{1(n-1)}\}$. Then the first coordinate of the representation of v_{in} is 1 and others are 2, first coordinate of the representation of v_n is 2 and others are 3, for $2 \leq i \neq j \leq n-1$, i^{th} and j^{th} coordinates of the representation of v_{ij} are 2 and $v_iv_j \in E(G)$ and for $2 \leq i \leq n-1$, i^{th} coordinate of the representation of v_{in} is 1 and $v_iv_n \in E(G)$. It follows that W is a resolving set of $S(G)$. Since $\langle W \rangle$ has no isolates, $tr(S(G)) \leq n-1$.

Case $2 : \Delta(G) \leq n-2$.

If $n = 3$ or 4, then we can easily verify that $tr(S(G)) \leq n-1$. So we may assume that $n \geq 5$. Let $V_1, V_2, \ldots, V_k$ be the partition of $V(G)$ and $|V_i| = r_i$, where $V_i = \{v_{i1}, v_{i2}, \ldots, v_{ir_i}\}$ and v_{i1} is adjacent to all vertices of V_i for all $1 \leq i \leq k$. Since $\Delta(G) \leq n-2$, $k \geq 2$. Now, let $|V_1| \geq 3$ and $|V_i| \geq 2$ for all $2 \leq i \leq k$. Let v_{ij} be the new vertex of the edge v_iv_j in $S(G)$. Let $W = \{v_{i1} \ / \ 1 \leq i \leq k\} \cup \{v_{(11)(1i)} \ / \ 2 \leq i \leq r_1 - 1\} \cup \{v_{(i1)(i2)}, v_{(i1)(i3)}, \ldots, v_{(i1)(ir_i)} \ / \ 2 \leq i \leq k\}$. We claim that W is a resolving set of $S(G)$. Let x, y be two distinct vertices of $V(S(G)) \setminus W$. Let $X_i = V(S\langle V_i \rangle)$, $1 \leq i \leq k$. If $x, y \in X_i$, then $r(x|W \cap X_i) \neq r(y|W \cap X_i)$ and hence $r(x|W) \neq r(y|W)$. If $x \in X_i$ and $y \in X_j$, $i \neq j$, let $A = X_i \cup X_j$. Then $r(x|W \cap A) \neq r(y|W \cap A)$ and hence $r(x|W) \neq r(y|W)$. Now, we assume that $x \notin X_i$ for all $1 \leq i \leq k$. We consider the following two subcases.

Subcase $2.1 : y \in X_i$ for some $1 \leq i \leq k$.

Without loss of generality, let $y \in X_1$. If $d(x, w) \neq d(y, w)$ for some $w \in W \cap X_1$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, w) = d(y, w)$ for all $w \in W \cap X_1$. Then x is the neighbor of v_{11} and $y = v_{1r_1}$. Since $d(x) = 2$, neighbor of x other than v_{11} belongs to X_i for some $2 \leq i \leq k$. Without loss of generality, let X_2 be such set. Then $d(x, w) > d(y, w)$ for all $w \in W \cap X_2$ and hence $r(x|W) \neq r(y|W)$.

Subcase $2.2 : y \notin X_i$ for all $1 \leq i \leq k$.

Since $d(x) = d(y) = 2$ in $S(G)$, let $N(x) = \{x_1, x_2\}$ and $N(y) = \{y_1, y_2\}$. Then x_1, x_2, y_2 and y_2 are in union of two, three or four partite sets of $V(G)$. Without loss of generality, let $x_1, x_2, y_1, y_2 \in V_1 \cup V_2 \cup V_3 \cup V_4$. If $x_1, y_1 \in V_1$, $x_2 \in V_2$ and $y_2 \in V_3$ or $x_1 \in V_1$, $y_1 \in V_2$, $x_2 \in V_3$ and $y_2 \in V_4$, then $r(x|W \cap X_1) \neq r(y|W \cap X_1)$. It follows that $r(x|W) \neq r(y|W)$. Let $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$ and $B = X_1 \cup X_2$. If $x_1 \neq y_1$ and $x_2 \neq y_2$, then $r(x|A) \neq r(y|A)$. If $x_1 = y_1$ and $x_2 \neq y_2$, then $r(x|W \cap X_2) \neq r(y|W \cap X_2)$ and hence $r(x|W) \neq r(y|W)$.

