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On the General Finite Sum of r^k and Its Applications

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ABSTRACT. This paper analyses the field of mathematics, number theory, around the divergence and convergence of series of varying forms such as P-series, geometric series and even more compound functions including the hyperbolic trigonometric function of sine, cosine and tangent. The purpose is typically to support and embed an additional step forward to great conjectures proposed by early mathematicians around the field, with more precision the paper aims to redefine the Riemann zeta functions by proposing new methods of concluding the finite sum of series, and while the hypotheses of Riemann discusses infinite series it remains possible to, with limits, evaluate the newly defined result to infinity. Mainly, a question that has drawn the attention of early mathematicians was the ordinary power series in which Bernoulli has proposed a theorem of his involving Bernoulli polynomials that was only limited to power series. This paper enhances upon this and suggests with proof new methods and applications. The methods used to investigate this dilemma came from previous research documents published by Bernoulli himself along with Euler and other more recent mathematicians (Details included in the references) and how they were able to approach and influence the problem. Fortunately, the research established has gained extensive success as with reasonable complexity new summations have been defined for the power series along with details on reduction and more composite functions that influence modern arithmetic and number theory. The results propose a new method of looking at functions and analysing their behaviour as series along with comparing them to the infamous zeta function and sequences such as the Fibonacci sequence and the Legendre sequence as well without direct approach. It is with all delight that these results have been previously placed into applications however with yet no direct proposal for publication.

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1 Introduction

The problem stated in the paper is simply to attempt to redefine the sum of series such as the sum of integers, squares, cubes etc. Yet it has been made apparent that current mathematics approaches such problems via the use of the Bernoulli polynomials where it is rather complex when getting to extensive powers of the power series. Current understanding is further only limited to power series and lacks grasp of more complex functions which is concluded in this paper. And despite the ability of modern mathematics to discuss the relevant problem is stand the question: Is it possible to redefine more compound function for use in complex analysis? With thorough pursuit, the details in the paper encompass this question and produce answers in the form of theorems, a conjecture and a table of systematic basis results that can be applied even further to form more specific summations. The approach uses the study of several polynomials and the idea behind proof by induction to prove the theorem that is reiterated from the polynomial with reference to previous research by outer sources along with independent research as well. The result while complex act as great support in the modules of number theory with necessity of rigorous computerized arithmetic calculations, which as it stands limits mathematical development.

2 General Formula

In this section we discuss the progress current mathematics has maintained in the field and detail on how considering the expansion of a certain function would influence forming a formula for the finite sum of power series as some basis for the upcoming mentionable applications.

2.1 Current Formulae

Modern mathematical approach to deriving the finite sum of a power series from one to infinity has been rather limited and, as mentioned, does not provide general access to all the power series in a systematic manner where the formula derived should be based on the exponent of , yet the following along with many more have been found,

$$\sum r^0 = n$$
$$\sum r^1 = \frac{1}{2}n(n+1)$$
$$\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$$
$$\sum r^3 = \frac{1}{4}n^2(n+1)^2$$
$$\sum r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$
$$\sum r^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$
$$\sum r^6 = \frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)$$
$$\sum r^7 = \frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)$$

$$\sum r^8 = \frac{1}{90}n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)$$
$$\sum r^9 = \frac{1}{20}n^2(n+1)^2(n^2+n-1)(2n^4+4n^3-n^2-3n+3)$$
$$\sum r^8 = \frac{1}{90}n(n+1)(2n+1)(n^2+n-1)(3n^6+9n^5+2n^4-11n^3+3n^2+10n-5)$$
$$\vdots$$

This paper investigates the origin of what equates such a sum to the series corresponding and analyses the connection to compute a formula for deriving the sum of a general power based primarily on the previous ones, the states enhancements of such a formula with possible reduction.

