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Some weaker forms of Continuity in Bitopological ordered spaces

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ABSTRACT. The main purpose of the present paper is to introduce and study some weaker forms of continuity in bitopological ordered spaces .Such as pairwise I-continuous maps, pairwise D-continuous maps, pairwise B-continuous maps, pairwise I-open maps, pairwise D-open maps, pairwise B-open maps, pairwise I-closed maps, pairwise D-closed maps and pairwise B-closed maps.

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1 Introduction

Singal, M. K. and Singal, A. R. [9] initiated the study of bitopological ordered spaces. Raghavan, T. G. [7], [8] and other authors have contributed to development and construction some properties of such spaces (see,[1] ,[4],[3], [2], [5]). In (2002) M.K.R.S. Veera Kumar [10] introduced I-continuous maps, D-continuous maps and B-continuous maps, I-open maps, D-open maps, B-open maps, I-closed maps, D-closed maps and B-closed maps for topological ordered spaces together with their characterizations. Leopoldo Nachbin [6] initiated the study of topological ordered spaces in (1965). A topological ordered space is a triple $(X, \tau, \leq)$, whereτ is a topology on X and $\leq$ is a partial order on X. In this paper we introduce pairwise I-continuous maps, pairwise D-continuous maps and pairwise B-continuous maps, pairwise I-open maps, pairwise D-open maps, pairwise B-open maps,

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pairwise I-closed maps, pairwise D-closed maps and pairwise B-closed maps for bitopological ordered spaces together with their characterizations as a generalization of that were studied for topological ordered spaces by M.K.R.S. Veera Kumar [10].

2 Preliminaries

Let $(X, \leq)$ be a partially ordered set (i.e. a set X together with a reflexive, antisymmetric and transitive relation). For a subset $A \subseteq X$, we write:

$$L(A) = \{y \in X : y \leq x \text{ for some } x \in A\},$$
$$M(A) = \{y \in X : x \leq y \text{ for some } x \in A\}.$$

In particular, if A is a singleton set, say $\{x\}$, then we write $L(x)$ and $M(x)$ respectively. A subset A of X is said to be decreasing (resp. increasing) if $A = L(A)$ (resp. $A = M(A)$). The complement of a decreasing (resp. an increasing) set is an increasing (resp. a decreasing) set. A mapping $f : (X, \leq) \to (X^*, \leq^*)$ from a partially ordered set $(X, \leq)$ to a partially ordered set $(X^*, \leq^*)$ is increasing (resp. a decreasing) if $x \leq y$ in X implies $f(x) \leq^* f(y)$ (resp. $f(y) \leq^* f(x)$). f is called an order isomorphism if it is an increasing bijection such that f^{-1} is also increasing.

A bitopological ordered space [9] is a quadruple consisting of a bitopological space (X, τ_1, τ_2), and a partial order $\leq$ on X; it is denoted as $(X, \tau_1, \tau_2, \leq)$. The partial order $\leq$ said to be closed (resp. weakly closed) [7] if its graph $G(\leq) = \{(x, y) : x \leq y\}$ is closed in the product topology $\tau_i \times \tau_j$ (resp.$\tau_1 \times \tau_2$) where $i, j = 1, 2; i \neq j$, or equivalently, if $L(x)$ and $M(x)$ are τ_1-closed, where $i = 1, 2$ (resp. $L(x)$ is τ_1-closed and $M(x)$ is τ_2-closed), for each $x \in X$.

For a subset A of a bitopological ordered space $(X, \tau_1, \tau_2, \leq)$,

$H_i^l(A) = \bigcap\{F \mid F$ is τ_i-decreasing closed subset of X containing $A\}$,

$H_i^m(A) = \bigcap\{F \mid F$ is τ_i-increasing closed subset of X containing $A\}$,

$H_i^b(A) = \bigcap\{F \mid F$ is a closed subset of X containing A with $F = L(F) = M(F)\}$,

$O_i^l(A) = \bigcup\{G \mid G$ is τ_i-decreasing open subset of X contained in $A\}$,

$O_i^m(A) = \bigcup\{G \mid G$ is τ_i-increasing open subset of X contained in $A\}$,

$O_i^b(A) = \bigcup\{G \mid G$ is both τ_i-increasing and τ_i-decreasing open subset of X contained in $A\}$.

