International Journal of Advances in Mathematics
Volume 2018, Number 2, Pages 17-41, 2018
eISSN 2456-6098

Eco-epidemiological prey-predator model for limited growth prey species and susceptible-infected species

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ABSTRACT. Presented dynamical behavior of a prey–predator system where both prey and predator populations are affected by diseases with susceptible–infected. We also analysis the system of equilibrium point and stability analysis. A system of four differential equation susceptible–infected prey species and predator species has been proposed and analyzed. Computer simulations are carried out to illustrate our analytical findings. Discussed, in population ecology, in particular, the predator–prey interaction in presence of an eco–epidemiological system of the biological implications of analytical and numerical findings.

1. Introduction

Many researches were carried out to observe the dynamics of a system under infected–susceptible in the prey–predator population only. Modifying the prey–predator model with disease in both populations. Observed commonly in the ecological species with a disease in the prey–predator population. But in recent times, a study of harvesting in a predator–prey model with disease in both populations [6]. Before presenting harvesting as a disease control measure in an eco–epidemiological system, it is well known that the predator–prey interaction. A study on example in Saltan sea of California, fish populations are infected by a Vibrio class of bacteria, Vibrio alginolyticus, which spreads from one infected fish to another susceptible fish, causing a large number of fishes to death [15]. The study of prey–predator effect of dynamics with an susceptible–infected prey–predator has a great importance in ecological prey–predator population. An infected place is one where germs or bacteria are causing a disease to spread among people or animals. But this area has been neglected for a long time in theoretical ecology. Recently a few researchers have cultured some prey–predator models for disease with susceptible–infected

* Corresponding Author.
Received September 09, 2017; revised February 03, 2018; accepted February 11, 2018.
2010 Mathematics Subject Classification: 35Q30, 34A34, 22E60,76D05, 76D06, 76D09, 76D55.
Key words and phrases: Prey predator model; Susceptible-infected; equilibrium point; Stability analysis.

[3]. Ecological populations suffer from various diseases. The effect of disease in ecological system is an important area from mathematical model[6]. So, in recent time ecologists and researchers are paying more and more affection to the development of important tools along with experimental ecology and describe how ecological species. The literature abounds with such evidences, in the last few decades, mathematical models have become extremely important tools in understanding and analyzing this spread and susceptible–infected control of infectious diseases [7]. They established that ecological population suffer from various diseases [1], [2] [14]. The importance of viruses for marine and especially phytoplankton ecology has been acknowledged in several recent publications using predation may defeat spatial spread of infection [8]. A prey–predator model with infection in both prey and predator for numerous models have been cultured by various researchers [10]. The dynamics in a harvested prey–predator model with susceptible–infected –susceptible epidemic diseases in the prey both the theoretical ecology and the epidemiology are developed research field and treated separately [12]. An epidemic model includes the property of population growth, the spread rules of infectious diseases and the related ecological factors to construct mathematical model reflecting the dynamic properties of infections diseases for effects of additional food in a susceptible exposed infected prey–predator model has been formulated [9]. Before presenting another extension involves treating the system with stage structure which is compartmentalizing a species class into mature and immature class. Many researchers study the prey and predator models where there are stage structures. [5]. The Leslie –Grower model with type II functional response is represented by switching from simple to complex dynamics in a predator–prey parasite model. For an interplay between infection rate and incubation delay [11]. Eco–epidemiology is a branch in mathematical biology which considers both the ecological and epidemiological issues simultaneously [13]. It is well known that the prey–predator harvesting as a disease control measure in an eco–epidemiological system. Some examples for the role of environmental disturbance in an eco–epidemiological model with diseases from external sources [8]. Viral, bacterial and parasitic microbes from recent times are harvesting an important issues in the predator–prey system where both species are infected by some transmissible diseases. A study of harvesting in a predator–prey model with disease in both populations [10]. As follows the next section prepared the mathematical model.

2. Method for selection of parameter values

In this section, we have mathematical model assumption. We make the following assumptions to formulate the mathematical model :

(i) We have considered "eco–epidemiology" a prey–predator ecosystem. It is assumed that both the prey and the predator are susceptible to some transmissible disease like viral disease and in the presence of disease.

(ii) We have considered an eco–epidemiology, a prey–predator populations are divided into two classes (1)Susceptible prey (W) and infected prey (X). (2) Susceptible predator (Y) and infected predator (Z).

(iii) We have assumed that the prey species is a commutative species and susceptible prey (W) is capable of reproducing with intra–specific competition coefficient a_1 and intrinsic growth rate r_1. Infected prey (X) is capable of reproducing with intra–specific competition coefficient a_2 and intrinsic growth rate r_2.

$$\frac{dW}{dt} = W\,(r_1 - a_1 W)\,, \frac{dX}{dt} = X\,(r_2 - a_2 X)$$

Assumption of the model of the prey–predator for susceptible–infected species are

$$\frac{dW}{dt} = W\left(r_1 - a_1 W - \frac{c_1\,Y}{mW+1} - \alpha\,X\right)$$
$$\frac{dX}{dt} = X\,(r_2 - a_2 X - c_2 Z + \alpha\,W)$$
$$\frac{dY}{dt} = Y\left(\frac{c_3 W}{mW+1} + a_3 X - d_1 - \beta\,Z\right)$$
$$\frac{dZ}{dt} = Z\left(\frac{c_4 W}{mW+1} + a_4 X - d_2 + \beta\,Y\right) \tag{2.1}$$

Let r_1, r_2 are the intrinsic growth rate of the susceptible prey and infected prey respectively. Let $c_1$and c_2 is the capture rate of the susceptible prey by the susceptible predator and the capture rate of the infected prey by infected predator respectively. Let a_1 and a_2 are intra–specific competition coefficient by susceptible prey species and intra–specific competition coefficient by infected prey species. Let c_3, c_4 are the conversion factors for the susceptible predator and the infected predator, respectively, due to consumption of the susceptible prey. Let a_3, a_4 are the conversion factors for the susceptible predator and the infected predator, respectively, due to consumption of the infected prey. Let d_1, d_2 are the natural death rate in the susceptible predator and infected predator respectively. Let m half-capturing saturation constant. Let α is rate of contact between susceptible prey and infected prey. Let β is rate of contact between susceptible predator and infected predator.

3. Positiveness of theorem

Theorem 0.1. *Given system of equations* (2.1) *is always nonnegative. Then all possible solutions of the system* (2.1) *are positive.*

Proof. Consider the first equations (2.1) of the system we can write

$$\frac{dW}{W} = \left(r_1 - a_1 W - \frac{c_1\,Y}{mW+1} - \alpha\,X\right)dt$$
$$\frac{dW}{W} = \phi(W, X, Y)dt \tag{1}$$

Where $\phi(W, X, Y) = \left(r_1 - a_1 W - \frac{c_1\,Y}{mW+1} - \alpha\,X\right)$
Taking integration in the region [0,t], we get

$$W(t) = W(0)e^{\int \phi(W,X,Y,Z)dt} > 0,\ \forall\ t\ as\ W(0) \geq 0 \tag{2}$$

Next consider the second set of equations (2.1) system we get

$$\frac{dX}{X} = (\alpha\,W - a_2 X - c_2 Z + r_2)\,dt$$
$$\frac{dX}{X} = \varphi(W, X, Z)dt \tag{3}$$

Where $\varphi(W, X, Z) = (\alpha\,W - a_2 X - c_2 Z + r_2)$
Taking integration in the region [0,t] we get

$$X(t) = X(0)e^{\int \varphi(W,X,Z)dt} > 0,\ \forall\ t\ as\ X(0) \geq 0 \tag{4}$$

Next consider the third set of equations (2.1) system we get

$$\begin{aligned} \frac{dY}{Y} &= \left(\frac{c_3 W}{mW+1} + a_3 X - d_1 - \beta Z\right) dt \\ \frac{dY}{Y} &= \chi(W, X, Z) dt \end{aligned} \tag{5}$$

Where $\chi(W, X, Z) = \left(\frac{c_3 W}{mW+1} + a_3 X - d_1 - \beta Z\right)$
Taking integration in the region [0, t] we get

$$Y(t) = Y(0)e^{\int \chi(W,X,Z)dt} > 0, \; \forall\; t \; as \; Y(0) \geq 0 \tag{6}$$

Next consider the fourth set of equations (2.1) system we get

$$\begin{aligned} \frac{dZ}{Z} &= \left(\frac{c_4 W}{mW+1} + a_4 X - d_2 + \beta Y\right) dt \\ \frac{dZ}{Z} &= \psi(W, X, Y) dt \end{aligned} \tag{7}$$

Where $\psi(W, X, Y) = \left(\frac{c_4 W}{mW+1} + a_4 X - d_2 + \beta Y\right)$
Taking integration in the region [0, t] we get

$$Z(t) = Z(0)e^{\int \psi(W,X,Y)dt} > 0, \; \forall\; t \; as \; Z(0) \geq 0 \tag{8}$$

Hence it may be concluded that all the solutions of the system (2.1) are always positive.

