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Compatible mappings of types using implicit relations in fuzzy metric spaces

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ABSTRACT. In this paper, we prove some fixed point theorems with the notions of compatible mappings of type(R), of type (χ) and of type (E) using implicit relations in fuzzy metric space.

1 Introduction

It proved a turning point in the development of fuzzy mathematics when the notion of fuzzy set was introduced by Zadeh [23]. Fuzzy set theory has many applications in applied science such as neural network theory, stability theory, mathematical programming, modelling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. There are many view points of the notion of the metric space in fuzzy topology, see, e.g., Erceg [2], Deng [1], Kaleva and Seikkala[11], Kramosil and Michalek [12], George and Veermani [3]. In this paper, we are considering the Fuzzy metric space in the sense of Kramosil and Michalek [12].

Definition 1.1: A binary operation $*$ on $[0,1]$ is a t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,

(ii) $a * 1 = a$ for every $a \in [0,1]$,

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(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Basics examples of t-norm are t-norm $\Delta_L, \Delta_L(a,b) = max(a+b-1,0)$, t-norm $\Delta_P, \Delta_P(a,b) = ab$ and t-norm $\Delta_M, \Delta_M(a,b) = min\{a,b\}$.

Definition 1.2: The 3- tuple(K, M, Δ) is called a fuzzy metric space (in the sence of Kramosil and Michalek) if K is an arbitrary set , Δ is a continuous t-norm and M is a fuzzy set on $K^2 \times [0,\infty)$ satisfying the following conditions for all $p,q,r \in K$ and $s,t > 0$

1. $M(p,q,0) = 0, M(p,q,t) > 0,$
2. $M(p,q,t) = 1$, all $t > 0$ if and only if $p = q$,
3. $M(p,q,t) = M(q,p,t),$
4. $M(p,r,t+s) \geq \Delta(M(p,q,t), M(q,r,s)),$
5. $M(p,q,.) : [0,\infty) \rightarrow [0,1]$ is left continuous.

Note that $M(p,q,t)$ can be thought of as the degree of nearness between p and q with respect to t.
We identify $p = q$ with $M(p,q,t) = 1$ for all $t > 0$ and $M(p,q,t) = 0$ with $t = 0$.

Definition 1.3: A sequence $\{p_n\}$ in (K, M, Δ) is said to be:

(i) Convergent with limit$p$ if $lim_{n\rightarrow\infty} M(p_n, p, t) = 1$ for all $t > 0$

(ii) Cauchy sequence in K if given $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that $M(p_n, p_m, \epsilon) > 1 - \lambda$ for all $n, m \geq N_{\epsilon,\lambda}$.

(iii) Complete if every Cauchy sequence in K is convergent in K.

Fixed point theory in fuzzy metric space has been developing since the paper of Grabiec [4]. Subramanyam [21] gave a generalization of Jungck[6] theorem for commuting mapping in the setting of fuzzy metric space.

In 1996, Jungck[8] introduced the notion of weakly compatible as follows :

Definition 1.4: Two maps f and g are said to be weakly compatible if they commute at their coincidence points.
In 1999, Vasuki [22] introduced the notion of weakly commuting as follows:

Definition 1.5: Two self-mapping f and g of a fuzzy metric space (K, M, Δ) are said to be weakly commuting if $M(fgp, gfp, t) \geq M(fp, gp, t)$ for each $p \in K$ and for each $t > 0$.

In 1994, Mishra [13] generalised the notion of weakly commuting to compatible mappings in fuzzy metric space akin to the concept of compatible mapping in metric space.

Definition 1.6[13]: Let f and g be self-mappings from a fuzzy metric space (K, M, Δ) into itself. A pair of map $\{f, g\}$ is said to be compatible if $lim_{n\rightarrow\infty} M(fgp_n, gfp_n, t) = 1$ whenever $\{p_n\}$ is a sequence in K such that $lim_{n\rightarrow\infty} fp_n = lim_{n\rightarrow\infty} gp_n = u_1$ for some $u_1 \in K$ and for all $t > 0$.

In 1994, Pant [15] introduced the concept of R-weakly commuting maps in metric space. Later on, Vasuki [22] initiated the concept of non-compatible mapping in fuzzy metric space and introduced the notion of R- weakly commuting mapping in fuzzy metric space and proved some common fixed point theorems for these mappings.

Definition 1.7: Let f and g be self- mapping from a fuzzy metric space (K, M, Δ) into itself. Then the mappings f and g are said to be non-compatible if $lim_{n\rightarrow\infty} M(fgp_n, gfp_n, t) \neq 1$, whenever $\{p_n\}$ is a sequence in K such that $lim_{n\rightarrow\infty} fp_n = lim_{n\rightarrow\infty} gp_n = u_1$ for some $u_1 \in K$ and for all $t > 0$.

In 1999, Pant [14] introduced a new continuity condition, known as reciprocal continuity as follows:

Definition 1.8[14]: Two self- mappings f and g of a fuzzy metric space (K, M, Δ) are called reciprocally continuous if $lim_{n\to\infty} fgp_n = fr$ and $lim_{n\to\infty} gfp_n = gr$ whenever $\{p_n\}$ is a sequence such that $lim_{n\to\infty} fp_n = lim_{n\to\infty} gp_n = r$ for some $r \in K$.

If f and g are both continuous then they are obviously reciprocally continuous but the converse is need not be true.