Thus W is a resolving set of $S(G)$. Since $\langle W \rangle$ has no isolates and $|W| = n - 1$, $tr(S(G)) \leq n - 1$. □

Theorem 3.4. *For $s, t \geq 2$ and $t \geq s$, $tr(S(K_{s,t})) = s + t - 2$.*

Proof. Let $V(K_{s,t}) = S \cup T$, where $S = \{u_i \ / \ 1 \leq i \leq s\}$, $T = \{v_j / 1 \leq j \leq t\}$ and $E(K_{s,t}) = \{u_i v_j \ / \ 1 \leq i \leq s, 1 \leq j \leq t\}$. Let $A = S \cup T$ and $B = E(K_{s,t})$. Then $V(S(K_{s,t})) = A \cup B$. Let W be a total resolving set of $S(K_{s,t})$. First, we claim that $tr(S(K_{s,t})) \geq s + t - 2$.

Suppose $tr(S(K_{s,t})) \leq s + t - 3$. Since W is a total resolving set, either $|W \cap V(A)| \leq s - 2$ and $|W \cap V(B)| \leq t - 1$ or $|W \cap V(A)| \leq s - 1$ and $|W \cap V(B)| \leq t - 2$. If $|W \cap V(A)| \leq s - 1$ and $|W \cap V(B)| \leq t - 2$, then at least two vertices of $V(T)$ are not in W. Let v_1 and v_2 be such vertices. Since W is a total resolving set and $|W \cap V(B)| \leq t - 2$, neighbors of v_1 and v_2 are not in W. Thus $r(v_1|W) = r(v_2|W)$, which is a contradiction. If $|W \cap V(A)| \leq s - 2$ and $|W \cap V(B)| \leq t - 1$, then at least two vertices of $V(T)$ and $V(S)$ are not in W. Let v_1, v_2 and u_1, u_2 be such vertices. Since W is a total resolving set and $|W \cap V(B)| \leq t - 1$, either neighbors of v_1 and v_2 or u_1 and u_2 are not in W. Without loss of generality, let u_1 and u_2 be such vertices. Thus $r(u_1|W) = r(u_2|W)$, which is a contradiction. Thus $tr(S(K_{s,t})) \geq s + t - 2$. By Theorem 3.3, $tr(S(K_{s,t})) \leq s + t - 2$ and hence $tr(S(K_{s,t})) = s + t - 2$. □

Theorem 3.5. *For $n \geq 3$, $tr(S(K_n)) = n - 1$.*

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(K_n) = \{v_i v_j \ / \ 1 \leq i \leq n, 2 \leq j \leq n, i < j\}$. Let $M = V(K_n)$ and $N = E(K_n)$. Then $V(S(K_n)) = M \cup N$. Let W be a total resolving set of $S(K_n)$. By Theorem 3.3, $tr(S(K_n)) \leq n - 1$. Next, we claim that $tr(S(K_n)) \geq n - 1$.

Suppose $tr(S(K_n)) \leq n - 2$. Since W is a total resolving set, $|W \cap V(M) \geq 1|$. Since $tr(S(K_n)) \leq n - 2$, $|W \cap V(N)| \leq n - 3$. If $|W \cap V(M)| = 1$, then without loss of generality, let $v_1 \in W$. Since W is a total resolving set and $|W \cap V(N)| \leq n - 3$, $(W \setminus \{v_1\}) \subset \{v_1 v_2, v_1 v_3, \ldots, v_1 v_n\}$ and $|W \setminus \{v_1\}| = n - 3$. Without loss of generality, let $W \setminus \{v_1\} = \{v_1 v_2, v_1 v_3, \ldots, v_1 v_{n-2}\}$. Then $r(v_{n-1}|W) = r(v_n|W)$, which is a contradiction. Similarly, we can prove other cases. Thus $tr(S(K_n)) \geq n - 1$.

Hence $tr(S(K_n)) = n - 1$. □

Remark 3.6. *By Observation 3.2, $tr(S(K_{1,n-1})) = n - 1$ and $tr(S(B_{s,t})) = s + t$.*

Theorem 3.7. *Let G be a graph of order $n \geq 3$. Then $tr(S(G)) = 2$ if and only if G is isomorphic to P_n or C_n or (k, l)-kite.*

Proof. Assume that $tr(S(G)) = 2$. Since $S(G)$ is a bipartite graph, by Theorem 2.7, we can easily verify that G is isomorphic to P_n or C_n or (k, l)-kite. Conversely, if $G \cong P_n$, then $S(G) \cong P_{2n-1}$ and hence by Theorem 2.2, $tr(S(G)) = 2$. If $G \cong C_n$, then $S(G) \cong C_{2n}$ and hence by Theorem 2.2, $tr(S(G)) = 2$. If $G \cong$(k, l)-kite, then $S(G) \cong (2k, 2l)$-kite and hence by Observation 2.4, $tr(S(G)) = 2$. □

Theorems 3.3 and 3.5 with the first part of Remark 3.6 suggest the following.