4. Range of the interval in susceptible and infected species

In this model, we consider biological phenomena parameters that are imprecise in nature.
An interval number A is a closed interval $[a_l, a_r]$ and is defined by

$$A = \{\alpha : a_l \leq (\alpha, \beta) \leq a_r, \alpha \in R\}$$

where R is the set of real numbers and a_l, a_r are the left and right limits of the interval number, respectively.
An interval–valued number $\widehat{a}$ on [0,1] is a closed subinterval of [0,1] that is $\widehat{a} = [a_l, a_u]$ such that $0 \leq a_l \leq a_u \leq 1$, where a_l and a_u are the lower and upper limits of $\widehat{a}$, respectively. In this notation, $\widehat{0} = [0, 0]$ and $\widehat{1} = [1, 1]$. For any two interval numbers $\widehat{a} = [a_l, a_u]$ and $\widehat{b} = [b_l, b_u]$ on $[0, 1]$ we define

$$\widehat{a} \leq \widehat{b} \Leftrightarrow a_l \leq b_l \; and \; a_u \leq b_u$$

$$\widehat{a} = \widehat{b} \Leftrightarrow a_l = b_l \; and \; a_u = b_u$$

Before the work is interval valued function defined. Present this work is very different and not for the function define just use in the range of interval [0, 1] is α for force of infection between the susceptible prey and infected prey spices.

5. Boundedness results

We have three results on the boundedness of the system (2.1). First we consider the both prey species

Theorem 0.2. *The both prey are always bounded above.*

If W(0)=0, then the result is trivial, if $W(0) > 0$, Then $W(t) > 0$ for all t on equation (2.1) we obtain

$$\frac{dW}{dt} \leq W(r_1 - b_1W)$$

$$\frac{dX}{dt} \leq X(r_2 - b_2X), \quad \limsup_{t\longrightarrow\infty} W(t) \leq \frac{r_1}{b_1}, \quad \limsup_{t\longrightarrow\infty} X(t) \leq \frac{r_2}{b_2}$$

Theorem 0.3. *The both predator are always bounded above.*

If Y(0)=0, then the result is trivial, if $Y(0) > 0$, Then $Y(t) > 0$ for all t on equation (2.1) we obtain

$$\frac{dY}{dt} \leq d_1$$

$$\frac{dZ}{dt} \leq d_2, \quad \limsup_{t\longrightarrow\infty} Y(t) \leq d_1, \quad \limsup_{t\longrightarrow\infty} Z(t) \leq d_2$$

Theorem 0.4. *The trajectories of system* (2.1) *are bounded.*

Define the function $L = W + X + Y + Z$ and take its time derivative along the solution of (2.1)

$$\frac{dL}{dt} = \frac{dW}{dt} + \frac{dX}{dt} + \frac{dY}{dt} + \frac{dZ}{dt}$$

now $\frac{dL}{dt} + \rho L = r_1W - b_1W^2 + r_2X - b_2X^2 + \rho W + \rho X + \rho Y + \rho Z - d_1Y - d_2Z$
where ρ is a positive constant for $r_1 + \rho - b_1W \geq 0, r_2 + \rho - b_2X \geq 0, \rho - d_1 \geq 0, \rho - d_2 \geq 0$ given $\epsilon > 0$ there exists to such that t on $t \geq t_0$.
$\frac{dL}{dt} + \rho L \leq m + \epsilon, m = \min\{\frac{r_1+\rho}{b_1}, \frac{r_2+\rho}{b_2}, (-d_1 + \rho), (-d_2 + \rho)\}$
Hence $\frac{d}{dt}(Le^{\rho t}) \leq (m + \epsilon)e^{\rho t}$ $L(t) \leq L(t_0)e^{-\rho(t-t_0)} + \frac{(m+\epsilon)}{\rho}(1 - e^{-\rho(t-t_0)})$.
Letting $t \longrightarrow 0$ then letting $\epsilon \longrightarrow 0$

$$\limsup_{t\longrightarrow\infty} L(t) \leq \frac{m}{\rho}$$

On the initial conditions. Hence the system (2.1) are bounded.

6. Critical point

In this section, we have the equilibrium point of the parametric model (2.1) is given by steady state equations $\frac{dW}{dt} = \frac{dX}{dt} = \frac{dY}{dt} = \frac{dZ}{dt} = 0$. The system has (12) equilibrium points, and after algebraic calculation we get the trivial, axial and non trivial equilibrium points as follows.

(1) The trivial equilibrium point are

$$\xi_1 \{W = 0, X = 0, Y = 0, Z = 0\}$$

(2) Both prey–free equilibrium point are

$$\xi_2 \left\{W = 0, X = 0, Y = \tfrac{d_2}{\beta}, Z = -\tfrac{d_1}{\beta}\right\}$$

(3) Susceptible prey–free and both predator–free equilibrium point are

$$\xi_3 \left\{W = 0, X = \tfrac{r_2}{a_2}, Y = 0, Z = 0\right\}$$

(4) Both susceptible prey–predator–free equilibrium point are

$$\xi_4 \left\{W = 0, X = \frac{d_2}{a_4}, Y = 0, Z = \frac{r_2\, a_4 - a_2 d_2}{a_4 c_2}\right\}$$

(5) Susceptible prey–free equilibrium point are

$$\xi_5 \left\{(W = 0, X = \psi_1, Y = \psi_2, Z = \psi_3)\right\}$$

where $\psi_1 = \frac{\beta\, r_2 + c_2 d_1}{\beta\, a_2 + a_3 c_2}, \psi_2 = -\frac{\beta\, r_2\, a_4 - \beta\, a_2 d_2 - a_3 c_2 d_2 + a_4 c_2 d_1}{(\beta\, a_2 + a_3 c_2)\beta}, \psi_3 = \frac{r_2\, a_3 - a_2 d_1}{\beta\, a_2 + a_3 c_2}$

(6) Infected prey–free equilibrium point are

$$\xi_6 \left\{W = \phi_1, X = 0, Y = \phi_2, Z = \phi_3\right\}$$

(7) Infected prey–free and both predator–free equilibrium point are

$$\xi_7 \left\{W = \frac{r_1}{a_1}, X = 0, Y = 0, Z = 0\right\}$$

(8) Both infected prey and predator–free equilibrium point are

$$\xi_8 \left\{W = -\frac{d_1}{md_1 - c_3}, X = 0, Y = -\frac{c_3(mr_1\, d_1 - r_1\, c_3 + a_1 d_1)}{(md_1 - c_3)^2 c_1}, Z = 0\right\}$$

(9) Both predator–free equilibrium point are

$$\xi_9 \left\{W = -\frac{\alpha\, r_2 - r_1\, a_2}{\alpha^2 + a_1 a_2}, X = \frac{\alpha\, r_1 + r_2\, a_1}{\alpha^2 + a_1 a_2}, Y = 0, Z = 0\right\}$$

(10) Susceptible predator–free equilibrium point are

$$\xi_{10} \left\{W = \phi_4, X = \phi_5, Y = 0, Z = \phi_6\right\}$$

(11) Infected predator–free equilibrium point are

$$\xi_{11} \left\{W = \phi_7, x = \phi_8, y = \phi_9, z = 0\right\}$$

(12) Non–trivial equilibrium point are

$$\xi_{12} \left\{W = \phi_{10}, X = \phi_{11}, Y = \phi_{12}, Z = \phi_{13}\right\}$$

The system of the equation (2.1) is Jacobian matrix given by

$$\begin{bmatrix} m_{11} & -\alpha\, W & -\frac{Wc_1}{Wm+1} & 0 \\ X\alpha & m_{22} & 0 & -Xc_2 \\ \frac{Yc_3}{Wm+1} - \frac{Yc_3 Wm}{(Wm+1)^2} & Ya_3 & m_{33} & -\beta\, Y \\ \frac{Zc_4}{Wm+1} - \frac{Zc_4 Wm}{(Wm+1)^2} & Za_4 & \beta\, Z & m_{44} \end{bmatrix}$$

Where

$m_{11} = -2\,a_1W + r_1 - \frac{c_1\,Y}{Wm+1} + \frac{Wc_1\,Ym}{(Wm+1)^2} - X\alpha,$

$m_{22} = \alpha\,W - 2\,Xa_2 - Zc_2 + r_2,$

$m_{33} = \frac{c_3W}{Wm+1} + a_3X - d_1 - \beta\,Z,$

$m_{44} = \frac{c_4W}{Wm+1} + a_4X - d_2 + \beta\,Y$

7. Stability analysis

In this section we shall discuss the stability properties of the critical point.