Recently, Pant et al.[16] generalized the notion of reciprocal continuity to weak reciprocal continuity as follows:

Definition 1.9: Two self- mappings f and g of a fuzzy metric space (K, M, Δ) are called weakly reciprocally continuous if $lim_{n\to\infty} fgp_n = fr$ and $lim_{n\to\infty} gfp_n = gr$ whenever $\{p_n\}$ is a sequence such that $lim_{n\to\infty} fp_n = lim_{n\to\infty} gp_n = r$ for some $r \in K$.

If f and g are reciprocally continuous, then they are obviously weak reciprocally continuous but the converse is not true.

In 2004, Rohan et al. [17] introduced the concept of compatible mappings of type (R) in a metric space as follows:

Definition 1.10[17]: Let f and g be mappings from metric space K, d into itself. Then f and g are said to be compatible of type (R) if $lim_{n\to\infty} d(fgp_n, gfp_n = 0$ and $lim_{n\to\infty} d(ffp_n, ggp_n = 0$, whenever $\{p_n\}$ is a sequence such that $lim_{n\to\infty} fp_n = lim_{n\to\infty} gp_n = u_1$ for some $u_1 \in K$.

In 2007, Singh and Singh et al. [20] introduced the concept of compatible mappings of type (E) in a metric space as follows:

Definition 1.11[20]: Let f and g be mappings from metric space (K, d) into itself. Then f and g are said to be compatible of type (E) if $lim_{n\to\infty} ffp_n = u_1$ and $lim_{n\to\infty} ggp_n = lim_{n\to\infty} gfp_n = fu_1$ whenever $\{p_n\}$ is a sequence in K such that $lim_{n\to\infty} fp_n = lim_{n\to\infty} gp_n = u_1$ for some $u_1 \in K$.

In 2014, Jha et al. [5] introduced the concept of compatible mappings of type (χ) in a metric space as follows:

Definition 1.12[5]: Let f and g be mappings from metric space (K, d) into itself. Then f and g are said to be compatible of type (χ) if $lim_{n\to\infty} d(ffp_n, gu_1 = 0$ and $lim_{n\to\infty} d(ggp_n, gu_1) = 0$, whenever $\{p_n\}$ is a sequence in K such that $lim_{n\to\infty} fp_n = u_1$ for some $u_1 \in K$.

2 Properties of Compatible mappings of types

Recently, Kang et al.[10] introduced the notion of compatible mappings and its variants in a multiplicative metric space. Now we introduce the notions of compatible mappings of types in the setting of a Fuzzy metric space as follows: **Definition 2.1:** Let f and g be self-mapping on fuzzy metric space (K, M, Δ). Then f and g are called:

1. Compatible of type (R) if $lim_{n\to\infty} M(fgp_n, gfp_n, t) = 1$, and $lim_{n\to\infty} M(ffp_n, ggp_n, t) = 1$, whenever $\{p_n\}$ is a sequence in K such that $lim_{n\to\infty} fp_n = lim_{n\to\infty} gp_n = u_1$ for some $u_1 \in K$ and for all $t > 0$.

2. Compatible of type (χ) if $lim_{n\to\infty} M(ffp_n, gp, t) = 1$, and $lim_{n\to\infty} M(ggp_n, fp, t) = 1$, whenever $\{p_n\}$ is a sequence in K such that $lim_{n\to\infty} fp_n = lim_{n\to\infty} gp_n = p$ for some $p \in K$ and for all $t > 0$.

3. Compatible of type (E) if $lim_{n\to\infty} ffp_n = lim_{n\to\infty} fgp_n = gu_1$, and $lim_{n\to\infty} ggp_n = lim_{n\to\infty} gfp_n = fu_1$, whenever $\{p_n\}$ is a sequence in K such that $lim_{n\to\infty} fp_n = lim_{n\to\infty} gp_n = u_1$ for some $u_1 \in K$.

Now we give some properties related to compatible mappings of type (R) and type (E).

Proposition 2.1: Let f and g be compatible mappings of type (R) of a Fuzzy metric space (K, M, Δ) into itself. If $fp = gp$ for some $p \in K$, then $fgp = ffp = ggp = gfp$.

Proof: Suppose that $\{p_n\}$ is a sequence K in defined by $p_n = p, n = 1, 2...$ for some $p \in K$ and $fp = gp$, then we have $fp_n, gp_n \to fp$ as $n \to \infty$. Since f and g are compatible of type (R), we have, $M(fgp, gfp, t) = lim_{n\to\infty} M(fgp_n, gfp_n, t) = 1$.

Hence we have $fgp = ffp$. Therefore, since $fp = gp$, we have $fgp = ffp = ggp = gfp$.

Proposition 2.2: Let f and g be compatible mappings of type (R) of Fuzzy metric space (K, M, Δ) into it self. Suppose that $lim_{n\to\infty} fp_n = p$ for some p in K. Then

1. $lim_{n\to\infty} gfp_n = fp$ if f is continuous at p.
2. $lim_{n\to\infty} fgp_n = gp$ if g is continuous at p.
3. $fgp = gfp$ and $fp = gp$ if f and g are continuous at p.

Proof:**(a)** Suppose that f is continuous at p. Since $lim_{n\to\infty} fp_n = lim_{n\to\infty} gp_n = p$ for some p in K, we have $ffp_n, fgp_n \to f_p$ as an $n \to \infty$. Since f and g are compatible mappings of type (R), we have

$$lim_{n\to\infty} M(gfp_n, fp, t) = lim_{n\to\infty} M(gfp_n, fgp_n, t) = 1.$$

Therefore, $lim_{n\to\infty} gfp_n = fp$.

