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Local Convergence of a multi-point family of high order methods in Banach spaces under Hölder continuous derivative

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ABSTRACT. We present a local convergence analysis for a multi-point family of high order methods in order to approximate a solution of a nonlinear equation in a Banach space setting. The convergence ball and error estimates are given for these methods under Hölder continuity conditions. Numerical examples are also provided in this study.

1 Introduction

In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

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Many problems in computational sciences and other disciplines can be brought in a form like (1.1) using mathematical modelling [2, 3, 4, 9, 15, 16]. The solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. In particular, the practice of Numerical Functional Analysis for finding solution x^* of equation (1.1) is essentially connected to variants of Newton's method. This method converges quadratically to x^* if the initial guess is close enough to the solution. Iterative methods of convergence order higher than two such as Chebyshev-Halley-type methods [1, 3, 4, 6]–[17] require the evaluation of the second Fréchet-derivative, which is very expensive in general. However, there are integral equations, where the second Fréchet-derivative is diagonal by blocks and inexpensive [8]–[11] or for quadratic equations the second Fréchet-derivative is constant [9, 10, 14]. Moreover, in some applications involving stiff systems [2], [4] high order methods are usefull. However, in general the use of the second Fréchet-derivative restricts the use of these methods as their informational efficiency is less than or equal to unity. That is why we study the local convergenve of multi-point methods defined for each $n = 0, 1, 2, \cdots$ by

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n), \\
z_n &= x_n + \theta(y_n - x_n), \quad \theta \in (0,2) \\
H_n &= H(x_n, y_n) = \frac{1}{\theta}F'(x_n)^{-1}[F'(z_n) - F'(x_n)] \\
Q_n &= Q(x_n, y_n) = -\frac{1}{2}H_n(I + \frac{1}{2}H_n)^{-1}, \\
x_{n+1} &= y_n + Q_n(y_n - x_n),
\end{aligned}
\tag{1.2}
$$

where x_0 is an initial point, I the identity operator and $F'(x)$ denotes the Fréchet derivative of the operator F. There is a plethora of semi-local convergence results for these methods under conditions $(\mathcal{C})$ [1]–[18]:

($\mathcal{C}_1$) $F : D \to Y$ is twice Fréchet-differentiable and $F'(x_0)^{-1} \in L(Y, X)$ for some $x_0 \in D$ such that

$$\|F'(x_0)^{-1}\| \le \beta;$$

($\mathcal{C}_2$)

$$\|F'(x_0)^{-1}F(x_0)\| \le \eta;$$

($\mathcal{C}_3$)

$$\|F''(x)\| \le \beta_1 \quad \text{for each} \quad x \in D;$$

($\mathcal{C}_4$)

$$\|F''(x) - F''(y)\| \le \beta_2\|x - y\|^p \quad \text{for each} \quad x, y \in D \quad \text{and some} \quad p \in (0, 1].$$

In particular, Parida and Gupta [18] provided a semilocal convergence analysis of method (1.2) but for $\theta \in (0, 1]$. If $p = 1$ method (1.2) is shown to be of order two [9] and if $p \in (0, 1)$ the order of method is $2 + p$ [18]. Conditions $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$ restrict the applicability of these methods. In our study we assume the conditions$(\mathcal{A})$:

($\mathcal{A}_1$) $F : D \to Y$ is Fréchet-differentiable and there exists $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in L(Y, X)$;

($\mathcal{A}_2$)

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|^p \text{ for each } x \in D \text{ and some } p \in (0, 1]$$

and

($\mathcal{A}_3$)

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|^p \text{ for each } x, y \in D \text{ and some } p \in (0, 1].$$

The convergence ball of method (1.2) and the error estimates on the distances $\|x_n - x^*\|$ are given in this paper.

The paper is organized as follows: In Section 2 we present the local convergence of the method (1.2). The numerical examples are given in the concluding Section 3.

In the rest of this study, $U(w, q)$ and $\overline{U}(w, q)$ stand, respectively, for the open and closed ball in X with center $w \in X$ and of radius $q > 0$.

2 Local convergence

We present the local convergence of method (1.2) in this section. It is convenient for the local convergence of method (1.2) to introduce some parameters and functions.

Define parameters r_0, r_A, r_1, r_2 and r_3 by

$$r_0 = \left(\frac{1}{L_0}\right)^{\frac{1}{p}}, \tag{2.1}$$

$$r_A = \left(\frac{1+p}{(1+p)L_0 + L}\right)^{\frac{1}{p}}, \tag{2.2}$$

$$r_1 = \left(\frac{(1+p)(1-|1-\theta|)}{L|\theta| + (1-|1-\theta|)(1+p)L_0}\right)^{\frac{1}{p}}, \tag{2.3}$$

$$r_2 = \left(\frac{|\theta|}{2^{p-1}L} + |\theta|L_0\right)^{\frac{1}{p}} \tag{2.4}$$

and

$$r_3 = \min\{r_1, r_2\}. \tag{2.5}$$

Notice tha $r_A < r_0,\ r_1 < r_0,\ r_2 < r_0$ and $r_3 < r_0$. Define function f on $[0, r_3)$. by

$$f(t) = \frac{Lt^p}{(1+p)(1-L_0t^p)} + \frac{2^{p-1}Lt^p}{\theta - (L_0\theta + L2^{p-1})t^p}[1 + \frac{Lt^p}{(1+p)(1-L_0t^p)}] - 1. \tag{2.6}$$