2.2 General Formula Derivation

Derivation of such a formula involves transferring functions into a finite series initially by binomial expansion and is generally computed be investigation of the following polynomial then analysis of possible further simplification,

$$r^{k+1}-(r-1)^{k+1} \tag{2.1}$$

Now take,

$$\begin{aligned}(r-1)^{k+1} &= \sum_{a=0}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a \\ &= r^{k+1}-(k+1)r^k\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a\end{aligned}$$

substitute into (1)

$$\begin{aligned}r^{k+1}-(r-1)^{k+1} &= r^{k+1}-[r^{k+1}-(k+1)r^k+\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a] \\ &= (k+1)r^k-\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a\end{aligned}$$

which implies,

$$(k+1)r^k = r^{k+1}-(r-1)^{k+1}+\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a$$
$$r^k = \frac{r^{k+1}-(r-1)^{k+1}}{k+1}+\frac{1}{k+1}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a \tag{2.2}$$

Take the Summation of Both Sides,

$$\sum_{r=1}^{n} r^k = \sum_{r=1}^{n}\frac{r^{k+1}-(r-1)^{k+1}}{k+1}+\frac{1}{k+1}\sum_{r=1}^{n}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a$$

Consider, $\sum_{r=1}^{n}\frac{r^{k+1}-(r-1)^{k+1}}{k+1}$ when $r=1$, $\frac{1}{k+1}-0$
when $r=2$, $\frac{2^{k+1}}{k+1}-\frac{1}{k+1}$
when $r=3$, $\frac{3^{k+1}}{k+1}-\frac{2^{k+1}}{k+1}$
$\vdots$

when $r = n$, $\frac{n^{k+1}}{k+1} - \frac{(n-1)^{k+1}}{k+1}$

All the terms in this summation cancel out and the series yields, $\sum_{r=1}^{n} \frac{r^{k+1}-(r-1)^{k+1}}{k+1} = \frac{(n)^{k+1}}{k+1}$. Therefore $\sum_{r=1}^{n} r^k = \frac{(n)^{k+1}}{k+1} + \frac{1}{k+1}\sum_{r=1}^{n}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a$

After Simplification,we get,

$$\sum_{r=1}^{n} r^k = \frac{1}{k+1}[(n)^{k+1} + \sum_{r=1}^{n}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a] \tag{2.3}$$

And after rather complex expansion and simplification it has been briefly stated that a formula is present yet proof may be essential as proposed in the following section.

2.3 Proof by Induction

Such rigorous derivation and unordinary result may require proof, and an essential method of approaching proofs involving series is proof by induction as mathematically stated in the following,

Theorem 1.1: $\sum_{r=1}^{n} r^k = \frac{1}{k+1}[(n)^{k+1} + \sum_{r=1}^{n}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a]$, for all r, k and $n \in \mathbb{Z}^+$. Basis, we show that this is true for $n = 1$ then, $\sum_{r=1}^{1} r^k = 1^k = 1$.

On the otherside, $\frac{1}{k+1}[1 + \sum_{r=1}^{n}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a] = \frac{1}{k+1}[1+k] = 1$.

Therefore, This is true for $n = 1$. Assumption, We Assume this is true for $n = x$. $\sum_{r=1}^{x} r^k = \frac{1}{k+1}[(x)^{k+1} + \sum_{r=1}^{x}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a]$ Induction, We show that if this is true for $n = x$ then it must be true for $n = x+1$, $\sum_{r=1}^{x+1} r^k = \frac{1}{k+1}[(x+1)^{k+1} + \sum_{r=1}^{x+1}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a]$

Consider,

$$(x+1)^{k+1} = (x)^{k+1} + \sum_{a=2}^{k}\binom{k+1}{a}x^a \tag{2.4}$$

$$\sum_{r=1}^{x+1}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a = \sum_{r=1}^{x}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a + \sum_{a=2}^{k+1}\binom{k+1}{a}(x+1)^{k+1-a}(-1)^a] \tag{2.5}$$