Clearly, $H_i^m(A)$ (resp. $H_i^l(A), H_i^b(A)$) is the smallest τ_i- increasing resp. τ_i- decreasing, both τ_i- increasing and τ_i- decreasing) closed set containing A. Moreover $\bar{A}_i \subseteq H_i^m(A) \subseteq H_i^b(A)$ and where $\bar{A}_i$ stands for the τ_i- closure of A in $(X, \tau_1, \tau_2, \leq), i = 1, 2$. Further A is τ_i-decreasing (resp. τ_i-increasing) closed if and only if $A = H_i^m(A) = H_i^l(A)$.

Clearly, $O_i^m(A)$ (resp. $O_i^l(A), O_i^b(A)$) is the largest τ_i-increasing resp. τ_i-decreasing, both τ_i-increasing and τ_i-decreasing) open set contained in A. Moreover $O_i^b(A) \subseteq O_i^m(A) \subseteq A_i^o$ and $O_i^b(A) \subseteq O_i^l(A)$, where A_i^o denotes the τ_i-interior of A in $(X, \tau_1, \tau_2, \leq), i \neq j$. If A and B are two τ_1 subsets of a bitopological ordered space $(X, \tau_1, \tau_2, \leq), i \neq j$ such that $A \subseteq B$, then $O_i^m(A) \subseteq O_i^m(B) \subseteq B_i^o$. $\Omega(O_i^m(X))$ resp. $\Omega(O_i^l(X)), \Omega(O_i^b(X))$ denotes the collection of all τ_i-increasing (resp. τ_i-decreasing, both τ_i-increasing and τ_i-decreasing) open subset of a bitopological ordered space $(X, \tau_1, \tau_2, \leq)$.

3 Pairwise I-continuous, Pairwise D-continuous and Pairwise B-continuous maps

Definition 3.1. *A function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$ is called a pairwise $I-$continuous (resp. a pairwise $D-$continuous, a pairwise $B-$continuous) map if $f^{-1}(G) \in \Omega(O_i^m(X))$ (resp. $f^{-1}(G) \in \Omega(O_i^l(X)), f^{-1}(G) \in \Omega(O_i^b(X))$), whenever G is a $i-$open subset of $(X^*, \tau_1^*, \tau_2^*, \leq)$, i= 1, 2.*

It is evident that every pairwise $x-$continuous map is pairwise continuous for $x = I, D, B$ and that every pairwise $B-$continuous map is both pairwise $I-$continuous and pairwise $D-$continuous.

Example 3.2. *Let $X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \tau_2 = \{\emptyset, X, \{c\}\}$ and $\leq = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$. Clearly $(X, \tau_1, \tau_2, \leq)$ is a bitopological ordered space. Let f be the identity map from $(X, \tau_1, \tau_2, \leq)$ onto itself. $\{b\}$ is τ_1-open and $\{c\}$ is τ_2-open but $f^{-1}(\{b\}) = \{b\}$ is neither a τ_1-increasing nor a τ_1-decreasing open set and also $f^{-1}(\{c\}) = \{c\}$ is neither a τ_2-increasing nor a τ_2-decreasing open set and . Thus f is not pairwise x-continuous for $x = I, D, B$. However f is continuous.*

The following Example supports that a pairwise $D-$continuous map need not be a pairwise $B-$continuous map.

Example 3.3. *Let $X = \{a, b, c\} = X^*, \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} = \tau_1^*, \tau_2 = \{\emptyset, X, \{c\}\} = \tau_2^*$ and $\leq = \{(a, a), (b, b), (c, c), (a, c)\}$ and $\leq^* = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}$. Let g be the identity map from $(X, \tau_1, \tau_2, \leq)$ onto $(X^*, \tau_1^*, \tau_2^*, \leq)$. g is not pairwise $B-$continuous. However g is a pairwise $D-$continuous map.*

The following Example supports that a pairwise $I-$continuous map need not be a pairwise $B-$continuous map.