Theorem 0.5. *Given the linearized system of equations* (2.1) *is the trivial equilibrium point. In which the equilibrium point* $\xi_1(0,0,0,0)$ *is a saddle point.*

Proof. The variation of the Jacobian matrix are

$$J_1 = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & -d_1 & 0 \\ 0 & 0 & 0 & -d_2 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = r_1, \lambda_2 = r_2, \lambda_3 = -d_1, \lambda_4 = -d_2$.

1. An equilibrium point $\xi_1(0,0,0,0)$ is called a saddle point. If all eigenvalues of matrix J_1 have nonzero real parts is called a hyperbolic equilibrium point exists. Then the eigenvalues of matrix J_1 has at least of eigenvalues with a positive real parts and at least one eigenvalues with a negative real part is called a saddle point. Therefore the eigenvalues $\lambda_1 = r_1 > 0, \lambda_2 = r_2 > 0, \lambda_3 = -d_1 < 0, \lambda_4 = -d_2 < 0$ and that is $r_1 > 0, r_2 > 0, d_1 < 0, d_2 < 0$ is a saddle point.

Theorem 0.6. *Given the linearized system of equations* (2.1) *is both prey–free equilibrium point. In which the equilibrium point* $\xi_2\left\{W = 0, X = 0, Y = \frac{d_2}{\beta}, Z = -\frac{d_1}{\beta}\right\}$ *are source and sink.*

Proof. The variation of the Jacobian matrix are

$$J_2 = \begin{bmatrix} r_1 - \frac{c_1\,d_2}{\beta} & 0 & 0 & 0 \\ 0 & \frac{c_2d_1}{\beta} + r_2 & 0 & 0 \\ \frac{d_2c_3}{\beta} & \frac{d_2a_3}{\beta} & 0 & -d_2 \\ -\frac{d_1c_4}{\beta} & -\frac{d_1a_4}{\beta} & -d_1 & 0 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = \sqrt{d_2d_1}, \lambda_2 = -\sqrt{d_2d_1}, \lambda_3 = \frac{\beta\,r_1 - c_1\,d_2}{\beta}, \lambda_4 = \frac{\beta\,r_2 + c_2d_1}{\beta}$. that is $\beta r_1 < c_1d_2$ and $\beta r_2 + c_2d_2 < 0$ is a sink. Otherwise $\beta r_1 > c_1d_2$ and $\beta r_2 + c_2d_2 > 0$ is a source.

Theorem 0.7. *Given the linearized system of equations* (2.1) *is susceptible prey–free and both predator–free equilibrium point. In which the equilibrium point* $\xi_3\left\{W = 0, X = \frac{r_2}{a_2}, Y = 0, Z = 0\right\}$ *are source and sink.*

Proof. The variation of the Jacobian matrix are

$$J_3 = \begin{bmatrix} r_1 - \frac{\alpha r_2}{a_2} & 0 & 0 & 0 \\ \frac{\alpha r_2}{a_2} & -r_2 & 0 & -\frac{r_2 c_2}{a_2} \\ 0 & 0 & \frac{a_3 r_2}{a_2} - d_1 & 0 \\ 0 & 0 & 0 & \frac{a_4 r_2}{a_2} - d_2 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = \frac{a_3 r_2 - a_2 d_1}{a_2}, \lambda_2 = \frac{a_4 r_2 - a_2 d_2}{a_2}, \lambda_3 = -r_2, \lambda_4 = -\frac{\alpha r_2 - r_1 a_2}{a_2}$ that is $a_3 r_2 < a_2 d_1, a_4 r_2 < a_2 d_2, r_2 > 0, \alpha r_2 < r_1 a_2$ is a sink. Otherwise $a_3 r_2 > a_2 d_1, a_4 r_2 > a_2 d_2, r_2 < 0, \alpha r_2 > r_1 a_2$ is a source.

Theorem 0.8. *Given the linearized system of equations* (2.1) *is susceptible prey–free and susceptible predator–free equilibrium point. In which the equilibrium point* $\xi_4 \left\{ W = 0, X = \frac{d_2}{a_4}, Y = 0, Z = \frac{r_2 a_4 - a_2 d_2}{a_4 c_2} \right\}$ *is a source and sink.*

Proof. The variation of the Jacobian matrix are

$$J_4 = \begin{bmatrix} r_1 - \frac{d_2 \alpha}{a_4} & 0 & 0 & 0 \\ \frac{d_2 \alpha}{a_4} & -2\,\frac{a_2 d_2}{a_4} - \frac{a_4 r_2 - a_2 d_2}{a_4} + r_2 & 0 & -\frac{d_2 c_2}{a_4} \\ 0 & 0 & \frac{d_2 a_3}{a_4} - d_1 - \frac{\beta\,(a_4 r_2 - a_2 d_2)}{a_4 c_2} & 0 \\ \frac{(a_4 r_2 - a_2 d_2) c_4}{a_4 c_2} & \frac{a_4 r_2 - a_2 d_2}{c_2} & \frac{\beta\,(a_4 r_2 - a_2 d_2)}{a_4 c_2} & 0 \end{bmatrix}$$

The corresponding eigenvalues are
$\lambda_1 = -\frac{d_2 \alpha - r_1 a_4}{a_4}$,
$\lambda_2 = -\frac{\beta r_2 a_4 - \beta a_2 d_2 - a_3 c_2 d_2 + a_4 c_2 d_1}{a_4 c_2}$,
$\lambda_3 = 1/2\,\frac{-a_2 d_2 + \sqrt{-4 r_2 a_4^2 d_2 + a_2^2 d_2^2 + 4 a_2 a_4 d_2^2}}{a_4}$
$\lambda_4 = -1/2\,\frac{a_2 d_2 + \sqrt{-4 r_2 a_4^2 d_2 + a_2^2 d_2^2 + 4 a_2 a_4 d_2^2}}{a_4}$
that is $d_2 \alpha < r_1 a_4, \beta r_2 a_4 + a_4 c_2 d_1 < d_2(\beta a_2 + a_3 c_2)$ is a sink. Otherwise $d_2 \alpha > r_1 a_4, \beta r_2 a_4 + a_4 c_2 d_1 > d_2(\beta a_2 + a_3 c_2)$ is a source.

Theorem 0.9. *Given the linearized system of equations* (2.1) *is susceptible prey–free equilibrium point. Therefore the equilibrium point* $\xi_5 \{(W = 0, X = \psi_1, Y = \psi_2, Z = \psi_3)\}$ *is a locally asymptotically stable where* $\psi_1 = \frac{\beta r_2 + c_2 d_1}{\beta a_2 + a_3 c_2}$,
$\psi_2 = -\frac{\beta r_2 a_4 - \beta a_2 d_2 - a_3 c_2 d_2 + a_4 c_2 d_1}{(\beta a_2 + a_3 c_2)\beta}, \psi_3 = \frac{r_2 a_3 - a_2 d_1}{\beta a_2 + a_3 c_2}$.