Substitute (4) and (5) Into RHS, $\frac{1}{k+1}[x^{k+1} + \sum_{a=2}^{k+1}\binom{k+1}{a}x^a + \sum_{r=1}^{x}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a + \sum_{a=2}^{k+1}\binom{k+1}{a}(x+1)^{k+1-a}(-1)^a]$ $= \frac{1}{k+1}[x^{k+1}\sum_{r=1}^{x}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a] + \frac{1}{k+1}[\sum_{a=2}^{k+1}\binom{k+1}{a}x^a + \sum_{a=2}^{k+1}\binom{k+1}{a}(x+1)^{k+1-a}(-1)^a]$ From (3) We get that,

$$\sum_{r=1}^{x} r^k = \frac{1}{k+1}[\sum_{a=0}^{k}\binom{k+1}{a}x^a + \sum_{a=2}^{k+1}\binom{k+1}{a}(x+1)^{k+1-a}(-1)^a] \tag{2.6}$$

Reconsider (2) when $r = x+1$

$$r^k = \frac{r^{k+1}-(r-1)^{k+1}}{k+1} + \frac{1}{k+1}\sum_{a=2}^{k+1}\binom{k+1}{a}(r)^{k+1-a}(-1)^a$$

$$(x+1)^k = \frac{(x+1)^{k+1}-x^{k+1}}{k+1} + \frac{1}{k+1}\sum_{a=2}^{k+1}\binom{k+1}{a}(x+1)^{k+1-a}(-1)^a \tag{2.7}$$

Recall from (4) that,

$$\sum_{a=0}^{k}\binom{k+1}{a}(x)^a = (x+1)^{k+1} - x^{k+1} \tag{2.8}$$

Substitute (8) into (7), and simplify,

$$(x+1)^k = \frac{1}{k+1}\sum_{a=0}^{k}\binom{k+1}{a}x^a + \frac{1}{k+1}\sum_{a=2}^{k+1}\binom{k+1}{a}(x+1)^{k+1-a}(-1)^a \tag{2.9}$$

Substitute (9) into (6), and simplify,

$$\sum_{r=1}^{x} r^k + \frac{1}{k+1}[\sum_{a=0}^{k} \binom{k+1}{a} x^a + \sum_{a=2}^{k+1} \binom{k+1}{a} (x+1)^{k+1-a}(-1)^a] = \sum_{r=1}^{x} r^k + (x+1)^k = \sum_{r=1}^{x+1} r^k$$

Hence proven.

If the summation formula is true for $n = x$ then it is shown to also be true for $n = x+1$. As the result is true for $n = 1$, it is now also true for all $n \in \mathbb{Z}^+$ by mathematical induction.

3 Enhancement

While it stands possible to compute various series with the for numerous power series, the process stands rather difficult and one m=begins to wonder: Is reduction to lower powers possible? An answer to so is proposed in the following section when proof lies in what was demonstrated above and in the derivation itself as is what the remaining composite functions proofs developed will be based on as it seems superfluous to include reputation.

3.1 Reduction Formula

It has been observed and proven that it is required to undergo the value of and form of the theorem above that knowledge the sum of all the previous powers is required. And despite that fact that the same formula could be used for the previous powers it is a thorough process. To avoid such a method, it is possible to expand the innermost series and create what could be identified as nested series this is shown below for the case of reducing the power required for the summation another step down.

$$\sum_{r=1}^{x} r^k = \frac{1}{k+1}[n^{k+1} + \sum_{r=1}^{n}\sum_{a=2}^{k+1} \binom{k+1}{a} (r)^{k+1-a}(-1)^a]$$