Example 3.4. *Let $X = \{a, b, c\} = X^*, \tau_1 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \tau_1^* = \{\emptyset, X^*, \{a\}\}, \tau_2 = \{\emptyset, X, \{c\}\}, \tau_2^* = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$ and $\leq = \{(a, a), (b, b), (c, c), (a, b), (a, c), (c, b)\} = \leq^*$. Define $h : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$ by $h(a) = b, h(b) = a$ and $h(c) = c$. h is pairwise $I-$continuous but not a pairwise $B-$continuous map.*

Thus we have the following diagram:

For a function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$

, where $P \to Q$(resp. $P \not\leftrightarrow Q$) represents P implies Q but$Q$ need not imply P(resp. P and Q are independent of

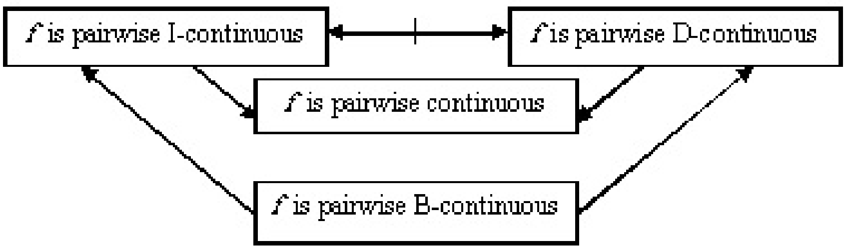

Figure 1:

each other).

The following Theorem characterizes pairwise $I-$continuous maps.

Theorem 3.5. *For a function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, the following statements are equivalent:*

(1) f is pairwise $I-$continuous.

(2) $f(H_i^l(A)) \subseteq \overline{(f(A))}_i$ for any $A \subseteq X, i = 1,2$.

(3) $H_i^l(f^{-1}(B) \subseteq f^{-1}(\bar{B})i$ for any $B \subseteq X^*, i = 1,2$.

(4) For every τ_i^*-closed subset K of $(X^*, \tau_1^*, \tau_2^*, \leq)$, $f^{-1}(K)$ is a τ_i-decreasing closed subset of $(X, \tau_1, \tau_2, \leq), i = 1,2$.

Proof. (1) $\Rightarrow$ (2): Since $X^* \setminus \overline{(f(A))}_i$ is τ_i-open in X^* and f is pairwise $I-$continuous, then $f^{-1}(X \setminus \overline{(f(A))}_i)$ is a τ_i-increasing open set in X. Then $X \setminus f^{-1}(X \setminus \overline{(f(A))}_i)$ is a τ_i-decreasing closed subset of X. Since $X \setminus f^{-1}(X \setminus \overline{(f(A))}_i) = f^{-1}(\overline{(f(A))}_i)$, then $f^{-1}(\overline{(f(A))}_i)$ is a τ_i-decreasing closed subset of X. Since $A \subseteq f^{-1}(\overline{(f(A))}_i)$ and is the smallest τ_i-decreasing closed set containing A, then $H_i^l(A) \subseteq f^{-1}(\overline{(f(A))}_i).f(f^{-1}(\overline{(f(A))}_i)$
$\subseteq \overline{(f(A))}_i$. Thus $H_i^l(A) \subseteq \overline{(f(A))}_i$.

(2) $\Rightarrow$ (3): Let $A = f^{-1}(B)$. Then $f(A) = f(f^{-1}(B)) \subseteq B$. This implies $(\overline{f(A)})_i \bar{B}_i$.
Now $H_i^l(f^{-1}(B)) \subseteq H_i^l(A) \subseteq f^{-1}(f(H_i^l(A))) \subseteq f^{-1}(\overline{f(A)})_i$ [By(2)in this theorem 3.5]. But $f^{-1}(\overline{f(A)})_i \subseteq f^{-1}(\bar{B}_i)$. Thus $H_i^l(f^{-1}(B)) \subseteq f^{-1}(\bar{B}_i)$.

(3) $\Rightarrow$ (4): $H_i^l(f^{-1}(K)) \subseteq f^{-1}(\bar{K}_i)$ for any τ_i^*-closed set K of $(X^*, \tau_1^*, \tau_2^*, \leq)$. Thus $f^{-1}(K)$ is a τ_i-decreasing closed in $(X, \tau_1, \tau_2, \leq)$ whenever K is a τ_i^*-closed set in $(X^*, \tau_1^*, \tau_2^*, \leq)$.

