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On Two General Integrals Involving Humbert's and Kummer's Hypergeometric Functions

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ABSTRACT. The aim of this paper is to apply generalized Kummer's theorem and generalized Dixon's theorem due to Lavoie et al. to establish two general integrals involving Humbert's functions of two variables Φ_2, Ψ_2 and Kummer's function ${}_1F_1$. Some interesting applications of our main results are also presented.

1. Introduction

In the usual notation, let ${}_pF_q$ denote generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters, defined by [8]

$$ {}_pF_q\left[\begin{array}{lc} a_1,\ldots,a_p & ; \\ & x \\ b_1,\ldots,b_q & ; \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \ \ldots \ (a_p)_n}{(b_1)_n \ \ldots \ (b_q)_n} \frac{x^n}{n!}, \tag{1.1} $$

where $(a)_n$ is the Pochhammer's symbol defined by

$$ (a)_n = \begin{cases} 1 & , \quad if \ n=0 \\ a(a+1)(a+2)\ldots(a+n-1) & , \quad if \ n=1,2,3,\ldots \end{cases} \tag{1.2} $$

For $p=q=1$, (1.1) reduces to Kummer's function ${}_1F_1(a;b;x)$ and for $p=2,\ q=1$, (1.1) reduces to the Gauss's function ${}_2F_1(a,b;c;x)$.

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The Humbert's confluent hypergeometric functions Φ_2 and Ψ_2 are defined and represented as follows [2,8]:

$$\Phi_2[a,a';b;x,y]=\sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n\,x^m\,y^n}{(b)_{m+n}m!\,n!} \tag{1.3}$$

and

$$\Psi_2[a;b,b';x,y]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\,x^m\,y^n}{(b)_m(b')_n m!\,n!}. \tag{1.4}$$

Exton [3,4] gave the definitions and the Laplace integral representations of the quadruple hypergeometric functions K_{10} and K_{13} as follows:

$$\begin{aligned}&K_{10}(a,a,a,a;b,b,c_1,c_2;d_1,d_2,d_3,d_4;x,y,z,t)\\&=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a)_{p+q+r+s}(b)_{p+q}(c_1)_r(c_2)_s\,x^p\,y^q\,z^r\,t^s}{(d_1)_p(d_2)_q(d_3)_r(d_4)_s\,p!q!r!s!}\\&=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-s}s^{a-1}\Psi_2(b;d_1,d_2;xs,ys)\,{}_1F_1(c_1;d_3;zs)\,{}_1F_1(c_2;d_4;ts)ds\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}&K_{13}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,d_1,d_2;x,y,z,t)\\&=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a)_{p+q+r+s}(b_1)_p(b_2)_q(b_3)_r(b_4)_s\,x^p\,y^q\,z^r\,t^s}{(c)_{p+q}(d_1)_r(d_2)_s\,p!q!r!s!}\\&=\frac{1}{\Gamma(a)}\int_0^{\infty}e^{-s}s^{a-1}\Phi_2(b_1,b_2;c;xs,ys)\,{}_1F_1(b_3;d_1;zs)\,{}_1F_1(b_4;d_2;ts)ds.\end{aligned} \tag{1.6}$$

The Kampé de Fériet function of two variables $F^{p:q;k}_{l:m;n}[x,y]$ is defined and represented as follows [8]:

$$F^{p:q;k}_{l:m;n}\left[\begin{array}{c}(a_p)\;:\;(b_q)\;;\;(c_k)\;;\\ \\(\alpha_l)\;:\;(\beta_m)\;;\;(\gamma_n)\;;\end{array}\;x\,,\,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}. \tag{1.7}$$

In order to obtain our main results, we require the following results:

Generalized Kummer's theorem [6]

$$\begin{aligned}{}_2F_1\left[\begin{array}{cc}a\,,\,b & ;\\ 1+a-b+i & ;\end{array}\;-1\right]&=\frac{\Gamma(\frac{1}{2})\Gamma(1+a-b+i)\Gamma(1-b)}{2^a\Gamma(1-b+\frac{1}{2}(i+|i|))}\\ \times\left\{\frac{A_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1+\frac{1}{2}a-b+\frac{1}{2}i)}\right.&+\left.\frac{B_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i)}\right\}\\ \text{for }(i=0,\pm1,\pm2,\pm3,\pm4,\pm5)&\end{aligned} \tag{1.8}$$

where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients A_i *and* B_i are given in [6]. For $i=0$, (1.8) reduces immediately to the classical Kummer's theorem (see [1,7])

$${}_2F_1\left[\begin{array}{cc}a\,,\,b & ;\\ 1+a-b & ;\end{array}\;-1\right]=\frac{\Gamma(1+a-b)\Gamma(\frac{1}{2})}{2^a\Gamma(1+\frac{1}{2}a-b)\Gamma(\frac{1}{2}a+\frac{1}{2})}. \tag{1.9}$$

Generalized Dixon's theorem [5]

$$
{}_3F_2\left[\begin{array}{ccc} a\,,\,b\,,\,c & ; & \\ & & 1 \\ 1+a-b+i, 1+a-c+i+j & ; & \end{array}\right]
$$
$$
= \frac{2^{-2c+i+j}\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\Gamma(b-\frac{1}{2}|i|-\frac{1}{2}i)\Gamma(c-\frac{1}{2}(i+j+|i+j|))}{\Gamma(b)\Gamma(c)\Gamma(1+a-2c+i+j)\Gamma(1+a-b-c+i+j)}
$$
$$
\times\left\{A_{i,j}\frac{\Gamma(\frac{1}{2}a-c+\frac{1}{2}+[\frac{i+j+1}{2}])\Gamma(\frac{1}{2}a-b-c+1+i+[\frac{j+1}{2}])}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}a-b+1+[\frac{i}{2}])}\right.
$$
$$
\left.+B_{i,j}\frac{\Gamma(\frac{1}{2}a-c+1+[\frac{i+j}{2}])\Gamma(\frac{1}{2}a-b-c+\frac{3}{2}+i+[\frac{j}{2}])}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b+\frac{1}{2}+[\frac{i+1}{2}])}\right\} \tag{1.10}
$$
$$
\{Re(a-2b-2c)>-2-2i-j\,;\ i=-3,-2,-1,0,1,2\,;\ j=0,1,2,3\}.
$$

The coefficients $A_{i,j}$ and $B_{i,j}$ are given in [5]. For $i=j=0$, (1.10) reduces immediately to the classical Dixon's theorem (see [1,7])

$$
{}_3F_2\left[\begin{array}{ccc} a\,,\,b\,,\,c & ; & \\ & & 1 \\ 1+a-b, 1+a-c & ; & \end{array}\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}. \tag{1.11}
$$

2. Main Results

First Integral:

$$
\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Psi_2(b;b',b'+i;xs,-xs)\,{}_1F_1(c-k;c';ys)\,{}_1F_1(c;c'+k+l;-ys)ds
$$
$$
= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p}(b)_{2m}(c-k)_{2p}\,x^{2m}\,y^{2p}}{(b')_{2m}(c')_{2p}(2m)!(2p)!}\times(A'_iA_{2m}+B'_iB_{2m})E'_{k,l}E_{2p}
$$
$$
+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p+1}(b)_{2m+1}(c-k)_{2p}\,x^{2m+1}\,y^{2p}}{(b')_{2m+1}(c')_{2p}(2m+1)!(2p)!}\times(A''_iA_{2m+1}+B''_iB_{2m+1})E'_{k,l}E_{2p}
$$
$$
+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p+1}(b)_{2m}(c-k)_{2p+1}\,x^{2m}\,y^{2p+1}}{(b')_{2m}(c')_{2p+1}(2m)!(2p+1)!}\times(A'_iA_{2m}+B'_iB_{2m})F'_{k,l}F_{2p+1}
$$
$$
+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p+2}(b)_{2m+1}(c-k)_{2p+1}\,x^{2m+1}\,y^{2p+1}}{(b')_{2m+1}(c')_{2p+1}(2m+1)!(2p+1)!}\times(A''_iA_{2m+1}+B''_iB_{2m+1})F'_{k,l}F_{2p+1} \tag{2.1}
$$
$$
for\{i=0,\pm1,\pm2,\pm3,\pm4,\pm5\,;\ k=-3,-2,-1,0,1,2\,;\ l=0,1,2,3\},
$$