Consider just,

$$\begin{aligned}
\sum_{r=1}^{n}\sum_{a=2}^{k+1} \binom{k+1}{a} (r)^{k+1-a}(-1)^a &= \sum_{r=1}^{n}[\binom{k+1}{2}(r)^{k-1} + \binom{k+1}{3}(r)^{k-2} + \binom{k+1}{4}(r)^{k-3} \\
&\quad .. + \binom{k+1}{k}(r)(-1)^k + (-1)^{k+1}] \\
&= \binom{k}{2}\frac{1}{k}[n^k + \sum_{r=1}^{n}\sum_{a=2}^{k+1}\binom{k}{a}(r)^{k-a}(-1)^a] \\
&\quad + \binom{k+1}{3}\frac{1}{k-1}[n^{k-1} + \sum_{r=1}^{n}\sum_{a=2}^{k-1}\binom{k-1}{a}(r)^{k-1-a}(-1)^a] \\
&\quad \vdots \\
&\quad + \binom{k+1}{k}\frac{(-1)^k(n)(n+1)}{2} + (-1)^{k+1} \\
&= \sum_{b=1}^{k-1}[\binom{k+1}{b+1}\frac{1}{k+1-b}[\sum_{r=1}^{n}\sum_{a=2}^{k+1-b}\binom{k+1-b}{a} \\
&\quad (r)^{k+1-b-a}(-1)^a]] + (-1)^{k+1}
\end{aligned} \tag{3.1}$$

Without extensive complexity we obtained a more rigorous equation that defines the finite sum where the maximum power required to be recomputed is $k-2$ and yet only half od the powers from 1 to $k-2$ are required to be computed using the same formula above. And despite the possibility of forming a substantially large equation that requires at most only computing linear sums (i.e. the sum of just r) it is rather difficult and thereby leads to the following conjecture.

3.2 Series Reduction Conjecture

After witnessing possibility of applying reduction to one degree less it stands reasonable to propose an assumption in the form of a conjecture stated simply as it is possible to express $\sum_{r=1}^{n}(r)^k$ as $k-1$ nested series and $k-1$ different variables such that the highest degree of r required to take the sum of is 1. Or in more mathematical terms,

Definition 1.1: A series is said to be nested if it is within another series that is dependent on its variables.

Conjecture 1.1,

$$\sum_{r=1}^{n}(r)^k = \sum_{a_k}[... \sum_{a_{k-1}}[... \sum_{a_{k-2}} ... \sum_{a_1}^{k+1-a_2} r \times c]]$$

Where c is a constant. The imperative difficulty of finding this reduction to degree $k-4$ (Where $k-3$ is rather simple to find) is substantial and may require computerized technology to develop such an equation however it stand, and up to degree 1 as well, possible.

4 Application

As previously mentioned it is very essential to consider the applications such a theorem attains along with further research upon composite functions, the following sections demonstrate the ability of the theorem to take place and be very systematically applied upon trigonometric, hyperbolic, and even exponential functions with use of both the similar Taylor and McLaurin series.

Corollary 4.1: The utility of the Derived formulae can be applied into a more general form of series, especially when considering summations with rational coefficients raised to exponents such as that of the Riemann zeta function. The applications can quite trivially be submitted when letting k assume only values of the form $-m$ where $m \in \mathbb{Z}^+$. And by direct substitution we may resemble the intended formula for the summation, yet, when considering components of the RHS such as the choose function with negative arguments, the necessity arises for complex analysis in the evaluation of such summation which may not lead to the desired number theoretic evaluation. Thus, we shall proceed to deriving the Taylor Expansion of the reciprocal of the previously stated theorem

$$\frac{1}{r^m} = 1 - m(r-1) + \frac{m(m+1)(r-1)^2}{2} + \frac{m(m+1)(m+2)(r-1)^3}{6} + ...$$