(b) Suppose that f is continuous at p. Since $lim_{n\to\infty} fp_n = p$ for some p in K, we have $gfp_n, ggp_n \to g_p$ as an $n \to \infty$. Since f and g are compatible mappings of type (R), we have

$$lim_{n\to\infty} M(fgp_n, gp, t) = lim_{n\to\infty} M(fgp_n, gfp_n, t) = 1.$$

Therefore, $lim_{n\to\infty} fgp_n = gp$.

(c) Suppose that f and g are continuous at p and $\{p_n\}$ is a sequence in K defined by $p_n = p, n = 1, 2..$ for some $p \in K$. Since $gp_n \to p$ as $n \to \infty$ and f is continuous at p, by (a), $gfp_n \to fp$ as $n \to \infty$. On the other hand, g is also continuous at $p, gfp_n \to gp$ as $n \to \infty$. Hence we have $fp = gp$ by the uniqueness of limit and so by Proposition2.1, $fgp = ffp = ggp = gfp$. This completes the proof.

Proposition 2.3: Let f and g be compatible mappings of type (E) of a Fuzzy metric space (K, M, Δ) into itself. Let one of f and g be continuous. Suppose that $lim_{n\to\infty} fp_n = lim_{n\to\infty} gp_n = p$ for some $p \in K$. Then

(a) $fp = gp$ and $lim_{n\to\infty} ffp_n = lim_{n\to\infty} ggp_n = lim_{n\to\infty} fgp_n = lim_{n\to\infty} gfp_n =$

(b) If there exists $u_1 \in K$ such that $fu_1 = gu_1 = p$, we have $fgu_1 = gfu_1$.

Proof: (a) Suppose that $\{p_n\}$ is a sequence in K such that $lim_{n\to\infty} fp_n = lim_{n\to\infty} gp_n = p$ for some $p \in K$. Then by definition of compatible of type (E), we have

$$lim_{n\to\infty} ffp_n = lim_{n\to\infty} fgp_n = gu_1.$$

If f is continuous, then we get

$$lim_{n\to\infty} ffp_n = fp = lim_{n\to\infty} ggp_n = lim_{n\to\infty} gfp_n,$$

which implies that $fp = gp$. Also, $lim_{n\to\infty} ffp_n = lim_{n\to\infty} ggp_n = lim_{n\to\infty} fgp_n = lim_{n\to\infty} gfp_n$.
Similarly, if g is continuous, then we get the same result.

(b) Next, suppose that $fu_1 = gu_1 = p$ for some u_1 in K. Then $fgu_1 = f(gu_1) = fp$ and $gfu_1 = g(fu_1) = gp$. And from (a), we have $fp = gp$. Hence $fgu_1 = gfu_1$.

Lemma 2.1[13]: Let $\{p_n\}$ be a sequence in a fuzzy metric space (K, M, Δ). If there exists $q \in (0, 1)$ such that $M(p_{n+2}, p_{n+1}, qt) \geq M(p_{n+1}, p_n, t), t > 0, n \in N$ then $\{p_n\}$ is a Cauchy sequence in K.

Lemma 2.2[13]: Let (K, M, Δ) be a fuzzy metric space. If there exists $q \in (0, 1)$ such that $M(p, q, kt) \geq M(p, q, t)$ for all $p, q \in K$ and $t > 0$, then $p = q$.

Implicit Relations:

Let $\mathcal{F}$ be set of all continuous functions $\psi(p_1, p_2, p_3, p_4, p_5, p_6) : \mathbb{R}^6 \to \mathbb{R}$ is a non-increasing in 6^{th} coordinate variable satisfying the following conditions:

1. $\psi(p, q, q, p, 1, \Delta(p, q)) \geq 1$ or $\psi(p, q, 1, 1, q, \Delta(q, p)) \geq 1$ implies that $p \geq q$.
2. $\psi(p, 1, q, 1, 1, 1) \geq 1$ or $\psi(p, 1, 1, p, 1, q) \geq 1$ implies that $p \geq q$.
3. $\psi(p, q, 1, 1, q, p) \geq 1$ or $\psi(p, 1, 1, q, 1, p) \geq 1$ implies that $p \geq q$.
4. $\psi(p, q, 1, 1, 1, p) \geq 1$ or $\psi(p, 1, q, 1, q, 1) \geq 1$ implies that $p \geq q$.
5. $\psi(p, q, 1, p, q, q) \geq 1$ implies that $p \geq q$.

3 Main Results

Recently, Kang et al. [10] proved the some common fixed point theorem in a complete multiplicative metric space. Now we prove the same in Fuzzy metric space using implicit relations.

Theorem 3.1. Let f, g, f_1 and g_1 be mappings of a complete Fuzzy metric space (K, M, Δ) into itself satisfying the following conditions:

(3.1) $g_1(K) \subset f(K), f_1(K) \subset g(K)$

(3.2)

$$\psi\left(\begin{array}{c} M(f_1p, g_1q, kt), M(fp, gq, t), M(fp, f_1p, t) \\ M(gq, g_1q, kt), M(f_1p, gq, t), M(fp, g_1q, (1+k)t) \end{array}\right) \geq 1$$

for all $p, q \in K$ where $k \in (0, 1), \psi \in \mathcal{F}, t > 0$.

(iii) one of the mappings f, g, f_1 and g_1 is continuous.