(4) $\Rightarrow$ (1): Let G be a τ_i^*-open set in $(X^*, \tau_1^*, \tau_2^*, \leq)$. Then $f^{-1}(X \setminus (G))$ is a τ_i-decreasing closed set in $(X, \tau_1, \tau_2, \leq)$, since $X^* \setminus (G)$ is a closed set in $(X^*, \tau_1^*, \tau_2^*, \leq)$. But $X \setminus (f^{-1}(G)) = f^{-1}(X \setminus G)$. Thus $X \setminus (f^{-1}(G))$ is a τ_i-decreasing closed set in $(X, \tau_1, \tau_2, \leq)$. So $f^{-1}(G)$ is a τ_i-increasing open set in $(X, \tau_1, \tau_2, \leq)$. Thus f is pairwise $I-$continuous. $\square$

The following two Theorems characterize pairwise $D-$continuous maps and pairwise $B-$continuous maps, whose proofs are similar to as that of the above Theorem 3.5.

Theorem 3.6. *For a function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, the following statements are equivalent:*

(1) f is pairwise $D-$continuous.

(2) $f(H_i^m(A)) \subseteq \overline{(f(A))}_i$ for any $A \subseteq X, i = 1, 2$.

(3) $H_i^m(f^{-1}(B) \subseteq f^{-1}(\bar{B})i$ for any $B \subseteq X^*, i = 1, 2$.

(4) For every τ_i^*-closed subset K of $(X^*, \tau_1^*, \tau_2^*, \leq)$, $f^{-1}(K)$ is a τ_i-increasing closed subset of $(X, \tau_1, \tau_2, \leq), i = 1, 2$.

Theorem 3.7. *For a function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, the following statements are equivalent:*

(1) f is pairwise $B-$continuous.

(2) $f(H_i^b(A)) \subseteq \overline{(f(A))}_i$ for any $A \subseteq X, i = 1, 2$.

(3) $H_i^b(f^{-1}(B) \subseteq f^{-1}(\bar{B})i$ for any $B \subseteq X^*, i = 1, 2$.

(4) For every τ_i^*-closed subset K of $(X^*, \tau_1^*, \tau_2^*, \leq)$, $f^{-1}(K)$ is both τ_i-increasing and τ_i-decreasing closed subset of $(X, \tau_1, \tau_2, \leq), i = 1, 2$.

Theorem 3.8. *Let $f : (X, \tau_1, \tau_2, \leq_1) \to (y, \nu_1, \nu_2, \leq_2)$ and $g : (y, \nu_1, \nu_2, \leq_2) \to (Z, \eta_1, \eta_2, \leq_3)$ be any two mappings. Then*

(1) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise $x-$continuous for $x = I, D, B$.

(2) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise $x-$continuous and g is pairwise continuous for $x = I, D, B$.

(3) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise $x-$continuous and g is pairwise $y-$continuouss for $x, y \in \{I, D, B\}$.

4 Pairwise I-open, Pairwise D-open and Pairwise B-open maps

Definition 4.1. *A function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ is called a pairwise $I-$open (resp. a pairwise $D-$open, a pairwise $B-$open) map if $f(G) \in \Omega(O_i^m(X^*))$ (resp. $f(G) \in \Omega(O_i^l(X^*)), f(G) \in \Omega(O_i^b(X^*))$) whenever G is a τ_i-open subset of $(X, \tau_1, \tau_2), i = 1, 2$.*

It is evident that every pairwise x-open map is a pairwise open map for $x = I, D, B$ and that every pairwise $B-$open map is both pairwise $I-$open and pairwise $D-$open.

The following Example shows that a pairwise open map need not be pairwise $x-$open for $x = I, D, B$.

Example 4.2. *Let $(X, \tau_1, \tau_2, \leq)$ and f be as in the Example 3.2. f is a pairwise open map but f is not pairwise $x-$open for $x = I, D, B$.*

The following Example shows that a pairwise $D-$open map need not be a pairwise $B-$open map.