This can be illustrated in sigma notation as the formula

$$\begin{aligned}
\frac{1}{r^m} &= \sum_{a=0}^{\infty} \frac{(-1)^a \prod^{a;\ b\geq a;\ b=1}(m+b-1)(r-1)^a}{a!} \\
&= \sum_{r=1}^{n} \frac{1}{r^m} = \sum_{r=1}^{n}\sum_{a=0}^{\infty} \frac{(-1)^a \prod_{a;\ b\geq a;\ b=1}(m+b-1)(r-1)^a}{a!} \\
&= \sum_{r=0}^{n-1} \frac{1}{r^m} = \sum_{r=1}^{n}\sum_{a=0}^{\infty} \frac{(-1)^a \prod_{a;\ b\geq a;\ b=1}(m+b-1)}{a!} \\
&= \sum_{a=0}^{\infty}\{[\frac{1}{a+1}[(n-1)^{a+1} + \sum_{r=1}^{n-1}\sum_{k=2}^{a+1}\binom{a+1}{k} r^{a+1-k}(-1)^k] - 1]\frac{(-1)^a \prod_{a;\ b\geq a;\ b=1}(m+b-1)}{a!}\}
\end{aligned} \tag{4.1}$$

And thus, we can rightfully conclude that

$$\sum_{r=1}^{n} \frac{1}{r^m} = \sum_{a=0}^{\infty}\{[\frac{1}{a+1}[(n-1)^{a+1} + \sum_{r=1}^{n-1}\sum_{k=2}^{a+1}\binom{a+1}{k} r^{a+1-k}(-1)^k] - 1]\frac{(-1)^a \prod_{ab\geq ab=1}(m+b-1)}{a!}\}$$

Corollary 4.2: As it is possible to also express basic trigonometric functions in a similar manner it is also possible to express them as finite series where r is now a measurement of the angle and is expressed in radians bounded between 0 and 2π for any n. Below is the McLaurin expansion for that of $sin(r)$ and $cos(r)$,

$$sin(r) = r - \frac{r^3}{3!} + \frac{r^5}{5!} - \frac{r^7}{7!} + \frac{r^9}{9!} - .....$$

$$cos(r) = r - \frac{r^2}{2!} + \frac{r^4}{4!} - \frac{r^6}{6!} + \frac{r^8}{8!} - .....$$

or in sigma notation,

$$sin(r) = \sum_{a=0}^{\infty} \frac{r^{2a+1}(-1)^a}{(2a+1)!}$$

$$cos(r) = \sum_{a=0}^{\infty} \frac{r^{2a}(-1)^a}{(2a)!}$$

We can then take the finite sum of both sides and while it may be a loss not to consider $r = 0$ for $cos(r)$ the user may well add 1 to the series at hand. For simplification the use of some properties of the commutativity of infinite series is applied,

$$\begin{aligned}
\sum_{r=1}^{\infty} sin(r) &= \sum_{r=1}^{\infty}[\sum_{a=0}^{\infty} \frac{r^{2a+1}(-1)^a}{(2a+1)!}] \\
&= \sum_{a=0}^{\infty}\sum_{r=1}^{\infty} \frac{r^{2a+1}(-1)^a}{(2a+1)!} \\
&= \sum_{a=0}^{\infty} \frac{(-1)^a}{(2a+1)!}\sum_{r=1}^{\infty} r^{2a+1}
\end{aligned} \tag{4.2}$$

The theorem stated as (3) could then be used where the k is replaced with $2a+1$,

$$\sum_{a=0}^{\infty}[\frac{(-1)^a}{(2a+1)!}\frac{1}{2a+2}[n^{2a+2} + \sum_{r=1}^{n}\sum_{b=2}^{2a+2}\binom{k=1}{b} r^{2a+2-b}(-1)^b]]$$

Or more simply, Theorem 3.1.

Therefore,

$$\sum_{r=0}^{n} sin(r) = \sum_{a=0}^{\infty}[\frac{(-1)^a}{(2a+1)!}\frac{1}{2a+2}[n^{2a+2} + \sum_{r=1}^{n}\sum_{b=2}^{2a+2}\binom{k=1}{b} r^{2a+2-b}(-1)^b]] \tag{4.3}$$

This could then be compared with $cos(r)$ and an equivalent statement could be deduced, Theorem 3.2

$$\sum_{r=0}^{n} cos(r) = 1 + \sum_{a=0}^{\infty}[\frac{(-1)^a}{(2a)!}\frac{1}{2a+1}[n^{2a+1} + \sum_{r=1}^{n}\sum_{b=2}^{2a+1}\binom{k=1}{b}r^{2a+1-b}(-1)^b]] \tag{4.4}$$