Open problem 3.8. *If G is a connected graph of order n, then characterize graphs for which* $tr(S(G)) = n-1$.

4 Total Graphs

In this section, we obtain the bounds for the total resolving number of total graph of a general graph and characterize the extremal graphs. Also, we determine the exact value of total resolving number of the total graph of cycles, spiders and bistars.

Definition 4.1. *The total graph* $T(G)$ *of a graph G is a graph whose vertex set is* $V(T(G)) = V(G) \cup E(G)$ *and two distinct vertices x and y of* $T(G)$ *are adjacent if x and y are adjacent vertices of G or adjacent edges of G or x is a vertex incident with edge y.*

Theorem 4.2. *Let G be a graph of order* $n \geq 3$. *Then* $tr(T(G)) = 2$ *if and only if* $G \cong P_n$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $V(T(G)) = V(G) \cup E(G)$.

Assume that $tr(T(G)) = 2$. Let $W = \{w_1, w_2\}$ be a total resolving set of $T(G)$. Then by Observation 2.1, $d(w_1) \leq 3$ and $d(w_2) \leq 3$. First, we claim that $\delta(G) = 1$. Suppose $\delta(G) \geq 2$. If $n = 4$, then $tr(T(G)) = 3$. If $n \geq 5$, then $\delta(G) \geq 4$. By Observation 2.1, $tr(T(G)) \geq 3$, which is a contradiction. Thus $\delta(G) = 1$. Now, we claim that $\Delta(G) = 2$. Suppose $\Delta(G) \geq 3$. Suppose $G \cong (3,l)$-kite. If $l = 1$ or 2, then $tr(T(G)) = 3$. Let $l \geq 3$. Let $v_1v_2v_3v_1$ be the cycle of $(3,l)$-kite, u be the pendant and v be its neighbor. Let $d(v_1) = 3$. Then by Observation 2.1, one vertex of W is u and another one is v. But $d(v_2, u) = d(v_3, u)$ and $d(v_2, v) = d(v_3, v)$. It follows that $r(v_2|W) = r(v_3|W)$, which is a contradiction. Suppose $G \cong (k,l)$-kite, $k \geq 4$. If $l = 1$ or 2, then we can easily verify that $tr(T(G)) \neq 2$. Let $l \geq 3$. Let $v_1v_2v_3 \ldots v_kv_1$ be the cycle C_k of (k, l)-kite and $v_kv_{k+1}v_{k+2} \ldots v_n$ be the path of (k,l)-kite. Then $d_{T(G)}(v_n) = 2$, $d_{T(G)}(v_{n-1}) = 3$ and $d(v_i) \geq 4$, $1 \leq i \leq n-2$. So $W = \{v_n, v_{n-1}\}$. But $d_{T(G)}(v_1, v_n) = d_{T(G)}(v_{k-1}, v_n)$ and $d_{T(G)}(v_1, v_{n-1}) = d_{T(G)}(v_{k-1}, v_{n-1})$. It follows that $r(v_1|W) = r(v_{k-1}|W)$, which is a contradiction. If $G \not\cong (k,l)$-kite, then we use the similar argument we get $tr(T(G)) \geq 3$. Thus $\Delta(G) = 2$. Since $\delta(G) = 1$, $G \cong P_n$.

The converse can be easily verified. □

Theorem 4.3. *For* $n \geq 3$, $tr(T(C_n)) = 3$.

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$ in which $e_i = v_iv_{i+1}$ for all $1 \leq i \leq n-1$ and $e_n = v_nv_1$. Then $V[T(C_n)] = \{v_1, v_2, \ldots, v_n,\ e_1, e_2, \ldots,\ e_n\}$ and $E[T(C_n)] = \{e_ie_{i+1}\ /\ 1 \leq i \leq n-1\} \cup e_ne_1 \cup \{v_iv_{i+1}\ /\ 1 \leq i \leq n-1\} \cup v_nv_1 \cup \{e_iv_{i+1}\ /\ 1 \leq i \leq n-1\} \cup e_nv_1 \cup \{v_ie_i\ /\ 1 \leq i \leq n\}$.