Assume that the pairs f, f_1 and g, g_1 are compatible of type (R), Then f, g, f_1 and g_1 have a unique common fixed point in K.

Proof. Since $f_1(K) \subset g(K)$. Now consider a point $p_0 \in K$, there exists $p_1 \in K$ such that $f_1p_0 = gp_1 = q_0$ for this point p_1 there exists $p_2 \in K$ such that $g_1p_1 = fp_2 = q_1$. Continuing in this way, we can define a sequence $\{q_n\}$ in K such that $q_{2n} = f_1p_{2n} = gp_{2n+1}; q_{2n+1} = g_1p_{2n+1} = fp_{2n+2}$.

Now we prove that $\{q_n\}$ is Cauchy sequence in K.

Putting $p = p_{2n}, q = p_{2n+1}$ in inequality (3.2), we have

$$1 \leq \psi \begin{pmatrix} M(f_1p_{2n}, g_1p_{2n+1}, kt), M(fp_{2n}, gp_{2n+1}, t), M(fp_{2n}, f_1p_{2n}, t) \\ M(gp_{2n+1}, g_1p_{2n+1}, kt), M(f_1p_{2n}, gp_{2n+1}, t), M(fp_{2n}, g_1p_{2n+1}, (1+k)t) \end{pmatrix}$$
$$\leq \psi \begin{pmatrix} M(q_{2n}, q_{2n+1}, kt), M(q_{2n-1}, q_{2n}, t), M(q_{2n-1}, q_{2n}, t) \\ M(q_{2n}, q_{2n+1}, kt), M(q_{2n}, q_{2n}, t), M(q_{2n-1}, q_{2n+1}, (1+k)t) \end{pmatrix}$$
$$\leq \psi \begin{pmatrix} M(q_{2n}, q_{2n+1}, kt), M(q_{2n-1}, q_{2n}, t), M(q_{2n-1}, q_{2n}, t) \\ M(q_{2n}, q_{2n+1}, kt), M(q_{2n}, q_{2n}, t), \Delta(M(q_{2n-1}, q_{2n}, t), M(q_{2n}, q_{2n+1}, kt)) \end{pmatrix}$$

Since the function ψ is non-increasing in the 6^{th} coordinate variable. Using properties of implicit relations $\mathcal{F}$, we get

$$M(q_{2n}, q_{2n+1}, kt) \geq M(q_{2n-1}, q_{2n}, t)$$

Again putting $p = p_{2n+1}, q = p_{2n+2}$, in inequality (3.2), we have

$$1 \leq \psi \begin{pmatrix} M(f_1p_{2n+1}, g_1p_{2n+2}, kt), M(fp_{2n+1}, gp_{2n+2}, t), M(fp_{2n+1}, f_1p_{2n+1}, t) \\ M(gp_{2n+2}, g_1p_{2n+2}, kt), M(f_1p_{2n+1}, gp_{2n+2}, t), M(fp_{2n+1}, g_1p_{2n+2}, (1+k)t) \end{pmatrix}$$
$$\leq \psi \begin{pmatrix} M(q_{2n+1}, q_{2n+2}, kt), M(q_{2n}, q_{2n+1}, t), M(q_{2n}, q_{2n+1}, t) \\ M(q_{2n+1}, q_{2n+2}, kt), M(q_{2n+1}, q_{2n+1}, t), M(q_{2n}, q_{2n+2}, (1+k)t) \end{pmatrix}$$
$$\leq \psi \begin{pmatrix} M(q_{2n+1}, q_{2n+2}, kt), M(q_{2n}, q_{2n+1}, t), M(q_{2n}, q_{2n+1}, t) \\ M(q_{2n+1}, q_{2n+2}, kt), M(q_{2n+1}, q_{2n+1}, t), \Delta(M(q_{2n}, q_{2n+1}, t), M(q_{2n+1}, q_{2n+2}, kt)) \end{pmatrix}$$

Since the function ψ is non-increasing in the 6^{th} coordinate variable. Using properties of implicit relations $\mathcal{F}$, we get

$$M(q_{2n+1}, q_{2n+2}, kt) \geq M(q_{2n}, q_{2n+1}, t)$$

Thus for all $n \in N$ and $t > 0$, we have

$$M(q_n, q_{n+1}, kt) \geq M(q_{n-1}, q_n, t)$$

Therefore, by Lemma 2.1, $\{q_n\}$ is a Cauchy sequence in K and hence it converges to some point $r \in K$. Consequently, the subsequence $\{f_1p_{2n}\}, \{gp_{2n+1}\}, \{g_1p_{2n+1}\}$ and $\{fp_{2n}\}$ of $\{q_n\}$ also converges to r.

Now, suppose that f is continuous. Since f and f_1 are compatible of type (R), by Proposition 2.1, ffp_{2n} and f_1fp_{2n} converges to fr as $n \to \infty$. We claim that $r = fr$.

Putting $p = fp_{2n}, q = p_{2n+1}$, in inequality (3.2), we have

$$1 \leq \psi \begin{pmatrix} M(f_1p_{2n}, g_1p_{2n+1}, kt), M(ffp_{2n}, gp_{2n+1}, t), M(ffp_{2n}, f_1p_{2n}, t) \\ M(gp_{2n+1}, g_1p_{2n+1}, kt), M(f_1fp_{2n}, gp_{2n+1}, t), M(ffp_{2n}, g_1p_{2n+1}, (1+k)t) \end{pmatrix}$$

Letting $n \to \infty$ we have

$$\leq \psi \begin{pmatrix} M(fr, r, kt), M(fr, r, t), M(fr, fr, t) \\ M(r, r, kt), M(fr, r, t), \Delta(M(fr, r, t), M(fr, r, kt)) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(fr, r, kt) \geq M(fr, r, t).$$

By lemma 2.2, $fr = r$.