And with so a very common question arises why has not the same been applied to tangent, this is because the Taylor expansion of tangent relates to the Bernoulli numbers which all the theorems stated above avoid, however using the sine-cosine identity for tangent a more significant formula could be deduced. Theorem 3.3.

$$\sum_{r=0}^{n} tan(r) = \frac{\sum_{a=0}^{\infty}[\frac{(-1)^a}{(2a+1)!}\frac{1}{2a+2}[n^{2a+2} + \sum_{r=1}^{n}\sum_{b=2}^{2a+2}\binom{k=1}{b}r^{2a+2-b}(-1)^b]]}{1 + \sum_{a=0}^{\infty}[\frac{(-1)^a}{(2a)!}\frac{1}{2a+1}[n^{2a+1} + \sum_{r=1}^{n}\sum_{b=2}^{2a+1}\binom{k=1}{b}r^{2a+1-b}(-1)^b]]} \tag{4.5}$$

While more composite functions that relate to the basic ones shown in (12), (13) and (14) such as secant and cosecant and cotangent it seems more worthwhile to consider more meaningful functions present in the following corollary.

Corollary.4.3: For the following series that will be defined accordingly for hyperbolic sine, cosine and tangent it is important to begin by looking into both the expansion of e^r as a Taylor series and also investigate its finite sum according to the theorem in (3),

$$e^r = 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} + \frac{r^4}{4!} + ... = \sum_{a=0}^{\infty}\frac{r^a}{a!}$$

And the finite sum is therefore,

$$\begin{aligned}\sum_{r=1}^{n} e^r &= \sum_{r=1}^{n}\sum_{a=0}^{\infty}\frac{r^a}{a!}\\ &= \sum_{a=0}^{\infty}\frac{1}{a!}\sum_{r=1}^{n} r^a \\ &= \sum_{a=0}^{\infty}[\frac{1}{a!}\frac{1}{a+1}[n^{a+1} + \sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^b]]\end{aligned} \tag{4.6}$$

$$\sum_{r=0}^{n} e^r = \sum_{a=0}^{\infty}[\frac{1}{a!}\frac{1}{a+1}[n^{a+1} + \sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^b]] \tag{4.7}$$

Theorem 4.2,

$$\sum_{r=0}^{n} e^r = \sum_{a=0}^{\infty}[\frac{(-1)^a}{a!}\frac{1}{a+1}[n^{a+1} + \sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^b]] \tag{4.8}$$

From this point on we can then consider the identities from each of three main hyperbolic functions then substitute the values present in (16) and (17) to establish their finite sum as well.

$$sinh(r) = \frac{e^r - e^{-r}}{2}$$
$$cosh(r) = \frac{e^r + e^{-r}}{2}$$
$$tanh(r) = \frac{e^r - e^{-r}}{e^r + e^{-r}}$$

When substituting the values this yields, Theorem 4.3,

$$\begin{aligned}\sum_{r=1}^{n} sinh(r) = \frac{1}{2}[\sum_{a=0}^{\infty}[\frac{1}{a!}\frac{1}{a+1}[n^{a+1} + \sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^{b}]] \\ - \sum_{a=0}^{\infty}[\frac{(-1)^{a}}{a!}\frac{1}{a+1}[n^{a+1} + \sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^{b}]]]\end{aligned} \tag{4.9}$$

Theorem 4.4,

$$\begin{aligned}\sum_{r=1}^{n} cosh(r) = 1 + \frac{1}{2}[\sum_{a=0}^{\infty}[\frac{1}{a!}\frac{1}{a+1}[n^{a+1} + \sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^{b}]] \\ + \sum_{a=0}^{\infty}[\frac{(-1)^{a}}{a!}\frac{1}{a+1}[n^{a+1} + \sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^{b}]]]\end{aligned} \tag{4.10}$$