Notice that function f is continuous on interval $[0, r_3)$ and $f(t) \to \infty$ as $t \to r_2$. We also have that $f(0) = -1 < 0$. Hence, it follows from the intermediate value theorem that function f has zeros in $[0, r_3)$. Denote by r_s the smallest such zero. If

$$r \in [0, r_s), \tag{2.7}$$

then

$$f(r) < 1. \tag{2.8}$$

Then, we can show the following local convergence result for method (1.2) under the ($\mathcal{A}$) conditions

THEOREM 2.1. *Suppose that the ($\mathcal{A}$) conditions and $\overline{U}(x^*,r) \subseteq D$, hold, where r is given by (2.7). Then, sequence $\{x_n\}$ generated by method (1.2) for some $x_0 \in U(x^*,r)$ is well defined, remains in $U(x^*,r)$ for each $n = 0,1,2,\cdots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0,1,2,\cdots$.*

$$\|x_{n+1} - x^*\| \leq f(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|. \tag{2.9}$$

Proof. We shall use induction to show that estimate (2.9) hold and $y_n, z_n, x_{n+1} \in U(x^*,r)$ for each $n = 0,1,2,\cdots$. Using ($\mathcal{A}_2$) and the hypothesis $x_0 \in U(x^*,r)$ we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\|^p < L_0 r^p < 1, \tag{2.10}$$

by the choice of r. It follows from (2.10) and the Banach Lemma on invertible operators [2, 5, 15] that $F'(x_0)^{-1} \in L(Y,X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|^p} < \frac{1}{1 - L_0 r^p}. \tag{2.11}$$

Using ($\mathcal{A}_2$), ($\mathcal{A}_3$), $F(x^*) = 0$, (2.10) and the choice of r_A we get from the first substep of method (1.2)

$$\begin{aligned} y_0 - x^* &= x_0 - F'(x_0)^{-1}F(x_0) \\ &= -[F'(x_0)^{-1}F'(x^*)][F'(x^*)^{-1} \\ &\quad \int_0^1 (F'(x^* + \tau(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\tau] \end{aligned} \tag{2.12}$$

that

$$\begin{aligned} \|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1} \\ &\quad \int_0^1 [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)]d\theta\|\|x_0 - x^*\| \\ &\leq \frac{1}{1 - L_0\|x_0 - x^*\|^p}\frac{L_0\|x_0 - x^*\|^p}{1+p}\|x_0 - x^*\| \\ &= \frac{L\|x_0 - x^*\|^{1+p}}{(1+p)(1 - L_0\|x_0 - x^*\|^p)} \\ &\leq \frac{Lr^p\|x_0 - x^*\|}{(1+p)(1 - L_0 r^p)} \leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.13}$$

which shows that $y_0 \in U(x^*,r)$. In view of the second substep of method (1.2) and (2.13) we get that

$$\begin{aligned} z_0 - x^* &= x_0 - x^* + \theta((y_0 - x^*) + (x^* - x_0)) \\ &= (1-\theta)(x_0 - x^*) + \theta(y_0 - x^*), \end{aligned} \tag{2.14}$$

so

$$\begin{aligned} \|z_0 - x^*\| &\leq |1-\theta|\|x_0 - x^*\| + \theta\|y_0 - x^*\| \\ &\leq |1-\theta|\|x_0 - x^*\| + \theta\frac{L\|x_0 - x^*\|^{1+p}}{(1+p)(1 - L_0\|x_0 - x^*\|^p)} \\ &\leq [|1-\theta| + \frac{L\theta r^p}{(1+p)(1 - L_0 r^p)}]\|x_0 - x^*\| \\ &\leq \|x_0 - x^*\| < r, \end{aligned} \tag{2.15}$$

which implies that $z_0 \in U(x^*, r)$. Hence, H_0 is well defined. Next, we shall show that $(I + \frac{1}{2}H_0)^{-1}$ exists. Using $(\mathcal{A}_3)$, (2.11) and the choice of r_2 we get in turn that

$$\begin{aligned} \|\frac{1}{2}H_0\| &\leq \frac{1}{2\theta}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}(F'(z_0) - F'(x_0))\| \\ &\leq \frac{1}{2\theta}\frac{L\|z_0 - x_0\|^p}{1 - L_0\|x_0 - x^*\|^p} \\ &\leq \frac{L(\|z_0 - x^*\| + \|x_0 - x^*\|)^p}{2\theta(1 - L_0\|x_0 - x^*\|^p)} \\ &\leq \frac{2^p L r^p}{2\theta(1 - L_0 r^p)} = \frac{2^{p-1}Lr^p}{\theta(1 - L_0 r^p)} < 1. \end{aligned} \tag{2.16}$$