Next we claim that $r = f_1 r$.

Putting $p = r$ and $q = p_{2n+1}$ in inequality (3.2) we have

$$\psi\begin{pmatrix} M(f_1 r, g_1 p_{2n+1}, kt), M(fr, gp_{2n+1}, t), M(fr, f_1 r, t) \\ M(r, r, kt), M(r, r, t), M(r, r, (1+k)t) \end{pmatrix} \geq 1$$

Letting $n \to \infty$ we have

$$\psi\begin{pmatrix} M(f_1 r, r, kt), M(r, r, t), M(r, f_1 r, t) \\ M(gp_{2n+1}, g_1 p_{2n+1}, kt), M(f_1 r, gp_{2n+1}, t), M(fr, g_1 p_{2n+1}, (1+k)t) \end{pmatrix} \geq 1$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(f_1 r, r, kt) \geq M(f_1 r, r, t).$$

By lemma 2.2, $f_1 r = r$.

Since $f_1(K) \subset g(K)$ and hence exists a point $u_1 \in K$ such that $r = f_1 r = gu_1$ We claim that $r = g_1 u_1$.

Putting $p = r$ and $q = u_1$ in inequality (3.2) we have

$$\begin{aligned} 1 &\leq \psi\begin{pmatrix} M(f_1 r, g_1 u_1, kt), M(fr, gu_1, t), M(fr, f_1 r, t) \\ M(gu_1, g_1 u_1, kt), M(f_1 r, gu_1, t), M(fr, g_1 u_1, (1+k)t) \end{pmatrix} \\ &\leq \psi\begin{pmatrix} M(r, g_1 u_1, kt), M(r, r, t), M(r, r, t) \\ M(r, g_1 u_1, kt), M(r, r, t), \Delta(M(r, r, kt), M(r, g_1 u_1, t)) \end{pmatrix} \\ &\leq \psi\begin{pmatrix} M(r, g_1 u_1, kt), M(r, r, t), M(r, r, t) \\ M(r, g_1 u_1, kt), M(r, r, t), M(r, g_1 u_1, t) \end{pmatrix} \end{aligned}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(r, g_1 u_1, kt) \geq M(r, g_1 u_1, t).$$

By lemma 2.2, $r = g_1 u_1$.

Since g and g_1 are compatible of type (R) and $gu_1 = g_1 u_1 = r$, by Proposition2.1, $gg_1 u_1 = g_1 gu_1$ and hence $gr = gg_1 u_1 = g_1 gu_1 = g_1 r$. Also, we have from the 3.2 we have

$$\begin{aligned} 1 &\leq \psi\begin{pmatrix} M(f_1 r, g_1 r, kt), M(fr, gr, t), M(fr, f_1 r, t) \\ M(gr, g_1 r, kt), M(f_1 r, gr, t), M(fr, g_1 r, (1+k)t) \end{pmatrix} \\ &\leq \psi\begin{pmatrix} M(r, gr, kt), M(r, gr, t), M(r, r, t) \\ M(gr, gr, kt), M(r, gr, t), \Delta(M(r, r, t), M(r, gr, kt)) \end{pmatrix} \end{aligned}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(r, gr, kt) \geq M(r, gr, t).$$

By lemma 2.2, $r = gr$.

Hence $r = gr = g_1 r = fr = f_1 r$. Therefore, r is a common fixed point of f, f_1, g and g_1.

Similarly, we can complete the proof when g is continuous. Next, suppose that f_1 is continuous. Since f and f_1 are compatible of type (R), by Proposition 2.1,$f_1 f_1 p_{2n}$ and $f_1 p_{2n}$ converges to $f_1 r$ as $n \to \infty$. We claim that $r = f_1 r$.

Putting $p = f_1 p_{2n}$ and $q = p_{2n+1}$, in inequality (3.2) we have

$$1 \leq \psi \begin{pmatrix} M(f_1 f_1 p_{2n}, g_1 p_{2n+1}, kt), M(f f_1 p_{2n}, g p_{2n+1}, t), M(f f_1 p_{2n}, f_1 f_1 p_{2n}, t) \\ M(g p_{2n+1}, g_1 p_{2n+1}, t), M(f_1 f_1 p_{2n}, g p_{2n+1}, t), M(f f_1 p_{2n}, g_1 p_{2n+1}, (1+k)t) \end{pmatrix}$$

Letting $n \to \infty$

$$\leq \psi \begin{pmatrix} M(f_1 r, r, kt), M(f_1 r, r, t), M(f_1 r, f_1 r, t) \\ M(r, r, kt), M(f_1 r, r, t), M(f_1 r, r, (1+k)t) \end{pmatrix}$$

$$\leq \psi \begin{pmatrix} M(f_1 r, r, kt), M(f_1 r, r, t), M(f_1 r, f_1 r, t) \\ M(r, r, kt), M(f_1 r, r, t), \Delta(M(f_1 r, f_1 r, t), M(f_1 r, r, kt) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(f_1 r, r, kt) \geq M(f_1 r, r, t).$$

By lemma 2.2, $f_1 r = r$.