Proof. The variation of the Jacobian matrix are

$$J_5 = \begin{bmatrix} m_{11} & 0 & 0 & 0 \\ \psi_1 \alpha & m_{22} & 0 & -\psi_1 c_2 \\ \psi_2 c_3 & \psi_2 a_3 & m_{33} & -\beta\,\psi_2 \\ \psi_3 c_4 & \psi_3 a_4 & \beta\,\psi_3 & m_{44} \end{bmatrix}$$

The corresponding eigenvalues are

$f_0(\lambda) = \lambda^4 + H_1\lambda^3 + H_2\lambda^2 + H_3\lambda + H_4$where

$H_1 = -m_{44} - m_{33} - m_{22} - m_{11}, H_2 = \beta^2\psi_2\psi_3 + a_4c_2\psi_1\psi_3 + m_{11}m_{22} + m_{11}m_{33} + m_{11}m_{44} + m_{22}m_{33} + m_{22}m_{44} + m_{33}m_{44}, H_3 = \beta a_3c_2\psi_1\psi_2\psi_3 - \beta^2 m_{11}\psi_2\psi_3 - \beta^2 m_{22}\psi_2\psi_3 - a_4c_2m_{11}\psi_1\psi_3 - a_4c_2m_{33}\psi_1\psi_3 - m_{11}m_{22}m_{33} - m_{11}m_{22}m_{44} - m_{11}m_{33}m_{44} - m_{22}m_{33}m_{44}, H_4 = -\beta a_3c_2m_{11}\psi_1\psi_2\psi_3 + \beta^2 m_{11}m_{22}\psi_2\psi_3 + a_4c_2m_{11}m_{33}\psi_1\psi_3 + m_{44}m_{33}m_{22}m_{11}.$

By Routh Hurwitzs criterion, all the eigenvalues of J_5 have negative real parts if $(i)H_0 > 0$,

$(ii)H_1 > 0$,

$(iii)H_3 > 0$,

$(vi)H_1H_2H_3 > H_3^2 + H_1^2H_4$. We observe that the system (2.1) is locally asymptotically stable around the positive equilibrium point (5) if the conditions stated in the theorem holds.

Theorem 0.10. *Given the linearized system of equations* (2.1) *is infected prey–free equilibrium. In Which the equilibrium point* $\xi_6\,\{W = \phi_1, X = 0, Y = \phi_2, Z = \phi_3\}$

Proof. The variation of the Jacobian matrix are

$$J_6 = \begin{bmatrix} m_{11} & -\alpha\phi_1 & -\frac{\phi_1 c_1}{m\phi_1+1} & 0 \\ 0 & m_{22} & 0 & 0 \\ m_{31} & \phi_2 a_3 & m_{33} & -\beta\phi_2 \\ m_{41} & \phi_3 a_4 & \beta\phi_3 & m_{44} \end{bmatrix}$$

where

$m_{11} = -2a_1\phi_1 + r_1 - \frac{c_1\phi_2}{m\phi_1+1} + \frac{\phi_1 c_1\phi_2 m}{(m\phi_1+1)^2}$,

$m_{22} = \alpha\phi_1 - c_2\phi_3 + r_2$,

$m_{31} = \frac{\phi_2 c_3}{m\phi_1+1} - \frac{\phi_2 c_3\phi_1 m}{(m\phi_1+1)^2}$,

$m_{33} = \frac{c_3\phi_1}{m\phi_1+1} - d_1 - \beta\phi_3$,

$m_{41} = \frac{\phi_3 c_4}{m\phi_1+1} - \frac{\phi_3 c_4\phi_1 m}{(m\phi_1+1)^2}$,

$m_{44} = \frac{c_4\phi_1}{m\phi_1+1} - d_2 + \beta\phi_2$

The characteristic equation $f_1(\lambda) = \lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4$ where

$A_1 = -m_{44} - m_{33} - m_{22} - m_{11}$

$A_2 = \frac{1}{m\phi_1+1}(\beta^2 m\phi_1\phi_2\phi_3 + \beta^2\phi_3\phi_2 + mm_{11}m_{22}\phi_1 + mm_{11}m_{33}\phi_1 + mm_{11}m_{44}\phi_1 + mm_{22}m_{33}\phi_1 + mm_{22}m_{44}\phi_1 + mm_{33}m_{44}\phi_1 + m_{31}\phi_1c_1 + m_{11}m_{22} + m_{11}m_{33} + m_{11}m_{44} + m_{22}m_{33} + m_{22}m_{44} + m_{33}m_{44})$

$A_3 = -\frac{1}{m\phi_1+1}(\beta^2 mm_{11}\phi_1\phi_2\phi_3 + \beta^2 mm_{22}\phi_1\phi_2\phi_3 + \beta^2 m_{11}\phi_2\phi_3 + \beta^2 m_{22}\phi_2\phi_3 + m_{41}\phi_1c_1\beta\phi_2 + mm_{11}m_{22}m_{33}\phi_1 + mm_{11}m_{22}m_{44}\phi_1 + mm_{11}m_{33}m_{44}\phi_1 + mm_{22}m_{33}m_{44}\phi_1 + m_{31}\phi_1c_1m_{22} + c_1m_{31}m_{44}\phi_1 + m_{11}m_{22}m_{33} + m_{11}m_{22}m_{44} + m_{11}m_{33}m_{44} + m_{22}m_{33}m_{44})$

$A_4 = \frac{m_{22}}{m\phi_1+1}(\beta^2 mm_{11}\phi_1\phi_2\phi_3 + \beta^2 m_{11}\phi_2\phi_3 + m_{41}\phi_1c_1\beta\phi_2 + mm_{11}m_{33}m_{44}\phi_1 + c_1m_{31}m_{44}\phi_1 + m_{11}m_{33}m_{44})$

By Routh Hurwitzs criterion, all the eigenvalues of J_6 have negative real parts if $(i)A_0 > 0$,

$(ii)A_1 > 0$,

$(iii)A_3 > 0$,

$(vi) A_1A_2A_3 > A_3^2 + A_1^2A_4$. we observe that the system (2.1) is locally asymptotically stable around the positive equilibrium point (6) if the conditions stated in the theorem holds.

Theorem 0.11. *Given the linearized system of equations* (2.1) *is infected prey–free and both predator–free equilibrium point. In which the equilibrium point* $\xi_7 \left\{W = \frac{r_1}{a_1}, X = 0, Y = 0, Z = 0\right\}$ *is a source and sink.*

Proof. The variation of the Jacobian matrix are

$$J_7 = \begin{bmatrix} -r_1 & -\frac{\alpha r_1}{a_1} & -\frac{r_1 c_1}{a_1}\left(\frac{mr_1}{a_1}+1\right)^{-1} & 0 \\ 0 & \frac{\alpha r_1}{a_1}+r_2 & 0 & 0 \\ 0 & 0 & \frac{r_1 c_3}{a_1}\left(\frac{mr_1}{a_1}+1\right)^{-1} - d_1 & 0 \\ 0 & 0 & 0 & \frac{c_4 r_1}{a_1}\left(\frac{mr_1}{a_1}+1\right)^{-1} - d_2 \end{bmatrix}$$

The corresponding eigenvalues are

$\lambda_1 = \frac{\alpha r_1 + r_2 a_1}{a_1}$

$\lambda_2 = -r_1$

$\lambda_3 = -\frac{mr_1 d_1 - r_1 c_3 + a_1 d_1}{mr_1 + a_1}$

$\lambda_4 = -\frac{mr_1 d_2 - c_4 r_1 + a_1 d_2}{mr_1 + a_1}$ If its satisfy the conditions $\alpha r_1 + r_2 a_1 < 0, r_1 > 0, d_1(mr_1 + a_1) < r_1 c_3, d_2(mr_1 + a_1) < r_1 c_4$ is a sink. Otherwise $\alpha r_1 + r_2 a_1 > 0, r_1 < 0, d_1(mr_1 + a_1) > r_1 c_3, d_2(mr_1 + a_1) > r_1 c_4$ is a source.