It follows from (2.16) and the Banach lemma that $(I + \frac{1}{2}H_0)^{-1}$ exists and

$$\begin{aligned} \|(I + \frac{1}{2}H_0)^{-1}\| &\leq \frac{1}{1 - \frac{L\|x_n - z_n\|^p}{2\theta(1 - L_0\|x_n - x^*\|^p)}} \\ &\leq \frac{1}{1 - \frac{2^{p-1}Lr^p}{\theta(1 - L_0 r^p)}}. \end{aligned} \tag{2.17}$$

Hence, x_1 is well defined. We shall show that (2.9) holds for $n = 0$ and $x_1 \in U(x^*, r)$. Using the last substep of method (1.2) for $n = 0$, (2.11), (2.13), (2.15), (2.16), (2.17) and (2.8) we obtain in turn that

$$\begin{aligned} \|x_1 - x^*\| &\leq \|y_0 - x^*\| + \|Q_0\|(\|y_0 - x^*\| + \|x_0 - x^*\|) \\ &\leq [\frac{L\|x_0 - x^*\|^p}{(1+p)(1 - L_0\|x_0 - x^*\|^p)} + \frac{L\|x_0 - x^*\|^p}{2\theta(1 - L_0\|x_0 - x^*\|^p) - L\|z_0 - x_0\|^p} \\ &\quad (1 + \frac{L\|x_0 - x^*\|^p}{(1+p)(1 - L_0\|x_0 - x^*\|^p)})]\|x_0 - x^*\| \\ &\leq f(\|x_0 - x^*\|)\|x_0 - x^*\| \leq f(r)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.18}$$

where, we also used the estimate

$$\begin{aligned} \|Q_0\| &\leq \frac{1}{2}\|H_0\|\|(I + \frac{1}{2}H_0)^{-1}\| \\ &\leq \frac{L\|z_0 - x_0\|^p}{2\theta(1 - L_0\|x_0 - x^*\|^p)}\frac{1}{1 - \frac{L\|z_0 - x_0\|^p}{2\theta(1 - L_0\|x_0 - x^*\|^p)}} \\ &\leq \frac{L\|z_0 - x_0\|^p}{2\theta(1 - L_0\|x_0 - x^*\|^p) - L\|z_0 - x_0\|^p} \\ &\leq \frac{2^p L r^p}{2\theta - (2\theta L_0 + 2^p L)r^p} \\ &\leq \frac{2^{p-1}Lr^p}{\theta - (\theta L_0 + 2^p L)r^p}. \end{aligned}$$

It then follows from (2.18) that (2.9) holds for $n = 0$ and $x_1 \in U(x^*, r)$. To complete the induction simply replace x_0, y_0, x_1, H_0, Q_0 by $x_k, y_k, x_{k+1}, H_k, Q_k$ in all the preceding estimates to arrive in particular at $\|x_{k+1} - x^*\| < \|x_k - x^*\| < r$, which imply that $x_{k+1} \in U(x^*, r)$ and that $\lim_{k\to\infty} x_k = x^*$. $\square$

REMARK 2.2. *(a) Condition $(\mathcal{A}_2)$ can be dropped, since this condition follows from $(\mathcal{A}_3)$. Notice, however that*

$$L_0 \leq L \tag{2.19}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [2]–[6].

(b) *It is worth noticing that it follows from (2.2) and (2.7) that r is such that*

$$r < r_A \tag{2.20}$$

The convergence ball of radius r_A was given by us in [2, 3, 5] for Newton's method under conditions $(\mathcal{A}_1)$- $(\mathcal{A}_3)$. Estimate (2.17) shows that the convergence ball of method (1.2) is smaller than the convergence ball of the quadratically convergent Newton's method.

(c) *The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2]– [5], [14, 15].*

(d) *The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [2]– [5], [14, 15]:*

$$F'(x) = T(F(x)),$$

where T is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $T(x) = x + 1$ and $x^* = 0$.

3 Numerical Examples

We present numerical examples where we compute the radii of the convergence balls.

EXAMPLE 3.1. *Let $X = Y = \mathbb{R}$. Define function F on $D = [1,3]$ by*

$$F(x) = \frac{2}{3}x^{\frac{3}{2}} - x. \tag{3.1}$$

Then, $x^* = \frac{9}{4} = 2.25$, $F'(x^*)^{-1} = 2$, $L_0 = 1 < L = 2$ and $p = 0.5$ Choose $\theta = 1$. Then, we get that $r \in [0, 0.1667)$ and $r_A = 0.5$.

EXAMPLE 3.2. *Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0,1)$. Define F on D for $v = x, y, z)$ by*

$$F(v) = (e^x - 1, \frac{e-1}{2}y^2 + y, z). \tag{3.2}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e - 1 < L = e$, and $p = 1$. Choose $\theta = 1$ then, we get that $r \in [0, 0.1175)$ and $r_A = 0.3249$.

EXAMPLE 3.3. *Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$be and equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function F on D by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\tau\varphi(\tau)^3 d\tau. \tag{3.3}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\tau\varphi(\tau)^2\xi(\tau)d\tau, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15$ *and* $p = 1$.

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