Since $f_1(K) \subset g(K)$ and hence exists a point $v_1 \in K$ such that $r = f_1 r = g v_1$. We claim that $r = g_1 v_1$.

Putting $p = f_1 p_{2n}$ and $q = v_1$, in inequality (3.2) we have

$$1 \leq \psi \begin{pmatrix} M(f_1 f_1 p_{2n}, g_1 v_1, kt), M(f f_1 p_{2n}, g v_1, t), M(f f_1 p_{2n}, f_1 p_{2n}, t) \\ M(g v_1, g_1 v_1, t), M(f_1 f_1 p_{2n}, g v_1, t), M(fr, g_1 v_1, (1+k)t) \end{pmatrix}$$

Letting $n \to \infty$

$$\leq \psi \begin{pmatrix} M(r, g_1 v_1, kt), M(r, r, t), M(r, r, t) \\ M(r, g_1 v_1, t), M(r, r, t), \Delta(M(r, r, t), M(g_1 v_1, r, kt) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(r, g_1 v_1, kt) \geq M(r, g_1 v_1, t).$$

By lemma 2.2, $r = g_1 v_1$.

Since g and g_1 are compatible of type (R) and $g v_1 = g_1 v_1 = r$, by Proposition 2.1,$g g_1 v_1 = g_1 g v_1$ and hence $gr = g g_1 v_1 = g_1 g v_1 = g_1 r\, f_1 p_{2n}$. We claim that $r = g_1 r$.

Putting $p = p_{2n}$ and $q = r$, in inequality (3.2) we have

$$1 \le \psi \begin{pmatrix} M(f_1p_{2n}, g_1r, kt), M(fp_{2n}, gr, t), M(fp_{2n}, f_1p_{2n}, t) \\ M(gr, g_1r, t), M(f_1p_{2n}, gr, t), M(fp_{2n}, g_1r, (1+k)t) \end{pmatrix}$$

$$Letting \quad n \to \infty$$

$$\le \psi \begin{pmatrix} M(r, g_1r, kt), M(r, g_1r, t), M(r, r, t) \\ M(g_1r, g_1r, t), M(r, g_1r, t), M(r, g_1r, (1+k)t) \end{pmatrix}$$

$$\le \psi \begin{pmatrix} M(r, g_1r, kt), M(r, g_1r, t), M(r, r, t) \\ M(g_1r, g_1r, t), M(r, g_1r, t), \Delta(M(r, r, t), M(r, g_1r, kt)) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(r, g_1r, kt) \ge M(r, g_1r, t).$$

By lemma 2.2, $g_1r = r$.

Since $g_1(K) \subset f(K)$ and hence exists a point $w \in K$ such that $r = f_1r = fw$. We claim that $r = f_1w$.

Putting $p = w$ and $q = r$, in inequality (3.2) we have

$$1 \le \psi \begin{pmatrix} M(f_1w, r, kt), M(fw, gr, t), M(fw, f_1w, t) \\ M(gr, g_1r, t), M(f_1w, gr, t), M(fw, g_1r, (1+k)t) \end{pmatrix}$$

$$= \psi \begin{pmatrix} M(f_1w, r, kt), M(r, r, t), M(r, f_1w, t) \\ M(g_1r, g_1r, t), M(f_1w, r, t), M(r, r, (1+k)t) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(f_1w, r, kt) \ge M(r, f_1w, t).$$

By Lemma 2.2, we get $r = f_1w$ Since f and f_1 are compatible of type (R) and $f_1w = fw = r$, by Proposition2.1, $ff_1w = f_1fw$ and hence $fr = ff_1w = f_1fw = f_1r$.

Hence $r = gr = g_1r = fr = f_1r$. Therefore, r is a common fixed point of f, f_1, g and g_1. Similarly, we can complete the proof when g_1 is continuous.

Next we prove the following theorem for compatible mappings of type (K) and type (E).

Theorem 3.2: Let f, g, f_1 and g_1 and be mappings of a complete Fuzzy metric space (K, M, Δ) into itself satisfying the following conditions (3.1), (3.2). Suppose that the pairs f, f_1 and g, g_1 are reciprocally continuous.

Assume that the pairs f, f_1 and g, g_1 are compatible of type (K). Then f, f_1 and g, g_1 have a unique common fixed point in K.

Proof. Now from the proof of Theorem 3.4 we can easily prove that $\{q_n\}$ is Cauchy sequence in K and hence it converges to some point $r \in K$. Consequently, the subsequence $\{f_1p_{2n}\}$, $\{gp_{2n+1}\}$, $\{g_1p_{2n+1}\}$ and $\{fp_{2n}\}$ of $\{q_n\}$ also converges to r.

Since the pairs f, f_1 and g, g_1 and are compatible of type (K), we have $ffp_{2n} \to f_1r, f_1f_1p_{2n} \to fr$ and $ggp_{2n} \to g_1r, g_1g_1p_{2n} \to gr$ as $n \to \infty$. We claim that $gr = fr$.