where

$$
A_m = \frac{\Gamma(\frac{1}{2})\Gamma(b'+i)\Gamma(b'+m)\Gamma(-\frac{1}{2}m+\frac{1}{2}i+\frac{1}{2}-[\frac{i+1}{2}])\Gamma(\frac{1}{2}m+b'+\frac{1}{2}i)}{2^{-m}\Gamma(b'+m+\frac{1}{2}(i+|i|))}
$$

$$
B_m = \frac{\Gamma(\frac{1}{2})\Gamma(b'+i)\Gamma(b'+m)\Gamma(-\frac{1}{2}m+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}m+b'-\frac{1}{2}+\frac{1}{2}i)}{2^{-m}\Gamma(b'+m+\frac{1}{2}(i+|i|))}
$$

$$E_p = \frac{2^{2(p+c'-1)+k+l}\Gamma(1-c+k-p)\Gamma(c'+k+l)\Gamma(c-\frac{1}{2}|k|-\frac{1}{2}k)\Gamma(1-c'-p-\frac{1}{2}(k+l+|k+l|))}{\Gamma(c)\Gamma(1-c'-p)\Gamma(2c'-1+k+l+p)\Gamma(c'-c+k+l)} \times\frac{\Gamma(\frac{1}{2}p+c'-\frac{1}{2}+[\frac{k+l+1}{2}])\Gamma(\frac{1}{2}p-c+c'+k+[\frac{l+1}{2}])}{\Gamma(\frac{1}{2}-\frac{1}{2}p)\Gamma(1-c-\frac{1}{2}p+[\frac{k}{2}])}$$

$$F_p = \frac{2^{2(p+c'-1)+k+l}\Gamma(1-c+k-p)\Gamma(c'+k+l)\Gamma(c-\frac{1}{2}|k|-\frac{1}{2}k)\Gamma(1-c'-p-\frac{1}{2}(k+l+|k+l|))}{\Gamma(c)\Gamma(1-c'-p)\Gamma(2c'-1+k+l+p)\Gamma(c'-c+k+l)} \times\frac{\Gamma(\frac{1}{2}p+c'+[\frac{k+l}{2}])\Gamma(\frac{1}{2}p-c+c'+\frac{1}{2}+k+[\frac{l}{2}])}{\Gamma(-\frac{1}{2}p)\Gamma(-\frac{1}{2}p-c+\frac{1}{2}+[\frac{k+1}{2}])}.$$

The coefficients A'_i and B'_i can be obtained from the tables of A_i and B_i given in [6] by replacing a and b by $-2m$ and $1-b'-2m$ and the coefficients A''_i and B''_i can be obtained from the tables of A_i and B_i by replacing a and b by $-2m-1$ and $-b'-2m$.

The coefficient $E'_{k,l}$ can be obtained from the table of $A_{i,j}$ given in [5] by replacing a, b and c by $-2p$, c and $1-c'-2p$ respectively and the coefficient $F'_{k,l}$ can be obtained from the table of $B_{i,j}$ given in [5] by replacing a, b and c by $-2p-1$, c and $-c'-2p$ respectively.

Second Integral:

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Phi_2(b-i,b;b';xs,-xs)_1F_1(c-k;c';ys)_1F_1(c;c'+k+l;-ys)ds$$

$$= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p}(b-i)_{2m}(c-k)_{2p}\, x^{2m}\, y^{2p}}{(b')_{2m}(c')_{2p}(2m)!(2p)!}\times(C'_iC_{2m}+D'_iD_{2m})E'_{k,l}E_{2p}$$

$$+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p+1}(b-i)_{2m+1}(c-k)_{2p}\, x^{2m+1}\, y^{2p}}{(b')_{2m+1}(c')_{2p}(2m+1)!(2p)!}\times(C''_iC_{2m+1}+D''_iD_{2m+1})E'_{k,l}E_{2p}$$