Theorem 4.5,

$$\sum_{r=1}^{n} tanh(r) = \frac{\left(\substack{\sum_{a=0}^{\infty}[\frac{1}{a!}\frac{1}{a+1}[n^{a+1}+\sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^{b}]] \\ -\sum_{a=0}^{\infty}[\frac{(-1)^{a}}{a!}\frac{1}{a+1}[n^{a+1}+\sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^{b}]]}\right)}{\left(\substack{2+\sum_{a=0}^{\infty}[\frac{1}{a!}\frac{1}{a+1}[n^{a+1}+\sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^{b}]] \\ +\sum_{a=0}^{\infty}[\frac{(-1)^{a}}{a!}\frac{1}{a+1}[n^{a+1}+\sum_{r=1}^{n}\sum_{b=2}^{a+1}\binom{a+1}{b}r^{a+1-b}(-1)^{b}]]}\right)} \tag{4.11}$$

These substantial results along with many more that may be deduced using these theorems have been established. This then leads to the temptation of grouping them in one table for often use.

5 Results

It proceeds the importance of concluding all these details in a scheme of tables that illustrates the established results where the name of the function if novel is indicated otherwise a dash is placed, other headings on the table describe the entries within each column respectively.

5.1 Tables of Formulae Derived

Can be found at after references section as Tables 1,2 and 3 respectively.

6 Conclusion

It has been recognized that there stands a connection between the different summations of consecutive degrees of r that has been generalized in a simple statement that varies from the use of Bernoulli Polynomials that have seemed to dominate the field. It has been later recognized that extensive difficult is required to compute the summation as the summation for every degree below k is required despite the ability to use the formula. It has the been shown that reduction is possible as stated for a degree less then conjectured that it is possible to reduce the equation to degree 1 for r, however with difficulty. Such theorems and conjectures serve number theory as a whole and influence modern mathematics greatly. Perhaps with these developed theorems the world of mathematics could broaden its understanding and embed steps into previous conjectures such as the Riemann hypotheses. **Acknowledgements:** It is a pleasure to mention the following people for aid upon the contents of the paper as education or as minor and major support in parts of the research performed in the various aspects of the

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Title	$f(r)$	$\sum_{r=1}^{n} f(r)$
Power Series	r^k	$\frac{1}{k+1}[n^{k+1} + \sum_{r=1}^{n} \sum_{a=2}^{k+1} \binom{k+1}{a} (r)^{k+1-a} (-1)^a]$
Reduction	r^k	$\sum_{b=1}^{k-1} [\binom{k+1}{b+1} \frac{1}{k+1-b} [n^{k+1-b} + \sum_{r=1}^{n} \sum_{a=2}^{k+1-b} \binom{k+1-b}{a}$
		$(r)^{k+1-b-a}(-1)^a]] + (-1)^{k+1}$
P-Series	$\frac{1}{r^k}$	$\approx \sum_{r=1}^{n} [1 + \sum_{a=1}^{\infty} [\sum_{b=0}^{a} [\binom{a}{b} (r)^{a-b} (-1)^b] \frac{1}{a! \prod_{a>m}^{a} (-m+a)}]]$