Putting $p = f_1 p_{2n}$ and $q = g_1 p_{2n+1}$ in inequality (3.2) we have

$$1 \leq \psi \begin{pmatrix} M(f_1 f_1 p_{2n}, g_1 g_1 p_{2n+1}, kt), M(f f_1 p_{2n}, g g_1 p_{2n+1}, t), M(f f_1 p_{2n}, f_1 f_1 p_{2n}, t) \\ M(g g_1 p_{2n+1}, g_1 g_1 p_{2n+1}, kt), M(f_1 f_1 p_{2n}, g g_1 p_{2n+1}, t), M(f f_1 p_{2n}, g_1 p_{2n+1}, (1+k)t) \end{pmatrix}$$

Letting $n \to \infty$,

$$\leq \psi \begin{pmatrix} M(fr, gr, kt), M(fr, gr, t), M(fr, fr, t) \\ M(gr, gr, kt), M(fr, fr, t), M(fr, gr, (1+k)t) \end{pmatrix}$$

$$\leq \psi \begin{pmatrix} M(fr, gr, kt), M(fr, gr, t), M(fr, fr, t) \\ M(gr, gr, kt), M(fr, fr, t), \Delta(M(fr, fr, t), M(fr, gr, kt)) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(fr, gr, kt) \geq M(fr, gr, t).$$

By Lemma 2.2, we get $fr = gr$.

Putting $p = r$ and $q = g_1 p_{2n+1}$ in inequality (3.2) we have

$$1 \leq \psi \begin{pmatrix} M(f_1 r, g_1 g_1 p_{2n+1}, kt), M(fr, g_1 p_{2n+1}, t), M(fr, f_1 r, t) \\ M(g g_1 p_{2n}, g_1 g_1 p_{2n+1}, kt), M(f_1 r, g g_1 p_{2n+1}, t), M(fr, g_1 g_1 p_{2n+1}, (1+k)t) \end{pmatrix}$$

Letting $n \to \infty$,

$$\leq \psi \begin{pmatrix} M(f_1 r, gr, kt), M(gr, gr, t), M(gr, f_1 r, t) \\ M(gr, gr, kt), M(f_1 r, gr, t), M(gr, gr, (1+k)t) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(f_1 r, gr, kt) \geq M(f_1 r, gr, t).$$

By Lemma 2.2, we get $f_1 r = gr$.

We claim that $f_1 r = g_1 r$ Putting $p = r$ and $q = r$ in inequality (3.2) we have

$$1 \leq \psi \begin{pmatrix} M(f_1 r, g_1 r, kt), M(fr, gr, t), M(fr, f_1 r, t) \\ M(gr, g_1 r, kt), M(f_1 r, gr, t), M(fr, g_1 r, (1+k)t) \end{pmatrix}$$

$$\leq \psi \begin{pmatrix} M(f_1 r, g_1 r, kt), M(gr, gr, t), M(fr, fr, t) \\ M(f_1 r, g_1 r, kt), M(f_1 r, f_1 r, t), \Delta(M(f_1 r, f_1 r, kt), M(f_1 r, g_1 r, t)) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(f_1 r, g_1 r, kt) \geq M(f_1 r, g_1 r, t).$$

By Lemma 2.2, we get $f_1 r = g_1 r$.

We claim that $r = g_1 r$ Putting $p = p_{2n}$ and $q = r$ in inequality (3.2) we have

$$1 \leq \psi \begin{pmatrix} M(f_1 p_{2n}, g_1 r, kt), M(f p_{2n}, gr, t), M(f p_{2n}, f_1 p_{2n}, t) \\ M(gr, g_1 r, kt), M(f_1 p_{2n}, gr, t), M(f p_{2n}, g_1 r, (1+k)t) \end{pmatrix}$$

$$\textit{Letting} \quad n \to \infty,$$

$$\leq \psi \begin{pmatrix} M(r, g_1 r, kt), M(r, g_1 r, t), M(r, r, t) \\ M(r, g_1 r, kt), M(r, g_1 r, kt), M(r, g_1 r, (1+k)t) \end{pmatrix}$$

$$\leq \psi \begin{pmatrix} M(r, g_1 r, kt), M(r, g_1 r, t), M(r, r, t) \\ M(r, g_1 r, kt), M(r, g_1 r, kt), \Delta(M(r, r, kt), M(r, g_1 r, t)) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(r, g_1 r, kt) \geq M(r, g_1 r, t).$$

By Lemma 2.2, we get $r = g_1 r$.

Hence $r = gr = g_1 r = fr = f_1 r$. Therefore, r is a common fixed point of f, f_1, g and g_1.

Theorem 3.6. Let f, f_1, g and g_1 be mappings of a complete Fuzzy metric space (K, M, Δ) into itself satisfying the following conditions (3.1), (3.2). Suppose that one of f and f_1 is continuous and one of g and g_1 is continuous. Assume that the pairs f, f_1 and g, g_1 are compatible of type (E), Then f, f_1, g and g_1 have a unique common fixed point in K.

Proof. Now from the proof of Theorem 3.4 we can easily prove that $\{q_n\}$ is Cauchy sequence in K and hence it converges to some point $r \in K$. Consequently, the subsequence $\{f_1 p_{2n}\}, \{g p_{2n+1}\}, \{g_1 p_{2n+1}\}$ and $\{f p_{2n}\}$ of $\{q_n\}$ also converges to r. Now, suppose that one of the mappings f and f_1 is continuous. Since f and f_1 are compatible of type E by Proposition 2.3, $fr = f_1 r$. Since $f_1(K) \subset g(K)$ and hence exists a point $w \in K$ such that $f_1 r = gw$.

We claim that $f_1 r = g_1 w$.