Theorem 0.12. *Given the linearized system of equations* (2.1) *is both infected prey–predator–free equilibrium point. In Which the equilibrium point* $\xi_8 \left\{W = -\frac{d_1}{md_1 - c_3}, X = 0, Y = -\frac{c_3(mr_1 d_1 - r_1 c_3 + a_1 d_1)}{(md_1 - c_3)^2 c_1}, Z = 0\right\}$ *is a locally asymptotically stable.*

Proof. The variation of the Jacobian matrix are

$$J_8 = \begin{bmatrix} \frac{(m^2 r_1 d_1 - mr_1 c_3 + ma_1 d_1 + a_1 c_3) d_1}{c_3(md_1 - c_3)} & \frac{\alpha d_1}{md_1 - c_3} & -\frac{d_1 c_1}{c_3} & 0 \\ 0 & -\frac{-mr_2 d_1 + \alpha d_1 + r_2 c_3}{md_1 - c_3} & 0 & 0 \\ -\frac{mr_1 d_1 - r_1 c_3 + a_1 d_1}{c_1} & -\frac{c_3(mr_1 d_1 - r_1 c_3 + a_1 d_1) a_3}{(md_1 - c_3)^2 c_1} & 0 & \frac{\beta c_3 (mr_1 d_1 - r_1 c_3 + a_1 d_1)}{(md_1 - c_3)^2 c_1} \\ 0 & 0 & 0 & m_{44} \end{bmatrix}$$

where

$m_{44} = -\frac{1}{c_3(md_1 - c_3)^2 c_1}(c_1 m^2 c_3 d_1^2 d_2 - c_1 m^2 c_4 d_1^3 + \beta m r_1 c_3^2 d_1 - 2 c_1 m c_3^2 d_1 d_2 + 2 c_1 m c_3 c_4 d_1^2 - \beta r_1 c_3^3 + \beta a_1 c_3^2 d_1 + c_1 c_3^3 d_2 - c_1 c_3^2 c_4 d_1)$ The corresponding eigenvalues are

$\lambda_1 = 1/2\, \frac{m^2 r_1 d_1^2 - mr_1 c_3 d_1 + ma_1 d_1^2 + a_1 d_1 c_3 \sqrt{A}}{(md_1 - c_3) c_3}$

$\lambda_2 = -1/2\, \frac{-m^2 r_1 d_1^2 + mr_1 c_3 d_1 - ma_1 d_1^2 - a_1 d_1 c_3 + \sqrt{A}}{(md_1 - c_3) c_3}$

$\lambda_3 = -\frac{1}{c_1 c_3 (m^2 d_1^2 - 2 m c_3 d_1 + c_3^2)}(c_1 m^2 c_3 d_1^2 d_2 - c_1 m^2 c_4 d_1^3 + \beta m r_1 c_3^2 d_1 - 2 c_1 m c_3^2 d_1 d_2 + 2 c_1 m c_3 c_4 d_1^2 - \beta r_1 c_3^3 + \beta a_1 c_3^2 d_1 + c_1 c_3^3 d_2 - c_1 c_3^2 c_4 d_1)$

$\lambda_4 = -\frac{-mr_2 d_1 + \alpha d_1 + r_2 c_3}{md_1 - c_3}$ where

$A = m^4 r_1^2 d_1^4 - 2 m^3 r_1^2 c_3 d_1^3 + 2 m^3 r_1 a_1 d_1^4 + 4 m^3 r_1 c_3 d_1^4 + m^2 r_1^2 c_3^2 d_1^2 - 12 m^2 r_1 c_3^2 d_1^3 + m^2 a_1^2 d_1^4 + 4 m^2 a_1 c_3 d_1^4 - 2 m r_1 a_1 c_3^2 d_1^2 + 12 m r_1 c_3^3 d_1^2 + 2 m a_1^2 c_3 d_1^3 - 8 m a_1 c_3^2 d_1^3 - 4 r_1 c_3^4 d_1 + a_1^2 c_3^2 d_1^2 + 4 a_1 c_3^3 d_1^2$

Its satisfied conditions $(\alpha\, d_1 + r_2\, c_3) < (m r_2\, d_1)$, *and* $(c_1\, m^2 c_3 {d_1}^2 d_2 + \beta\, m r_1\, {c_3}^2 d_1 + 2\, c_1\, m c_3 c_4 {d_1}^2 + \beta\, a_1 {c_3}^2 d_1 + c_1\, {c_3}^3 d_2) < (c_1\, m^2 c_4 {d_1}^3 + 2\, c_1\, m {c_3}^2 d_1 d_2 + \beta\, r_1\, {c_3}^3 + c_1\, {c_3}^2 c_4 d_1)$. Hence its locally asymptotically stable.

Theorem 0.13. *Given the linearized system of equations* (2.1) *is both predator–free equilibrium point. In which the equilibrium point* $\xi_9 \left\{ W = -\frac{\alpha\, r_2 - r_1\, a_2}{\alpha^2 + a_1 a_2}, X = \frac{\alpha\, r_1 + r_2\, a_1}{\alpha^2 + a_1 a_2}, Y = 0, Z = 0 \right\}$ *is locally asymptotically stable.*

Proof. The variation of the Jacobian matrix are

$$J_9 = \begin{bmatrix} m_{11} & \frac{\alpha\,(\alpha\, r_2 - r_1\, a_2)}{\alpha^2 + a_1 a_2} & m_{13} & 0 \\ \frac{(\alpha\, r_1 + r_2\, a_1)\alpha}{\alpha^2 + a_1 a_2} & m_{22} & 0 & -\frac{(\alpha\, r_1 + r_2\, a_1) c_2}{\alpha^2 + a_1 a_2} \\ 0 & 0 & m_{33} & 0 \\ 0 & 0 & 0 & m_{44} \end{bmatrix}$$