Example 4.3. *Let $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq$ and $\leq^*$ be as in the Example 3.3. Let θ be the identity map from $(X, \tau_1, \tau_2, \leq)$ onto $(X^*, \tau_1^*, \tau_2^*, \leq^*)$. θ is pairwise $D-$open but not a pairwise $B-$open map.*

The following Example shows that a pairwise $I-$open map need not be a pairwise $B-$open map

Example 4.4. *Let $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq$ and $\leq^*$ be as in the Example 3.4. Define $\varphi : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$. by $\varphi(a) = b, \varphi(b) = a$ and $\varphi(c) = c$. φ is a pairwise $I-$open map but not a pairwise $B-$open map.*

Thus we have the following diagram:

For a function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$
,where $P \to Q$(resp. $P \nleftrightarrow Q$) represents P implies Q but$Q$ need not imply P(resp. P and Q are independent of

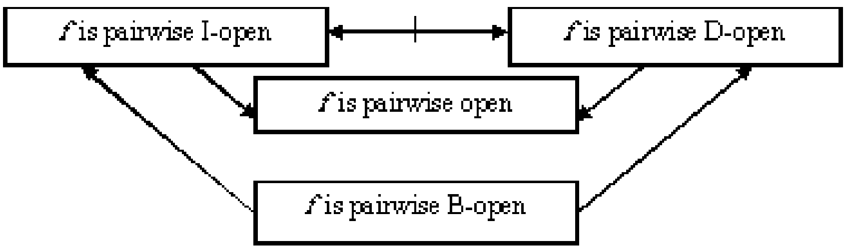

Figure 2:

each other).

Before characterizing pairwise $I-$open (resp. pairwise $D-$open, pairwise $B-$open) maps, we establish the following useful Lemma.

Lemma 4.5. *Let A be any subset of a bitopological ordered space $(X, \tau_1, \tau_2, \leq)$. Then*

(1) $X \setminus H_i^l(A) = O_i^m(X \setminus A), i = 1, 2$

(2) $X \setminus H_i^m(A) = O_i^l(X \setminus A), i = 1, 2$

(31) $X \setminus H_i^b(A) = O_i^b(X \setminus A), i = 1, 2$

Proof. (1) $X \setminus H_i^l(A) = X \setminus (\cap\{F|F$ is a τ_i-decreasing closed subset of X containing $A\}$ = $\bigcup\{X \setminus F|F$ is a τ_i-decreasing closed subset of X containing $A\}$ =$\bigcup\{G|G$ is an τ_i-increasing open subset of X contained in $X \setminus A\} = O_i^m(X \setminus A)$.

The proofs for (2) and (3) are analogous to that of (1) and so omitted. □

The following Theorem characterizes pairwise I-open functions.

Theorem 4.6. *For any function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq)$, the following statements are equivalent:*

(1) f is pairwise $I-$open map.

(2) $f((A_i^0)) \subseteq O_i^m(f(A)$ for any $A \subseteq X, i = 1, 2$.

(3) $(f^{-1}(B)_i^o \subseteq f^{-1}(O_i^m(B)$ for any $B \subseteq X^*, i = 1, 2$.

(4) $f^{-1}(H_i^l(B)) \subseteq H_i^l(f^{-1}(B))$ for any $B \subseteq X^*, i = 1, 2$.

Proof. (1) $\Rightarrow$ (3): Since $(f^{-1}(B))_i^o$ is τ_i-open in X and f is pairwise $I-$open, then $f((f^{-1}(B))_i^o)$ is an τ_i-increasing open set in X^*. Also $f(f^{-1}(B))_i^o) \subseteq f(f^{-1}(B)) \subseteq B$. Then $f(f^{-1}(B))_i^o \subseteq O_i^m(B)$ since $O_i^m(B)$ is the largest τ_i-increasing open set contained in B. Therefore $(f^{-1}(B))_i^0 \subseteq f^{-1}(O_i^m(B))$.

(3) $\Rightarrow$ (4): Replacing B by $X \setminus B$ in (3), we get $(f^{-1}(X \setminus B))_i^o \subseteq f^{-1}(O_i^m(X \setminus B))$. Since $f^{-1}(X \setminus B)) = X \setminus (f^{-1}(B))$, then $(X \setminus (f^{-1}(B)))_i^o \subseteq f^{-1}(O_i^m(X \setminus B))$. Now $X \setminus (H_i^l(f^{-1}(B))) = O_i^m(X \setminus (f^{-1}(B))) \subseteq (X \setminus$

$(f^{-1}(B)))^o_i \subseteq f^{-1}(O^m_i(X \setminus (B))) = f^{-1}(X \setminus (H^l_i(B))) = X \setminus (f^{-1}(H^l_i(B)))$ using the above Lemma 4.5. Therefore $f^{-1}(H^l_i(B))) \subseteq H^l_i(f^{-1}(B))$.