By Theorem 4.2, $tr(T(C_n)) \geq 3$. We claim that $W = \{v_1, e_1, v_2\}$ is a resolving set of $T(C_n)$. Let x, y be two distinct vertices of $V(T(C_n)) \setminus W$. If either $d(x, v_1) \neq d(y, v_1)$ or $d(x, v_2) \neq d(y, v_2)$, then $r(x|W) \neq r(y, W)$. So we may assume that $d(x, v_1) = d(y, v_1)$ and $d(x, v_2) = d(y, v_2)$. Clearly, $x \in V(C_n)$, $y \in E(C_n)$ or $x \in E(C_n)$ and $y \in V(C_n)$. Without loss of generality, let $x \in V(C_n)$, $y \in E(C_n)$. Then $xy \in E(T(C_n))$. Clearly, $d(x, e_1) = d(y, e_1) - 1$. It follows that $r(x|W) \neq r(y|W)$. Thus $tr(T(C_n)) \leq 3$ and hence $tr(T(C_n)) = 3$. □

Theorem 4.4. *Let G be a spider. Then $tr(T(G)) = \Delta(G)$.*

Proof. Let $V(G) = \{v, v_{i1}, v_{i2}, \ldots, v_{ir_i} / 1 \leq i \leq t\}$, where $d(v) = t \geq 3$ in G and $E(G) = \{vv_{i1}, v_{i1}v_{i2}, v_{i2}v_{i3}, \ldots, v_{i(r_i-1)}v_{ir_i}\ / 1 \leq i \leq t\}$, where $|V(G)| = r_1 + r_2 + \ldots + r_t + 1$. Then $V(T(G)) = V(G)$ and $E(T(G)) = E(G) \cup \{vv_{i2}, v_{i1}v_{i3}, v_{i2}v_{i4}, v_{i3}v_{i5}, v_{i4}v_{i6}, \ldots v_{ir_i}$ $i \leq t\}$.

Let W be a minimum total resolving set of $T(G)$. Then we claim that W contains at least one vertex from the set $\{v_{i1}, v_{i2}, \ldots, v_{it_i}\}$ for all $1 \leq i \leq t$ with one exception. Suppose no vertex of $\{v_{11}, v_{12}, \ldots, v_{1t_1}\}$ and $\{v_{21}, v_{22}, \ldots, v_{2t_2}\}$ belongs to W. Then $r(v_{1i}|W) = r(v_{2j}|W)$ for $i = j$, which is a contradiction. Since W is a minimum total resolving set, $t-1$ vertices from the set $\{v_{11}, v_{21}, \ldots, v_{t1}\}$ belong to W. Without loss of generality, let $v_{11}, v_{21}, \ldots, v_{(t-1)1}$ belongs to W. But each coordinate of the representation of v_{t1} and v_1 is 1. It follows that $r(v_{t1}|W) = r(v|W)$. Therefore v or v_{t1} belongs to W. Thus $tr(T(G)) \geq t$. Let $W = \{v, v_{11}, v_{21}, \ldots, v_{(t-1)1}\}$. We claim that W is a resolving set of $T(G)$. Let x, y be two distinct vertices of $V(T(G)) \setminus W$. We consider the following two cases.

Case 1 : x lies on v_{i1}-v_{ir_i} path of G for some $1 \leq i \leq t-1$.

Then $r(x|W) \neq r(y|W)$ for all $x, y \in V(T(G)) \setminus W$ with respect to $\{v, v_{i1}\}$, $1 \leq i \leq t-1$.

Case 2 : x lies on v_{t1}-v_{tr_t} path of G.

For $1 \leq i \leq t-1$, if x lies on v_{i1}-v_{ir_i} path of G, then by Case 1, $r(x|W) \neq r(y|W)$ for all $x, y \in V \setminus W$. So we may assume that y lies on v_{t1}-v_{tr_t} path of G. If $d(x, v) \neq d(y, v)$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v) = d(y, v)$. If x lies on y-v path of G, then $d(y, v_{11}) = d(x, v_{11}) + 1$ and if y lies on x-v path of G, then $d(x, v_{11}) = d(y, v_{11}) + 1$. So $r(x|W) \neq r(y|W)$ for all $x, y \in V(T(G)) \setminus W$. Therefore each vertex of $V(T(G)) \setminus W$ have distinct representations and $\langle W \rangle$ has no isolates. Thus $tr(T(G)) \leq t$ and hence $tr(T(G)) = t = \Delta(G)$. □