where $m_{11} = r_1 + 2\, \frac{a_1(\alpha\, r_2 - r_1\, a_2)}{\alpha^2 + a_1 a_2} - \frac{(\alpha\, r_1 + r_2\, a_1)\alpha}{\alpha^2 + a_1 a_2}$,
$m_{13} = \frac{(\alpha\, r_2 - r_1\, a_2) c_1}{\alpha^2 + a_1 a_2} \left(-\frac{m(\alpha\, r_2 - r_1\, a_2)}{\alpha^2 + a_1 a_2} + 1 \right)^{-1}$,
$m_{22} = -\frac{\alpha\,(\alpha\, r_2 - r_1\, a_2)}{\alpha^2 + a_1 a_2} - \frac{(2\,\alpha\, r_1 + 2\, r_2\, a_1) a_2}{\alpha^2 + a_1 a_2} + r_2$,
$m_{33} = -\frac{c_3(\alpha\, r_2 - r_1\, a_2)}{\alpha^2 + a_1 a_2} \left(-\frac{m(\alpha\, r_2 - r_1\, a_2)}{\alpha^2 + a_1 a_2} + 1 \right)^{-1} + \frac{a_3(\alpha\, r_1 + r_2\, a_1)}{\alpha^2 + a_1 a_2} - d_1$,
$m_{44} = -\frac{c_4(\alpha\, r_2 - r_1\, a_2)}{\alpha^2 + a_1 a_2} \left(-\frac{m(\alpha\, r_2 - r_1\, a_2)}{\alpha^2 + a_1 a_2} + 1 \right)^{-1} + \frac{a_4(\alpha\, r_1 + r_2\, a_1)}{\alpha^2 + a_1 a_2} - d_2$
$\lambda_1 = 1/2\, \frac{-\alpha\, r_1\, a_2 + \alpha\, r_2\, a_1 - r_1\, a_1 a_2 - r_2\, a_1 a_2 + \sqrt{U}}{\alpha^2 + a_1 a_2}$
$\lambda_2 = -1/2\, \frac{\alpha\, r_1\, a_2 - \alpha\, r_2\, a_1 + r_1\, a_1 a_2 + r_2\, a_1 a_2 + \sqrt{U}}{\alpha^2 + a_1 a_2}$
$\lambda_3 = -\frac{1}{B}(-\alpha^3 m r_2\, d_1 + \alpha^2 m r_1\, r_2\, a_3 + \alpha^2 m r_1\, a_2 d_1 - \alpha\, m {r_1}^2 a_2 a_3 + \alpha\, m {r_2}^2 a_1 a_3 - \alpha\, m r_2\, a_1 a_2 d_1 - m r_1\, r_2\, a_1 a_2 a_3 + m r_1\, a_1 {a_2}^2 d_1 + \alpha^4 d_1 - \alpha^3 r_1\, a_3 + \alpha^3 r_2\, c_3 - \alpha^2 r_1\, a_2 c_3 - \alpha^2 r_2\, a_1 a_3 + 2\,\alpha^2 a_1 a_2 d_1 - \alpha\, r_1\, a_1 a_2 a_3 + \alpha\, r_2\, a_1 a_2 c_3 - r_1\, a_1 {a_2}^2 c_3 - r_2\, {a_1}^2 a_2 a_3 + {a_1}^2 {a_2}^2 d_1)$
$\lambda_4 = -\frac{1}{B}(-\alpha^3 m r_2\, d_2 + \alpha^2 m r_1\, r_2\, a_4 + \alpha^2 m r_1\, a_2 d_2 - \alpha\, m {r_1}^2 a_2 a_4 + \alpha\, m {r_2}^2 a_1 a_4 - \alpha\, m r_2\, a_1 a_2 d_2 - m r_1\, r_2\, a_1 a_2 a_4 + m r_1\, a_1 {a_2}^2 d_2 + \alpha^4 d_2 - \alpha^3 r_1\, a_4 + \alpha^3 r_2\, c_4 - \alpha^2 r_1\, a_2 c_4 - \alpha^2 r_2\, a_1 a_4 + 2\,\alpha^2 a_1 a_2 d_2 - \alpha\, r_1\, a_1 a_2 a_4 + \alpha\, r_2\, a_1 a_2 c_4 - r_1\, a_1 {a_2}^2 c_4 - r_2\, {a_1}^2 a_2 a_4 + {a_1}^2 {a_2}^2 d_2)$
$U = 4\,\alpha^4 r_1\, r_2 - 4\,\alpha^3 {r_1}^2 a_2 + 4\,\alpha^3 {r_2}^2 a_1 + \alpha^2 {r_1}^2 {a_2}^2 - 2\,\alpha^2 r_1\, r_2\, a_1 a_2 + \alpha^2 {r_2}^2 {a_1}^2 - 2\,\alpha\, {r_1}^2 a_1 {a_2}^2 - 2\,\alpha\, r_1\, r_2\, {a_1}^2 a_2 + 2\,\alpha\, r_1\, r_2\, a_1 {a_2}^2 + 2\,\alpha\, {r_2}^2 {a_1}^2 a_2 + {r_1}^2 {a_1}^2 {a_2}^2 - 2\, r_1\, r_2\, {a_1}^2 {a_2}^2 + {r_2}^2 {a_1}^2 {a_2}^2$,
$B = -\alpha^3 m r_2 + \alpha^2 m r_1\, a_2 - \alpha\, m r_2\, a_1 a_2 + m r_1\, a_1 {a_2}^2 + \alpha^4 + 2\,\alpha^2 a_1 a_2 + {a_1}^2 {a_2}^2$, then its conditions satisfy the $\alpha^2 m r_1\, a_2 d_1 + \alpha\, m {r_2}^2 a_1 a_3 + m r_1\, a_1 {a_2}^2 d_1 + \alpha^4 d_1 + \alpha^3 r_2\, c_3 + 2\,\alpha^2 a_1 a_2 d_1 + \alpha\, r_2\, a_1 a_2 c_3 + {a_1}^2 {a_2}^2 d_1 < (\alpha^3 m r_2\, d_1 + \alpha^2 m r_1\, r_2\, a_3 + \alpha\, m {r_1}^2 a_2 a_3 + \alpha\, m r_2\, a_1 a_2 d_1 + m r_1\, r_2\, a_1 a_2 a_3 + \alpha^3 r_1\, a_3 + \alpha^2 r_1\, a_2 c_3 + \alpha^2 r_2\, a_1 a_3 + \alpha\, r_1\, a_1 a_2 a_3 + r_1\, a_1 {a_2}^2 c_3 + r_2\, {a_1}^2 a_2 a_3)$ and $(\alpha^2 m r_1\, r_2\, a_4 + \alpha^2 m r_1\, a_2 d_2 + \alpha\, m {r_2}^2 a_1 a_4 + m r_1\, a_1 {a_2}^2 d_2 + \alpha^4 d_2 + \alpha^3 r_2\, c_4 + 2\,\alpha^2 a_1 a_2 d_2 + \alpha\, r_2\, a_1 a_2 c_4 + {a_1}^2 {a_2}^2 d_2) < (r_2\, {a_1}^2 a_2 a_4 + r_1\, a_1 {a_2}^2 c_4 + \alpha\, r_1\, a_1 a_2 a_4 + \alpha^2 r_1\, a_2 c_4 + \alpha^2 r_2\, a_1 a_4 + \alpha^3 r_1\, a_4 + \alpha\, m r_2\, a_1 a_2 d_2 + m r_1\, r_2\, a_1 a_2 a_4 + \alpha\, m {r_1}^2 a_2 a_4 + \alpha^3 m r_2\, d_2)$.
Hence the equilibrium point is locally asymptotically stable.

Theorem 0.14. *Given the linearized system of equations* (2.1) *is susceptible predator–free equilibrium point. In which the equilibrium point* $\xi_{10}\, \{W = \phi_4, X = \phi_5, Y = 0, Z = \phi_6\}$ *is locally asymptotically stable.*

Proof. The variation of the Jacobian matrix are

$$J_{10} = \begin{bmatrix} m_{11} & -\alpha\,\phi_4 & -\frac{\phi_4 c_1}{m\phi_4+1} & 0 \\ \phi_5\alpha & m_{22} & 0 & -\phi_5 c_2 \\ 0 & 0 & m_{33} & 0 \\ m_{41} & \phi_6 a_4 & \beta\,\phi_6 & m_{44} \end{bmatrix}$$

$m_{11} = -\phi_5\alpha - 2\,a_1\phi_4 + r_1,$

$m_{22} = \alpha\,\phi_4 - 2\,a_2\phi_5 - c_2\phi_6 + r_2,$

$m_{33} = \frac{c_3\phi_4}{m\phi_4+1} + a_3\phi_5 - d_1 - \beta\,\phi_6,$

$m_{41} = \frac{\phi_6 c_4}{m\phi_4+1} - \frac{\phi_6 c_4 \phi_4 m}{(m\phi_4+1)^2},$

$m_{44} = \frac{c_4\phi_4}{m\phi_4+1} + a_4\phi_5 - d_2$. The characteristic function are $f_2(\lambda) = B_0\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4$

Where

$B_0 = 1,$

$B_1 = (-m_{44} - m_{33} - m_{22} - m_{11})\,,$

$B_2 = (\alpha^2\phi_4\phi_5 + a_4c_2\phi_5\phi_6 + m_{11}m_{22} + m_{11}m_{33} + m_{11}m_{44} + m_{22}m_{33} + m_{22}m_{44} + m_{33}m_{44}),$

$B_3 = (-\alpha^2 m_{33}\phi_4\phi_5 - \alpha^2 m_{44}\phi_4\phi_5 - \alpha\,c_2m_{41}\phi_4\phi_5 - a_4c_2m_{11}\phi_5\phi_6 - a_4c_2m_{33}\phi_5\phi_6 - m_{11}m_{22}m_{33} - m_{11}m_{22}m_{44} - m_{11}m_{33}m_{44} - m_{22}m_{33}m_{44}),$

$B_4 = (\alpha^2 m_{33}m_{44}\phi_4\phi_5 + \alpha\,c_2m_{33}m_{41}\phi_4\phi_5 + a_4c_2m_{11}m_{33}\phi_5\phi_6 + m_{11}m_{22}m_{33}m_{44})$. By Routh Hurwitz criterion, all the eigenvalues of J_{10} have negative real parts if $(i)B_0 > 0,$

$(ii)B_1 > 0,$

$(iii)B_3 > 0,$

$(vi)B_1B_2B_3 > B_3^2 + B_1^2B_4$. we observe that the system (2.1) is locally asymptotically stable around the positive equilibrium point (10) if the conditions stated in the theorem holds.