$$+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p+1}(b-i)_{2m}(c-k)_{2p+1}\, x^{2m}\, y^{2p+1}}{(b')_{2m}(c')_{2p+1}(2m)!(2p+1)!}\times(C'_iC_{2m}+D'_iD_{2m})F'_{k,l}F_{2p+1}$$

$$+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p+2}(b-i)_{2m+1}(c-k)_{2p+1}\, x^{2m+1}\, y^{2p+1}}{(b')_{2m+1}(c')_{2p+1}(2m+1)!(2p+1)!}\times(C''_iC_{2m+1}+D''_iD_{2m+1})F'_{k,l}F_{2p+1} \quad (2.2)$$

$$for\{i=0,\pm1,\pm2,\pm3,\pm4,\pm5\ ;\ k=-3,-2,-1,0,1,2\ ;\ l=0,1,2,3\}$$

where

$$C_m = \frac{\Gamma(\frac{1}{2})\Gamma(1-m-b+i)\Gamma(1-b)\Gamma(-\frac{1}{2}m+\frac{1}{2}i+\frac{1}{2}-[\frac{i+1}{2}])\Gamma(1-\frac{1}{2}m-b+\frac{1}{2}i)}{2^{-m}\Gamma(1-b+\frac{1}{2}(i+|i|))}$$

$$D_m = \frac{\Gamma(\frac{1}{2})\Gamma(1-m-b+i)\Gamma(1-b)\Gamma(-\frac{1}{2}m+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}-\frac{1}{2}m-b+\frac{1}{2}i)}{2^{-m}\Gamma(1-b+\frac{1}{2}(i+|i|))}.$$

The coefficients C_i' and D_i' can be obtained from the tables of A_i and B_i given in [6] by replacing a by $-2m$ and the coefficients C_i'' and D_i'' can be also obtained from the tables of A_i and B_i by replacing a by $-2m-1$.

Proof of the first integral: Denoting the left hand side of (2.1) by S, then from the definition (1.5), we have

$$S=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(b)_{m+n}(c-k)_p(c)_q\; x^m(-x)^n y^p(-y)^q}{(b')_m(b'+i)_n(c')_p(c'+k+l)_q m!n!p!q!}$$

Using the well-known results [8]

$$\begin{aligned}(a)_{m+n} &= (a)_m(a+m)_n\\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A(n,m) &= \sum_{m=0}^{\infty}\sum_{n=0}^{m}A(n,m-n),\\ (a)_{m-n} &= \frac{(-1)^n(a)_m}{(1-a-m)_n},\; 0\le n\le m\\ (m-n)! &= \frac{(-1)^n m!}{(-m)_n},\; 0\le n\le m,\end{aligned}$$

then after a little simplification, we have

$$S=\sum_{m,p=0}^{\infty}\frac{(a)_{m+p}(b)_m(c-k)_p\; x^m\; y^p}{(b')_m(c')_p\; m!p!}\times{}_2F_1\left[\begin{matrix}-m\,,\,1-b'-m\;;\\ & -1\\ b'+i\;;\end{matrix}\right]{}_3F_2\left[\begin{matrix}-p\,,\,c\,,1-c'-p\;;\\ & 1\\ 1-c-p+k\,,\,c'+k+l\;;\end{matrix}\right]$$