Putting $p = r$ and $q = w$ in inequality (3.2) we have

$$1 \leq \psi \begin{pmatrix} M(f_1 r, g_1 w, kt), M(fr, gw, t), M(fr, f_1 r, t) \\ M(gw, g_1 w, kt), M(f_1 r, gw, t), M(fr, g_1 w, (1+k)t) \end{pmatrix}$$

$$\leq \psi \begin{pmatrix} M(f_1 r, g_1 w, kt), M(fr, f_1 r, t), M(f_1 r, f_1 r, t) \\ M(f_1 r, g_1 w, kt), M(f_1 r, gw, t), \Delta(M(f_1 r, f_1 r, kt), M(f_1 r, g_1 w, kt)) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(f_1 r, g_1 w, kt) \geq M(f_1 r, g_1 r, t).$$

By Lemma 2.2, we get $f_1 r = g_1 w$.

Thus we have $fr = f_1 r = g_1 w = gw$. We claim that $f_1 r = r$ Putting $p = r$ and $q = p_{2n+1}$ in inequality (3.2) we have

$$1 \le \psi \begin{pmatrix} M(f_1 r, g_1 p_{2n+1}, kt), M(fr, gp_{2n+1}, t), M(fr, f_1 r, t) \\ M(gp_{2n+1}, g_1 p_{2n+1}, kt), M(f_1 r, gp_{2n+1}, t), M(fr, g_1 p_{2n+1}, (1+k)t) \end{pmatrix}$$

$$\le \psi \begin{pmatrix} M(f_1 r, r, kt), M(f_1 r, r, t), M(r, r, t) \\ M(r, r, kt), M(f_1 r, r, t), \Delta(M(f_1 r, f_1 r, kt), M(f_1 r, r, t)) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(f_1 r, r, kt) \ge M(f_1 r, r, t).$$

By Lemma 2.2, we get $r = f_1 r$.

Hence $r = gr = g_1 r = f_1 r = fr$. Therefore, r is a common fixed point of f, f_1, g and g_1.

Again, suppose g and g_1 are compatible of type (E) and one of the mappings g and g_1 is continuous. Then $gw = g_1 w = r$. By Proposition 2.3, we have $ggw = gg_1 w = g_1 gw = g_1 g_1 w$. Hence $gr = g_1 r$.

We claim that $r = g_1 r$.

Putting $p = p_{2n}$ and $q = r$ in inequality (3.2) we have

$$1 \le \psi \begin{pmatrix} M(f_1 p_{2n}, g_1 r, kt), M(fp_{2n}, gr, t), M(fp_{2n}, f_1 p_{2n}, t) \\ M(gr, g_1 r, kt), M(f_1 p_{2n}, gr, t), M(fp_{2n}, g_1 r, (1+k)t) \end{pmatrix}$$

Letting $n \to \infty$,

$$\le \psi \begin{pmatrix} M(f_1 r, r, kt), M(f_1 r, r, t), M(f_1 r, f_1 r, t) \\ M(r, r, kt), M(f_1 r, r, t), \Delta(M(f_1 r, f_1 r, kt), M(f_1 r, r, t)) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(f_1 r, r, kt) \ge M(f_1 r, r, t).$$

By lemma 2.2, $f_1 r = r$.

Since $f_1(K) \subset g(K)$ and hence exists a point $v \in K$ such that $r = f_1 r = gv$. We claim that $r = g_1 v$.

Putting $p = f_1 p_{2n}$ and $q = v$, in inequality (3.2) we have

$$1 \le \psi \begin{pmatrix} M(f_1 f_1 p_{2n}, g_1 v, kt), M(ff_1 p_{2n}, gv, t), M(ff_1 p_{2n}, f_1 p_{2n}, t) \\ M(gv, g_1 v, kt), M(f_1 f_1 p_{2n}, gv, t), M(fr, g_1 v, (1+k)t) \end{pmatrix}$$

Letting $n \to \infty$,

$$\le \psi \begin{pmatrix} M(r, g_1 v, kt), M(r, r, t), M(r, r, t) \\ M(r, g_1 v, kt), M(r, r, t), \Delta(M(r, r, kt), M(r, g_1 v, t)) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(r, g_1 v, kt) \ge M(r, g_1 v, t).$$

By Lemma 2.2, we get $r = f_1 r$.

Hence $r = g_1 v$. Since g and g_1 are compatible of type (R) and $gv = g_1 v = r$. By Proposition 2.3, we have

$gg_1v = g_1gv$ and hence $gr = gg_1v = g_1gv = g_1r$.

We claim that $r = g_1r$.

Putting $p = p_{2n}$ and $q = r$ in inequality (3.2) we have

$$1 \leq \psi \begin{pmatrix} M(f_1p_{2n}, g_1r, kt), M(fp_{2n}, gr, t), M(fp_{2n}, f_1p_{2n}, t) \\ M(gr, g_1r, kt), M(f_1p_{2n}, gr, t), M(fp_{2n}, g_1r, (1+k)t) \end{pmatrix}$$

$$Letting \quad n \to \infty,$$

$$\leq \psi \begin{pmatrix} M(r, g_1r, kt), M(r, g_1r, t), M(r, r, t) \\ M(g_1r, g_1r, kt), M(r, g_1r, t), \Delta(M(r, r, kt), M(r, g_1r, t)) \end{pmatrix}$$

Using properties of implicit relations $\mathcal{F}$, we get

$$M(r, g_1r, kt) \geq M(r, g_1r, t).$$

By Lemma 2.2, we get $g_1r = r$.

Hence $r = gr = g_1r$. Therefore, r is a common fixed point of f, f_1, g and g_1. Similarly, we can complete the proof when g_1 is continuous. Uniqueness follows easily. This completes the proof.

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