Theorem 0.15. *Given the linearized system of equations* (2.1) *is susceptible predator–free equilibrium point. In which the equilibrium point* $\xi_{11}\,\{W = \vartheta_7, x = \vartheta_8, y = 0, z = \vartheta_9\}$ *is locally asymptotically stable.*

Proof. The variation of the Jacobian matrix are

$$J_{11} = \begin{bmatrix} m_{11} & -\alpha\,\phi_7 & -\frac{\phi_7 c_1}{m\phi_7+1} & 0 \\ \phi_8\alpha & m_{22} & 0 & -\phi_8 c_2 \\ m_{31} & \phi_9 a_3 & m_{33} & -\beta\,\phi_9 \\ 0 & 0 & 0 & m_{44} \end{bmatrix}$$

Where

$m_{44} = \frac{c_4\phi_7}{m\phi_7+1} + a_4\phi_8 - d_2 + \beta\,\phi_9,$

$m_{11} = -2\,a_1\phi_7 + r_1 - \frac{c_1\,\phi_9}{m\phi_7+1} + \frac{\phi_7 c_1\,\phi_9 m}{(m\phi_7+1)^2} - \phi_8\alpha,$

$m_{22} = \alpha\,\phi_7 - 2\,a_2\phi_8 + r_2,$

$m_{31} = \frac{\phi_9 c_3}{m\phi_7+1} - \frac{\phi_9 c_3\phi_7 m}{(m\phi_7+1)^2},$

$m_{33} = \frac{c_3\phi_7}{m\phi_7+1} + a_3\phi_8 - d_1$. The characteristic function are $f_3(\lambda) = C_0\lambda^4 + C_1\lambda^3 + C_2\lambda^2 + C_3\lambda + C_4$

Where

$C_0 = 1,$

$C_1 = (-m_{44} - m_{33} - m_{22} - m_{11})$,
$C_2 = \frac{1}{m\phi_7+1}(\alpha^2 m\phi_7{}^2\phi_8 + \alpha^2\phi_7\phi_8 + mm_{11}m_{22}\phi_7 + mm_{11}m_{33}\phi_7 + mm_{11}m_{44}\phi_7 + mm_{22}m_{33}\phi_7 + mm_{22}m_{44}\phi_7 + mm_{33}m_{44}\phi_7 + c_1 m_{31}\phi_7 + m_{11}m_{22} + m_{11}m_{33} + m_{11}m_{44} + m_{22}m_{33} + m_{22}m_{44} + m_{33}m_{44})$,
$C_3 = -\frac{1}{m\phi_7+1}(\alpha^2 mm_{33}\phi_7{}^2\phi_8 + \alpha^2 mm_{44}\phi_7{}^2\phi_8 - \alpha\, c_1\, a_3\phi_7\phi_8\phi_9 + \alpha^2 m_{33}\phi_7\phi_8 + \alpha^2 m_{44}\phi_7\phi_8 + mm_{11}m_{22}m_{33}\phi_7 + mm_{11}m_{22}m_{44}\phi_7 + mm_{11}m_{33}m_{44}\phi_7 + mm_{22}m_{33}m_{44}\phi_7 + c_1\, m_{22}m_{31}\phi_7 + c_1\, m_{31}m_{44}\phi_7 + m_{11}m_{22}m_{33} + m_{11}m_{22}m_{44} + m_{11}m_{33}m_{44} + m_{22}m_{33}m_{44})$,
$C_4 = \frac{m_{44}}{m\phi_7+1}(\alpha^2 mm_{33}\phi_7{}^2\phi_8 - \alpha\, c_1\, a_3\phi_7\phi_8\phi_9 + \alpha^2 m_{33}\phi_7\phi_8 + mm_{11}m_{22}m_{33}\phi_7 + c_1\, m_{22}m_{31}\phi_7 + m_{11}m_{22}m_{33})$.
By Routh Hurwitzs criterion, all the eigenvalues of J_{11} have negative real parts if $(i)C_0 > 0$,
$(ii)C_1 > 0$,
$(iii)C_3 > 0$,
$(vi)C_1C_2C_3 > C_3^2 + C_1^2C_4$. we observe that the system (2.1) is locally asymptotically stable around the positive equilibrium point (11) if the conditions stated in the theorem holds.

Theorem 0.16. *Given the linearized system of equations* (2.1) *is nontrivial equilibrium point. In which the equilibrium point* $\xi_{12}\,\{W = \phi_{10}, x = \phi_{11}, y = \phi_{12}, z = \phi_{13}\}$ *is locally asymptotically stable.*

Proof. The variation of the Jacobian matrix are

$$J_{12} = \begin{bmatrix} m_{11} & -\alpha\,\phi_{10} & -\frac{\phi_{10}c_1}{m\phi_{10}+1} & 0 \\ \phi_{11}\alpha & m_{22} & 0 & -\phi_{11}c_2 \\ \frac{\phi_{12}c_3}{m\phi_{10}+1} - \frac{\phi_{12}c_3\phi_{10}m}{(m\phi_{10}+1)^2} & \phi_{12}a_3 & m_{33} & -\beta\,\phi_{12} \\ \frac{\phi_{13}c_4}{m\phi_{10}+1} - \frac{\phi_{13}c_4\phi_{10}m}{(m\phi_{10}+1)^2} & \phi_{13}a_4 & \beta\,\phi_{13} & m_{44} \end{bmatrix}$$

Where $m_{11} = -2\,a_1\phi_{10} + r_1 - \frac{c_1\,\phi_{12}}{m\phi_{10}+1} + \frac{\phi_{10}c_1\,\phi_{12}m}{(m\phi_{10}+1)^2} - \phi_{11}\alpha$,
$m_{22} = \alpha\,\phi_{10} - 2\,a_2\phi_{11} - c_2\phi_{13} + r_2$,
$m_{33} = \frac{c_3\phi_{10}}{m\phi_{10}+1} + a_3\phi_{11} - d_1 - \beta\,\phi_{13}$,
$m_{44} = \frac{c_4\phi_{10}}{m\phi_{10}+1} + a_4\phi_{11} - d_2 + \beta\,\phi_{12}$
The characteristic function are $\Lambda_4(\lambda) = E_0\lambda^4 + E_1\lambda^3 + E_2\lambda^2 + E_3\lambda + E_4$
where
$E_0 = 1$
$E_1 = (-m_{44} - m_{33} - m_{22} - m_{11})$
$E_2 = \frac{1}{m\phi_{10}+1}(\beta^2 m\phi_{10}\phi_{12}\phi_{13} + \beta^2\phi_{13}\phi_{12} + ma_4c_2\phi_{10}\phi_{11}\phi_{13} + \phi_{13}a_4\phi_{11}c_2 + mm_{11}m_{44}\phi_{10} + mm_{22}m_{44}\phi_{10} + mm_{33}m_{44}\phi_{10} + m_{11}m_{44} + m_{22}m_{44} + m_{33}m_{44} + \alpha^2 m\phi_{10}{}^2\phi_{11} + \phi_{11}\alpha^2\phi_{10} + mm_{11}m_{22}\phi_{10} + mm_{11}m_{33}\phi_{10} + mm_{22}m_{33}\phi_{10} + m_{31}\phi_{10}c_1 + m_{22}m_{11} + m_{11}m_{33} + m_{22}m_{33})$
$E_3 = -\frac{1}{m\phi_{10}+1}(ma_4c_2m_{11}\phi_{10}\phi_{11}\phi_{13} + ma_4c_2m_{33}\phi_{10}\phi_{11}\phi_{13} + \alpha\, c_2m_{41}\phi_{10}\phi_{11} + \beta\, c_1\, m_{41}\phi_{10}\phi_{12} + m_{11}m_{22}m_{33} + m_{44}m_{11}m_{33} + m_{44}m_{22}m_{33} + m_{44}m_{22}m_{11} + \alpha^2 m_{33}\phi_{10}\phi_{11} + m_{31}\phi_{10}c_1\, m_{22} + c_1\, m_{31}m_{44}\phi_{10} + m_{44}\phi_{11}\alpha^2\phi_{10} + \beta^2 m_{11}\phi_{12}\phi_{13} + \beta^2 m_{22}\phi_{12}\phi_{13} + m_{44}mm_{11}m_{33}\phi_{10} + m_{44}mm_{22}m_{33}\phi_{10} + m_{44}\alpha^2 m\phi_{10}{}^2\phi_{11} + m_{44}mm_{11}m_{22}\phi_{10} + a_4c_2m_{11}\phi_{11}\phi_{13} + a_4c_2m_{33}\phi_{11}\phi_{13} + \alpha^2 mm_{33}\phi_{10}{}^2\phi_{11} + mm_{11}m_{22}m_{33}\phi_{10} - \beta\, a_3c_2\phi_{11}\phi_{12}\phi_{13} + \beta^2 mm_{11}\phi_{10}\phi_{12}\phi_{13} + \beta^2 mm_{22}\phi_{10}\phi_{12}\phi_{13} - \beta\, ma_3c_2\phi_{10}\phi_{11}\phi_{12}\phi_{13} - \phi_{12}a_3\phi_{11}\alpha\,\phi_{10}c_1 + \alpha\, mc_2m_{41}\phi_{10}{}^2\phi_{11})$
$E_4 = \frac{1}{m\phi_{10}+1}(m_{33}\alpha\, c_2m_{41}\phi_{10}\phi_{11} + m_{22}\beta\, c_1\, m_{41}\phi_{10}\phi_{12} + \phi_{13}ma_4c_2m_{11}m_{33}\phi_{10}\phi_{11} + c_1\, m_{22}m_{31}m_{44}\phi_{10} + m_{44}m_{11}m_{22}m_{33} +$