Now, separating into its even and odd terms, we have

$$\begin{aligned}S=&\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p}(b)_{2m}(c-k)_{2p}\; x^{2m}\; y^{2p}}{(b')_{2m}(c')_{2p}\;(2m)!(2p)!}\\ &\times{}_2F_1\left[\begin{matrix}-2m\,,\,1-b'-2m\;;\\ & -1\\ b'+i\;;\end{matrix}\right]{}_3F_2\left[\begin{matrix}-2p\,,\,c\,,1-c'-2p\;;\\ & 1\\ 1-c-2p+k\,,\,c'+k+l\;;\end{matrix}\right]\\ &+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p+1}(b)_{2m+1}(c-k)_{2p}\; x^{2m+1}\; y^{2p}}{(b')_{2m+1}(c')_{2p}\;(2m+1)!(2p)!}\\ &\times{}_2F_1\left[\begin{matrix}-2m-1\,,\,-b'-2m\;;\\ & -1\\ b'+i\;;\end{matrix}\right]{}_3F_2\left[\begin{matrix}-2p\,,\,c\,,1-c'-2p\;;\\ & 1\\ 1-c-2p+k\,,\,c'+k+l\;;\end{matrix}\right]\\ &+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p+1}(b)_{2m}(c-k)_{2p+1}\; x^{2m}\; y^{2p+1}}{(b')_{2m}(c')_{2p+1}\;(2m)!(2p+1)!}\\ &\times{}_2F_1\left[\begin{matrix}-2m\,,\,1-b'-2m\;;\\ & -1\\ b'+i\;;\end{matrix}\right]{}_3F_2\left[\begin{matrix}-2p-1\,,\,c\,,-c'-2p\;;\\ & 1\\ -c-2p+k\,,\,c'+k+l\;;\end{matrix}\right]\\ &+\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a)_{2m+2p+2}(b)_{2m+1}(c-k)_{2p+1}\; x^{2m+1}\; y^{2p+1}}{(b')_{2m+1}(c')_{2p+1}\;(2m+1)!(2p+1)!}\end{aligned}$$

$$\times {}_2F_1\left[\begin{array}{ccc} -2m-1\,,\ -b'-2m & ; & \\ & & -1 \\ b'+i & ; & \end{array}\right] {}_3F_2\left[\begin{array}{ccc} -2p-1\,,\ c\,, -c'-2p & ; & \\ & & 1 \\ -c-2p+k\,,\ c'+k+l & ; & \end{array}\right]$$

Finally, if we use generalized Kummer's theorem (1.8) and generalized Dixon's theorem (1.10), then we readily arrive at the right hand side of (2.1). This completes the proof of the first integral. The proof of the second integral is similar to that of the first integral and we use here the results (1.6), (1.8) and (1.10).

3. Applications

In our investigations in this section, we require the following results [8]:

$$\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n, \quad \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n} \tag{3.1}$$

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1+\alpha) \;=\; 2^{\alpha}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}\alpha\right)\Gamma\left(1+\frac{1}{2}\alpha\right) \tag{3.2}$$

$$(\alpha)_{2n} = 2^{2n}\left(\frac{1}{2}\alpha\right)_n \left(\frac{1}{2}\alpha+\frac{1}{2}\right)_n \tag{3.3}$$

$$(2n)! \;=\; 2^{2n}\left(\frac{1}{2}\right)_n n! \quad and \quad (2n+1)! \;=\; 2^{2n}\left(\frac{3}{2}\right)_n n!. \tag{3.4}$$

(i) Setting $i=k=l=0$ in (2.1) and (2.2) and using the results (3.1)–(3.4), we get the following integrals:

$$\frac{1}{\Gamma(a)}\int_0^{\infty} e^{-s}s^{a-1}\Psi_2(b;b',b';xs,-xs)\,{}_1F_1(c;c';ys)\,{}_1F_1(c;c';-ys)ds$$

$$= F^{\,2:2;2}_{\,0:3;3}\left[\begin{array}{ccccccc} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} & ; & c, c'-c & ; & \\ & & & & & & -4x^2, y^2 \\ - & : & b', \frac{1}{2}b', \frac{1}{2}b'+\frac{1}{2} & ; & c', \frac{1}{2}c', \frac{1}{2}c'+\frac{1}{2} & ; & \end{array}\right] \tag{3.5}$$

and

$$\frac{1}{\Gamma(a)}\int_0^{\infty} e^{-s}s^{a-1}\Phi_2(b,b;b';xs,-xs)\,{}_1F_1(c;c';ys)\,{}_1F_1(c;c';-ys)ds$$

$$= F^{\,2:1;2}_{\,0:2;3}\left[\begin{array}{ccccccc} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & : & b & ; & c, c'-c & ; & \\ & & & & & & x^2, y^2 \\ - & : & \frac{1}{2}b', \frac{1}{2}b'+\frac{1}{2} & ; & c', \frac{1}{2}c', \frac{1}{2}c'+\frac{1}{2} & ; & \end{array}\right]. \tag{3.6}$$