Theorem 4.5. *For $s, t \geq 2$, $tr(T(B_{s,t})) = s + t - 1$.*

Proof. Let $V(B_{s,t}) = \{u_0, u_1, \ldots, u_s\} \cup \{v_0, v_1, \ldots, v_t\}$ and $E(B_{s,t}) = \{e_i = u_0u_i\ /\ 1 \leq i \leq s\} \cup \{e_{s+j} = v_0v_j\ /\ 1 \leq j \leq t\} \cup \{e_{s+t+1} = u_0v_0\}$. Then $V[T(B_{s,t})] = \{u_0, u_1, \ldots, u_s\} \cup \{v_0, v_1, \ldots, v_t\} \cup \{e_1, e_2, \ldots, e_{s+t+1}\}$. If $s = t = 2$, then we can easily verify that $tr(B_{s,t}) = 3 = s + t - 1$. So we may assume that either $s \geq 3$ or $t \geq 3$. Without loss of generality, let $t \geq 3$. Let $W = \{e_1, e_2, \ldots, e_{s-1}, u_0, e_{s+1}, \ldots, e_{s+t-1}\}$. Then each coordinate of the representation of e_{s+t+1} is 1 and that of any other vertex is not 1;

first coordinate and last $t-1$ coordinates of the representation of v_0 are 1 and that of any other vertex is not 1;

only first coordinate and i^{th} coordinate of the reprenentation of $u_i (1 \leq i \leq s-1)$ are 1 and that of any other vertex is not 1;

only $(s+j)^{th}$ coordinate of the representation of $v_j (1 \leq j \leq t-1)$ is 1 and that of any other vertex is not 1;

last $t-1$ coordinates of the representation of u_s are 3 and that of any other vertex is not 3;

first s coordinates and last $t-1$ coordinates of the representation of e_s are 1 and 2 respectively and that of any other vertex is not 1 and 2;

first coordinate and last $t-1$ coordinates of the representation of v_t are 2 and that of any other vertex is not 2;

first s coordinates and last $t-1$ coordinates of the representation of e_{s+t} are 2 and 1 respectively and that of any other vertex is not 2 and 1;

Thus W is a resolving set of $T(B_{s,t})$. Since $\langle W \rangle$ has no isolates, W is a total resolving set of $T(B_{s,t})$ and hence

$tr(T(B_{s,t})) \leq s+t-1$.

Next, we claim that $tr(T(B_{s,t})) \geq s+t-1$. Suppose that $tr(T(B_{s,t})) \leq s+t-2$. Let W be a total resolving set of $T(B_{s,t})$. Since $d(u_i, v) = d(u_j, v)$ for all $v \in V(B_{s,t}) \setminus \{e_1, e_2, \ldots, e_s\}$ and $d(v_r, v) = d(v_s, v)$ for all $v \in V(B_{s,t}) \setminus \{e_{s+1}, e_{s+2}, \ldots, e_{s+t}\}$, W contains at least one vertex from $\{u_i, e_i\}$ for all $1 \leq i \leq s$ with one exception and at least one vertex from $\{v_j, e_{s+j}\}$ for all $1 \leq j \leq t$ with one exception. Therefore $W \subset E(B_{s,t})$. Without loss of generality, let $W = \{e_1, e_2, \ldots, e_{s-1}\} \cup \{e_{s+1}, e_{s+2}, \ldots, e_{s+t-1}\}$. Then $r(e_s|W) = r(u_0|W)$ and $r(e_{s+t+1}|W) = r(v_0|W)$, which is a contradiction.

Thus $tr(T(B_{s,t})) \geq s+t-1$ and hence $tr(T(B_{s,t})) = s+t-1$. □

Theorem 4.6. *Let G be a graph of order $n \geq 3$ and $\delta(G) \geq 2$. Then $tr(T(G)) \leq n-1$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. For $i, j \in \{1, 2, \ldots, n\}$. Let v_{ij} be the new vertex of the edge $v_i v_j$.

By Theorems 4.2 and 4.3, $tr(T(P_n)) = 2$ and $tr(T(C_n)) = 3$. So we may assume that $\Delta(G) \geq 3$. Let $d(v_n) = r \geq 3$. If G is 1-connected, then we assume v_n is a cut vertex of G. Let $W = \{v_1, v_2, \ldots, v_{n-1}\}$. Then first r coordinates of the representation of v_n is 1 and that of any other vertex is 1, i^{th} and j^{th} coordinates of the representation of v_{ij}, $1 \leq i \leq n-1$, $1 \leq j \leq n-2$ are 1 and that of any other vertex are not 1, i^{th} coordinates of the representation of v_{in}, $1 \leq i \leq n-1$ is 1 and that of any other vertex is not 1. Thus each vertex of $V(T(G)) \setminus W$ have distinct representations and we can easily verify $\langle W \rangle$ has no isolates. Thus $tr(T(G)) \leq n-1$. □

Open problem 4.7. *If G is a connected graph of order $n \geq 3$ without pendant vertices, then characterize G for which $tr(T(G)) = n-1$.*

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