(4) $\Rightarrow$ (3): All the steps in (3) $\Rightarrow$ (4) are reversible.

(3) $\Rightarrow$ (2): Replacing B by $f(A)$in (3), we get $(f^{-1}(f(A)))^o_i \subseteq f^{-1}(O^m_i(f(A)))$. Since $A^o_i \subseteq (f^{-1}(f(A)))^o_i$, then we have $A^o_i \subseteq f^{-1}(O^m_i(f(A)))$.This implies that $f(A^o_i) \subseteq f(f^{-1}(O^m_i(f(A)))) \subseteq O^m_i(f(A))$. Hence $f(A^o_i) \subseteq O^m_i(f(A))$.

(2) $\Rightarrow$ (1): Let G be any τ_i-open subset of X. Then $f(G) = f(G^o_i) \subseteq O^m_i(f(G))$. So $f(G)$ is a τ^*_i-increasing open set in X^*. Therefore f is a pairwise $I-$open map. □

The following two Theorems give characterizations for $D-$open maps and $B-$open maps, whose proofs are similar to as that of the above Theorem 4.6.

Theorem 4.7. *For any function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau^*_1, \tau^*_2, \leq)$, the following statements are equivalent:*

(1) f is pairwise $D-$open map.

(2) $f((A^0_i)) \subseteq O^l_i(f(A)$ for any $A \subseteq X, i = 1, 2$.

(3) $(f^{-1}(B)^o_i \subseteq f^{-1}(O^l_i(B)$ for any $B \subseteq X^*, i = 1, 2$.

(4) $f^{-1}(H^m_i(B)) \subseteq H^m_i(f^{-1}(B))$ for any $B \subseteq X^*, i = 1, 2$.

Theorem 4.8. *For any function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau^*_1, \tau^*_2, \leq)$, the following statements are equivalent:*

(1) f is an $B-$open map.

(2) $f((A^0_i)) \subseteq O^b_i(f(A)$ for any $A \subseteq X, i = 1, 2$.

(3) $(f^{-1}(B)^o_i \subseteq f^{-1}(O^b_i(B)$ for any $B \subseteq X^*, i = 1, 2$.

(4) $f^{-1}(H^b_i(B)) \subseteq H^b_i(f^{-1}(B))$ for any $B \subseteq X^*, i = 1, 2$.

Theorem 4.9. *Let $f : (X, \tau_1, \tau_2, \leq_1) \to (y, \nu_1, \nu_2, \leq_2)$ and $g : (y, \nu_1, \nu_2, \leq_2) \to (Z, \eta_1, \eta_2, \leq_3)$ be any two mappings. Then*

(1) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise $x-$open if f is pairwise open and g is pairwise $x-$open for $x = I, D, B$.

(2) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise $x-$open if both f and g are pairwise $x-$open for $x = I, D, B$.

(3) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise x-open if f is pairwise y-open and g is pairwise x-open for $x, y \in \{I, D, B\}$.

Proof. Omitted. □

5 Pairwise I-closed, Pairwise D-closed and Pairwise B-closed maps

Definition 5.1. *A function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau^*_1, \tau^*_2, \leq^*)$ is called a pairwise $I-$closed (resp. a pairwise $D-$closed, a pairwise $B-$closed) map if $f(G) \in \Omega(H^m_i(X^*))$ (resp. $f(G) \in \Omega(H^l_i(X^*)), f(G) \in \Omega(H^b_i(X^*)))$ whenever G is a τ_i-open subset of (X, τ_1, τ_2), where $\Omega(H^m_i(X^*))$ (resp. COmega$(H^l_i(X^*)), \Omega(H^b_i(X^*))$ is the collection of all τ_i-increasing (resp. τ_i-decreasing, both τ_i-increasing and τ_i-decreasing) closed subsets of $(X^*, \tau^*_1, \tau^*_2, \leq^*), i = 1, 2$.*

Clearly every pairwise x-closed map is a pairwise closed map for $x = I, D, B$ and every pairwise B-closed map is both pairwise I-closed and pairwise D-closed. The following Example shows that a pairwise closed map need not be pairwise x-closed for $x = I, D, B$.