(ii) Setting $i=k=0,\ l=1$ in (2.1) and (2.2) and using the results (3.1)–(3.4), we get the following integrals:

$$\frac{1}{\Gamma(a)}\int_0^{\infty} e^{-s}s^{a-1}\Psi_2(b;b',b';xs,-xs)\,{}_1F_1(c;c';ys)\,{}_1F_1(c;c'+1;-ys)ds$$

$$= F^{\,2:2;2}_{\,0:3;3}\left[\begin{array}{ccccccc} \frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & : & \frac{1}{2}b, \frac{1}{2}b+\frac{1}{2} & ; & c, c'-c+1 & ; & \\ & & & & & & -4x^2, y^2 \\ - & : & b', \frac{1}{2}b', \frac{1}{2}b'+\frac{1}{2} & ; & c', \frac{1}{2}c'+\frac{1}{2}, \frac{1}{2}c'+1 & ; & \end{array}\right]$$

$$+\frac{acy}{c'(c'+1)}F\begin{array}{c}2:2;2\\0:3;3\end{array}\left[\begin{array}{ccccc}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1 & : & \frac{1}{2}b,\frac{1}{2}b+\frac{1}{2} & ; & c+1,c'-c+1 \; ; \\ & & & & & -4x^2,y^2\\ - & : & b',\frac{1}{2}b',\frac{1}{2}b'+\frac{1}{2} & ; & c'+1,\frac{1}{2}c'+1,\frac{1}{2}c'+\frac{3}{2} \; ;\end{array}\right] \tag{3.7}$$

and

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Phi_2(b,b;b';xs,-xs)_1F_1(c;c';ys)_1F_1(c;c'+1;-ys)ds$$

$$=F\begin{array}{c}2:1;2\\0:2;3\end{array}\left[\begin{array}{ccccc}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2} & : & b & ; & c,c'-c+1 \; ; \\ & & & & & x^2,y^2\\ - & : & \frac{1}{2}b',\frac{1}{2}b'+\frac{1}{2} & ; & c',\frac{1}{2}c'+\frac{1}{2},\frac{1}{2}c'+1 \; ;\end{array}\right]$$

$$+\frac{aby}{c'(c'+1)}F\begin{array}{c}2:1;2\\0:2;3\end{array}\left[\begin{array}{ccccc}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1 & : & b & ; & c+1,c'-c+1 \; ; \\ & & & & & x^2,y^2\\ - & : & \frac{1}{2}b',\frac{1}{2}b'+\frac{1}{2} & ; & c'+1,\frac{1}{2}c'+1,\frac{1}{2}c'+\frac{3}{2} \; ;\end{array}\right]. \tag{3.8}$$

(iii) Setting $i=k=1,\ l=0$ in (2.1) and (2.2) and using the results (3.1)–(3.4), we get the following integrals:

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Psi_2(b;b',b'+1;xs,-xs)_1F_1(c-1;c';ys)_1F_1(c;c'+1;-ys)ds$$

$$=F\begin{array}{c}2:2;2\\0:3;3\end{array}\left[\begin{array}{ccccc}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2} & : & \frac{1}{2}b,\frac{1}{2}b+\frac{1}{2} & ; & c,c'-c+1 \; ; \\ & & & & & -4x^2,y^2\\ - & : & b',\frac{1}{2}b'+\frac{1}{2},\frac{1}{2}b'+1 & ; & c',\frac{1}{2}c'+\frac{1}{2},\frac{1}{2}c'+1 \; ;\end{array}\right]$$

$$+\frac{abx}{b'(b'+1)}F\begin{array}{c}2:2;2\\0:3;3\end{array}\left[\begin{array}{ccccc}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1 & : & \frac{1}{2}b+\frac{1}{2},\frac{1}{2}b+1 & ; & c,c'-c+1 \; ; \\ & & & & & -4x^2,y^2\\ - & : & b'+1,\frac{1}{2}b'+1,\frac{1}{2}b'+\frac{3}{2} & ; & c',\frac{1}{2}c'+\frac{1}{2},\frac{1}{2}c'+1 \; ;\end{array}\right]$$