Table 1: Power series, P-series and Reduction

Title	$f(r)$	$\sum_{r=1}^{n} f(r)$
Sine	sin(r)	$\sum_{a=0}^{\infty} [\frac{(-1)^a}{(2a+1)!} \frac{1}{2a+2} [n^{2a+2} + \sum_{r=1}^{n} \sum_{b=2}^{2a+2} \binom{k=1}{b} r^{2a+2-b} (-1)^b]]$
Cosine	cos(r)	$\sum_{a=0}^{\infty} [\frac{(-1)^a}{(2a)!} \frac{1}{2a+1} [n^{2a+1} + \sum_{r=1}^{n} \sum_{b=2}^{2a+1} \binom{k=1}{b} r^{2a+1-b} (-1)^b]]$
Tangent	tan(r)	$\frac{\sum_{a=0}^{\infty} [\frac{(-1)^a}{(2a+1)!} \frac{1}{2a+2} [n^{2a+2} + \sum_{r=1}^{n} \sum_{b=2}^{2a+2} \binom{k=1}{b} r^{2a+2-b} (-1)^b]]}{1 + \sum_{a=0}^{\infty} [\frac{(-1)^a}{(2a)!} \frac{1}{2a+1} [n^{2a+1} + \sum_{r=1}^{n} \sum_{b=2}^{2a+1} \binom{k=1}{b} r^{2a+1-b} (-1)^b]]}$
Exponential	e^r	$\sum_{a=0}^{\infty} [\frac{1}{a!} \frac{1}{a+1} [n^{a+1} + \sum_{r=1}^{n} \sum_{b=2}^{a+1} \binom{a+1}{b} r^{a+1-b} (-1)^b]]$
-	e^{-r}	$\sum_{a=0}^{\infty} [\frac{(-1)^a}{a!} \frac{1}{a+1} [n^{a+1} + \sum_{r=1}^{n} \sum_{b=2}^{a+1} \binom{a+1}{b} r^{a+1-b} (-1)^b]]$

Table 2: Trigonometric and Exponential Functions

Title	$f(r)$	$\sum_{r=1}^{n} f(r)$
Hyperbolic sine	sinh(r)	$\frac{1}{2} [\sum_{a=0}^{\infty} [\frac{1}{a!} \frac{1}{a+1} [n^{a+1} + \sum_{r=1}^{n} \sum_{b=2}^{a+1} \binom{a+1}{b} r^{a+1-b} (-1)^b]]$
		$- \sum_{a=0}^{\infty} [\frac{(-1)^a}{a!} \frac{1}{a+1} [n^{a+1} + \sum_{r=1}^{n} \sum_{b=2}^{a+1} \binom{a+1}{b} r^{a+1-b} (-1)^b]]]$
Hyperbolic cosine	cosh(r)	$1 + \frac{1}{2} [\sum_{a=0}^{\infty} [\frac{1}{a!} \frac{1}{a+1} [n^{a+1} + \sum_{r=1}^{n} \sum_{b=2}^{a+1} \binom{a+1}{b} r^{a+1-b} (-1)^b]]$
		$+ \sum_{a=0}^{\infty} [\frac{(-1)^a}{a!} \frac{1}{a+1} [n^{a+1} + \sum_{r=1}^{n} \sum_{b=2}^{a+1} \binom{a+1}{b} r^{a+1-b} (-1)^b]]]$
Hyperbolic tangent	tanh(r)	$\frac{\left(\begin{matrix} \sum_{a=0}^{\infty} [\frac{1}{a!} \frac{1}{a+1} [n^{a+1} + \sum_{r=1}^{n} \sum_{b=2}^{a+1} \binom{a+1}{b} r^{a+1-b} (-1)^b]] \\ - \sum_{a=0}^{\infty} [\frac{(-1)^a}{a!} \frac{1}{a+1} [n^{a+1} + \sum_{r=1}^{n} \sum_{b=2}^{a+1} \binom{a+1}{b} r^{a+1-b} (-1)^b]] \end{matrix}\right)}{\left(\begin{matrix} 2 + \sum_{a=0}^{\infty} [\frac{1}{a!} \frac{1}{a+1} [n^{a+1} + \sum_{r=1}^{n} \sum_{b=2}^{a+1} \binom{a+1}{b} r^{a+1-b} (-1)^b]] \\ + \sum_{a=0}^{\infty} [\frac{(-1)^a}{a!} \frac{1}{a+1} [n^{a+1} + \sum_{r=1}^{n} \sum_{b=2}^{a+1} \binom{a+1}{b} r^{a+1-b} (-1)^b]] \end{matrix}\right)}$

Table 3: Four months plan: where,what how