Example 5.2. *Let $(X, \tau_1, \tau_2, \leq)$ and f be as in the Example 3.2. f is a pairwise closed map but f is not pairwise $x-$ closed for $x = I, D, B$.*

The following Example shows that a pairwise I-closed map need not be a pairwise B-closed map.

Example 5.3. *Let $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq$ and $\leq^*$ be as in the Example 4.3. θ is pairwise $I-$closed but not a pairwise $B-$closed map.*

The following Example shows that a pairwise $I-$closed map need not be a pairwise $B-$closed map.

Example 5.4. *Let $X, X^*, \tau_1, \tau_2, \tau_1^*, \tau_2^*, \leq, \leq^*$ and φ be as in the Example 4.4. φ is a pairwise $D-$closed map but not a pairwise $B-$closed map.*

Thus we have the following diagram:

For a function $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$
,where $P \to Q$(resp. $P \not\leftrightarrow Q$) represents P implies Q but$Q$ need not imply P(resp. P and Q are independent of

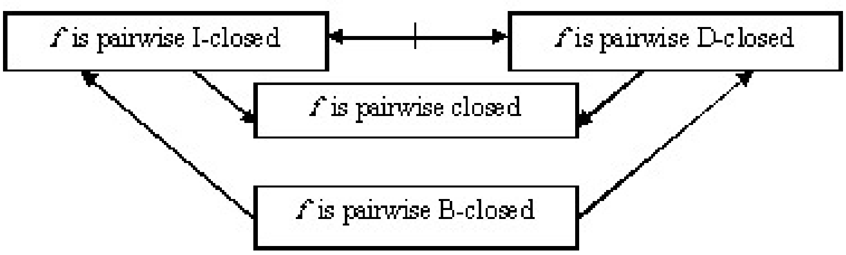

Figure 3:

each other).

The following Theorem characterizes $I-$closed maps.

Theorem 5.5. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be any map. Then f is pairwise $I-$closed if and only if $H_i^m(f(A)) \subseteq f(\bar{A}_i)$ for every $A \subseteq X, i = 1, 2$.*

Proof. Necessity: Since f is pairwise $I-$closed, then $f(\bar{A}_i)$ is a τ_i-increasing closed subset of X and $f(A) \subseteq f(\bar{A}_i)$. Therefore $H_i^m(f(A)) \subseteq f(\bar{A}_i))$ since $H_i^m(f(A))$ is the smallest τ_i-increasing closed set in X^* containing $f(A)$.

Sufficiency: Let F be any τ_i-closed subset of X. Then $f(F) \subseteq H_i^m(f(F)) \subseteq f(\bar{F}_i) = f(F)$. Thus $f(F) = H_i^m(f(F))$. So $f(F)$ is a τ_i-increasing closed subset of X^*. Therefore f is a pairwise $I-$closed map. □

The following two Theorems characterize pairwise $D-$closed maps and pairwise $B-$closed maps.

Theorem 5.6. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be any map. Then f is pairwise $D-$closed if and only if $H_i^l(f(A)) \subseteq f(\bar{A}_i)$ for every $A \subseteq X, i = 1, 2$.*

Proof. Omitted □

Theorem 5.7. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be any map. Then f is pairwise $B-$closed if and only if $H_i^b(f(A)) \subseteq f(\bar{A}_i)$ for every $A \subseteq X, i = 1, 2$.*

Proof. Omitted □

Theorem 5.8. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be a pairwise bijection map. Then*

(1) f is pairwise $I-$open if and only if f is pairwise $D-$closed.

(2) f is pairwise $I-$closed if and only if f is pairwise $D-$open.

(3) f is pairwise $B-$open if and only if f is pairwise $B-$closed.

Proof. (1) Necessity: Let F be any τ_i-closed subset of X. Then $f(X \setminus F)$ is a τ_i^*-increasing open subset of X^* since f is a pairwise $I-$open map and $(X\ F)$ is a τ_i-open subset of X. Since f is a pairwise bijection, then we have $f(X \setminus F) = X \setminus (f(F))$. So $f(F)$ is a τ_i^*-decreasing closed subset of X^*. Therefore f is pairwise $D-$closed.