$$+\frac{a(c-c'-1)y}{c'(c'+1)}F\begin{array}{c}2:2;2\\0:3;3\end{array}\left[\begin{array}{ccccc}\frac{1}{2}a+\frac{1}{2},\frac{1}{2}a+1 & : & \frac{1}{2}b,\frac{1}{2}b+\frac{1}{2} & ; & c,c'-c+2 \; ; \\ & & & & & -4x^2,y^2\\ - & : & b',\frac{1}{2}b'+\frac{1}{2},\frac{1}{2}b'+1 & ; & c'+1,\frac{1}{2}c'+1,\frac{1}{2}c'+\frac{3}{2} \; ;\end{array}\right]$$

$$+\frac{ab(a+1)(c-c'-1)xy}{b'c'(b'+1)(c'+1)}$$

$$\times F\begin{array}{c}2:2;2\\0:3;3\end{array}\left[\begin{array}{ccccc}\frac{1}{2}a+1,\frac{1}{2}a+\frac{3}{2} & : & \frac{1}{2}b+\frac{1}{2},\frac{1}{2}b+1 & ; & c,c'-c+2 \; ; \\ & & & & & -4x^2,y^2\\ - & : & b'+1,\frac{1}{2}b'+1,\frac{1}{2}b'+\frac{3}{2} & ; & c'+1,\frac{1}{2}c'+1,\frac{1}{2}c'+\frac{3}{2} \; ;\end{array}\right] \tag{3.9}$$

and

$$\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\Phi_2(b-1,b;b';xs,-xs)_1F_1(c-1;c';ys)_1F_1(c;c'+1;-ys)ds$$

$$
\begin{aligned}
&= F\begin{smallmatrix}2:1;2\\0:2;3\end{smallmatrix}\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2} & : & b & ; & c, c'-c+1 & ; \\ & & & & & & x^2, y^2 \\ - & : & \frac{1}{2}b', \frac{1}{2}b'+\frac{1}{2} & ; & c', \frac{1}{2}c'+\frac{1}{2}, \frac{1}{2}c'+1 & ;\end{matrix}\right]\\
&-\frac{ax}{b'}F\begin{smallmatrix}2:1;2\\0:2;3\end{smallmatrix}\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 & : & b & ; & c, c'-c+1 & ; \\ & & & & & & x^2, y^2 \\ - & : & \frac{1}{2}b'+\frac{1}{2}, \frac{1}{2}b'+1 & ; & c', \frac{1}{2}c'+\frac{1}{2}, \frac{1}{2}c'+1 & ;\end{matrix}\right]\\
&-\frac{a(c'-c+1)y}{c'(c'+1)}F\begin{smallmatrix}2:1;2\\0:2;3\end{smallmatrix}\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2}, \frac{1}{2}a+1 & : & b & ; & c, c'-c+2 & ; \\ & & & & & & x^2, y^2 \\ - & : & \frac{1}{2}b', \frac{1}{2}b'+\frac{1}{2} & ; & c'+1, \frac{1}{2}c'+1, \frac{1}{2}c'+\frac{3}{2} & ;\end{matrix}\right]\\
&+\frac{a(a+1)(c'-c+1)xy}{b'c'(c'+1)}F\begin{smallmatrix}2:1;2\\0:2;3\end{smallmatrix}\left[\begin{matrix}\frac{1}{2}a+1, \frac{1}{2}a+\frac{3}{2} & : & b & ; & c, c'-c+2 & ; \\ & & & & & & x^2, y^2 \\ - & : & \frac{1}{2}b'+\frac{1}{2}, \frac{1}{2}b'+1 & ; & c'+1, \frac{1}{2}c'+1, \frac{1}{2}c'+\frac{3}{2} & ;\end{matrix}\right].
\end{aligned}
\tag{3.10}
$$

The other special cases of the integrals (2.1) and (2.2) can also be obtained in the similar manner.

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