Sufficiency: Let G be any τ_i-open subset of X. Then $f(X \setminus G)$ is a τ_i-decreasing closed subset of X^* since f is a pairwise $D-$closed map and $X \setminus G$ is a τ_i-closed subset of X. Since f is a pairwise bijection, then we have that $f(X \setminus G) = X \setminus f(G)$. So $f(G)$ is a τ_i-increasing open subset ofX^*. Therefore f is a pairwise $I-$open map. The proofs for (2) and (3) are similar to that of (1). □

Theorem 5.9. *Let $f : (X, \tau_1, \tau_2, \leq_1) \to (y, \nu_1, \nu_2, \leq_2)$ and $g : (y, \nu_1, \nu_2, \leq_2) \to (Z, \eta_1, \eta_2, \leq_3)$ be any two mappings. Then*

(1) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise $x-$closed if f is pairwise closed and g is pairwise $x-$closed for $x = I, D, B$.

(2) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise $x-$closed if both f and g are pairwise $x-$closed for $x = I, D, B$.

(3) $g \circ f : (X, \tau_1, \tau_2, \leq_1) \to (Z, \eta_1, \eta_2, \leq_3)$ is pairwise $x-$closed if f is pairwise $y-$closed and g is pairwise $x-$closed for $x, y \in \{I, D, B\}$.

Theorem 5.10. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be a pairwise bijection map. Then the following statements are equivalent:*

(1) f is a pairwise $I-$open map.

(2) f is a pairwise $D-$closed map.

(3) f^{-1} is a pairwise $I-$continuous.

Theorem 5.11. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be a pairwise bijection map. Then the following statements are equivalent:*

(1) f is a pairwise $D-$open map.

(2) f is a pairwise $I-$closed map.

(3) f^{-1} is a pairwise $D-$continuous.

Theorem 5.12. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be a pairwise bijection map. Then the following statements are equivalent:*

(1) f is a pairwise $B-$open map.

(2) f is a pairwise $B-$closed map.

(3) f^{-1} is a pairwise $B-$continuous.

Theorem 5.13. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be a pairwise $I-$closed map and $B, C \subseteq X^*$. Then*

(1) If U is an τ_i-open neighborhood of $f^{-1}(B)$, then there exists a τ_i-decreasing open neighborhood V of B such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U, i = 1, 2$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint τ_i-neighborhoods, then $f^{-1}(B)$ and $f^-1(C)$ have disjoint τ_i-decreasing open neighborhoods, $i = 1, 2$.

Proof. (1) Let U be a τ_i-open neighborhood of $f^{-1}(B)$. Take $X^* \setminus V = f(X \setminus U)$. Since f is a pairwise $I-$closed map and $X \setminus U$ is a τ_i-closed set, then $X^* \setminus V = f(X \setminus U)$ is a τ_i-increasing closed subset of X*. Thus V is a i-decreasing open subset of X*. Since f -1(B) U, then $X^* \setminus V = f(X \setminus U) \subseteq f(f^{-1}(X^* \setminus B)) \subseteq X^* \setminus B$. So $B \subseteq V$. Thus V is a $^{\tau*}_{i}-$ decreasing open neighborhood of B. Further $X \setminus U \subseteq f^{-1}(f(X \setminus U)) = f^{-1}(X^* \setminus V) = X^* \setminus (f^{-1}(V))$. Thus $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U$. □

Theorem 5.14. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be a pairwise $D-$closed map and $B, C \subseteq X^*$. Then*

(1) If U is an τ_i-open neighborhood of $f^{-1}(B)$, then there exists a τ_i-decreasing open neighborhood V of B such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U, i = 1, 2$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint τ_i-neighborhoods, then $f^{-1}(B)$ and $f^-1(C)$ have disjoint τ_i-increasing open neighborhoods, $i = 1, 2$.

Theorem 5.15. *Let $f : (X, \tau_1, \tau_2, \leq) \to (X^*, \tau_1^*, \tau_2^*, \leq^*)$ be a pairwise $B-$closed map and $B, C \subseteq X^*$. Then*

(1) If U is an τ_i-open neighborhood of $f^{-1}(B)$, then there exists a τ_i-open neighborhood V of B which are both τ_i- increasing and τ_i-decreasing. such that $f^{-1}(B) \subseteq f^{-1}(V) \subseteq U, i = 1, 2$.

(2) If $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint τ_i-neighborhoods, then $f^{-1}(B)$ and $f^-1(C)$ have disjoint τ_i- open neighborhoods which are both τ_i- increasing and τ_i-decreasing, $i = 1, 2$.

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