$\phi_{13}\beta^2 m_{11} m_{22}\phi_{12} + m_{44}\alpha^2 m_{33}\phi_{10}\phi_{11} - \phi_{13} a_4 \phi_{11}\alpha\,\beta\, c_1\,\phi_{10}\phi_{12} - \phi_{13}\beta\, m a_3 c_2 m_{11}\phi_{10}\phi_{11}\phi_{12} + m_{33}\alpha\, m c_2 m_{41}{\phi_{10}}^2\phi_{11} + c_1\, a_4 c_2 m_{31}\phi_{10}\phi_{11}$
$c_1\, a_3 c_2 m_{41}\phi_{10}\phi_{11}\phi_{12} - \alpha\,\beta\, c_2 m_{31}\phi_{10}\phi_{11}\phi_{13} + \phi_{13}\alpha^2\beta^2 m{\phi_{10}}^2\phi_{11}\phi_{12} + \phi_{13}\beta^2 m m_{11} m_{22}\phi_{10}\phi_{12} - m_{44}\phi_{12} a_3\phi_{11}\alpha\,\phi_{10} c_1 -$
$\phi_{13}\beta\, a_3 c_2 m_{11}\phi_{11}\phi_{12} - \alpha\,\beta\, m c_2 m_{31}{\phi_{10}}^2\phi_{11}\phi_{13} + \phi_{13} a_4 c_2 m_{11} m_{33}\phi_{11} + \phi_{13}\alpha^2\beta^2\phi_{10}\phi_{11}\phi_{12} + m_{44}\alpha^2 m m_{33}{\phi_{10}}^2\phi_{11} + m_{44} m m_{11} m_{22} m_{33}\phi_{10}$

By Routh Hurwitzs criterion, all the eigenvalues of J_{12} have negative real parts if $(i) E_0 > 0$,
$(ii) E_1 > 0$,
$(iii) E_3 > 0$,
$(vi) E_1 E_2 E_3 > E_3^2 + E_1^2 E_4$. We observe that the system (2.1) is locally asymptotically stable around the positive equilibrium point (12) if the conditions stated in the theorem holds.

8. Numerical solution

In this section, we have performed numerical solution are equally important beside the analytical findings to verify them. In this section we present computer simulation of different solutions of the system (2.1) using maple 18 programming. First we take the parameters of the system as $\rho_1 = (\alpha = 1, \beta = 2, c_1 = 1, m = 1, r_1 = 1, r_2 = 1, a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, c_2 = 1, c_3 = 1, c_4 = 1, d_1 = 1, d_2 = 1)$. Then the initial conditions satisfied $W(0) = 0, X(0) = 0, Y(0) = 0, Z(0) = 1$ is infected predator population (see Figure 1) that is a periodic point is 0.367879356307219.

If we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 0, X(0) = 1, Y(0) = 0, Z(0) = 0$ is infected prey population (see Figure 2) that is a periodic point is 1.

If we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 0, X(0) = 0, Y(0) = 1, Z(0) = 0$ is susceptible predator population (see Figure 3) that is a periodic point is 0.367879356307219

If we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 1, X(0) = 0, Y(0) = 0, Z(0) = 0$ is susceptible prey population (see Figure 4) that is a periodic point is 1.

Now we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 0, X(0) = 0, Y(0) = 1, Z(0) = 1$ is both susceptible–infected predator population (see Figure 5).

Now we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = .5, X(0) = 0, Y(0) = 0, Z(0) = .5$ is susceptible prey and infected predator population (see Figure 6).

Now we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 0, X(0) = 1, Y(0) = 1, Z(0) = 0$ is susceptible predator and infected prey population (see Figure 7) that is a periodic solution.

Now we take the parameters of the system as ρ_1. Then the initial conditions satisfied $W(0) = 1, X(0) = 1, Y(0) = 0, Z(0) = 0$ is both susceptible–infected prey population (see Figure 8).

If we take the parameters of the system as $\rho_2 = (\alpha, \beta = 0, 0.2, 0.5, 0.8, 1, c_1 = 1, m = 1, r_1 = 1, r_2 = 1, a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1, c_2 = 1, c_3 = 1, c_4 = 1, d_1 = 1, d_2 = 1)$. Then the initial conditions satisfied $W(0) = 1, X(0) = 1, Y(0) = 1, Z(0) = 1$ is both susceptible–infected prey and predator population. Obtain $\alpha, \beta = 0, 0.2, 0.5, 0.8, 1$ are different type of the figure. Susceptible–infected predator range is decreasing and susceptible–infected prey range is increasing (see Figure 9, 10, 11, 12, 13).

If we take the parameters of the system as ρ_2. Then the initial conditions satisfied $W(0) = 0.1, X(0) = 0.1, Y(0) =$

$0.1, Z(0) = 0.1$ is both susceptible–infected prey and predator population. Obtain $\alpha, \beta = 0, 0.2, 0.5, 0.8, 1$ are different type of the figure. Susceptible–infected predator range is decreasing and susceptible–infected prey range is increasing (see Figure 14,15, 16, 17, 18).

Conclusion

The present investigation is being carried out to observe the eco-epidemiological model with the assumption that both prey species diseases with susceptible–infection and both predator species diseases susceptible–infection. Thus, we may conclude that sometimes variable observe stability and equilibrium point, positive and boundedness. Moreover, our numerical simulation suggests that in the presence of the environmental fluctuation, the stability analysis is locally asymptotically stable with different equilibrium point.

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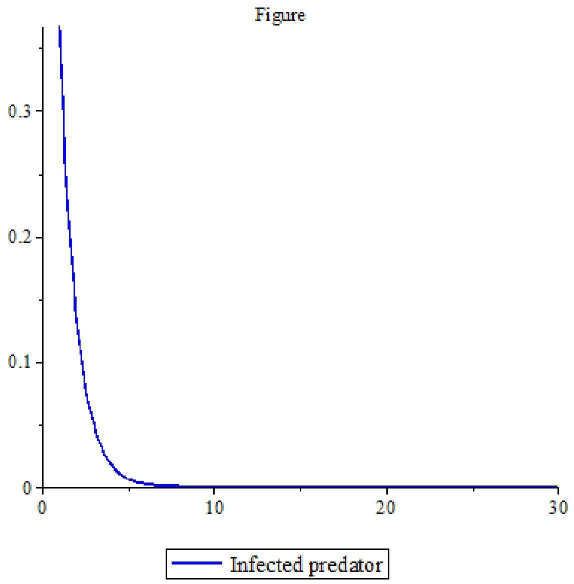

Figure 1: The infected predator population.

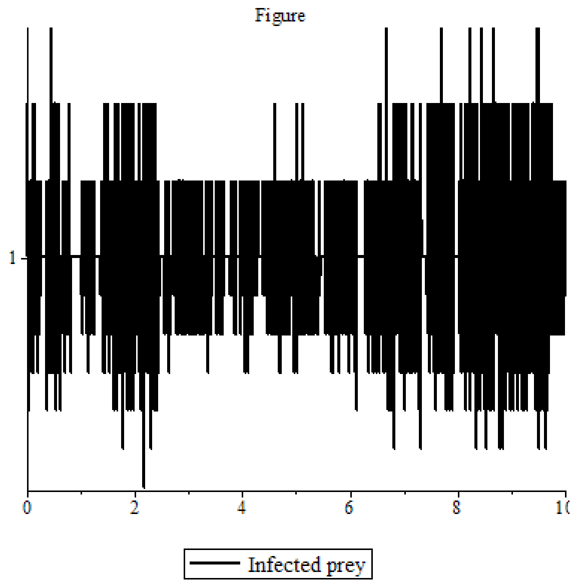

Figure 2: The infected prey population.

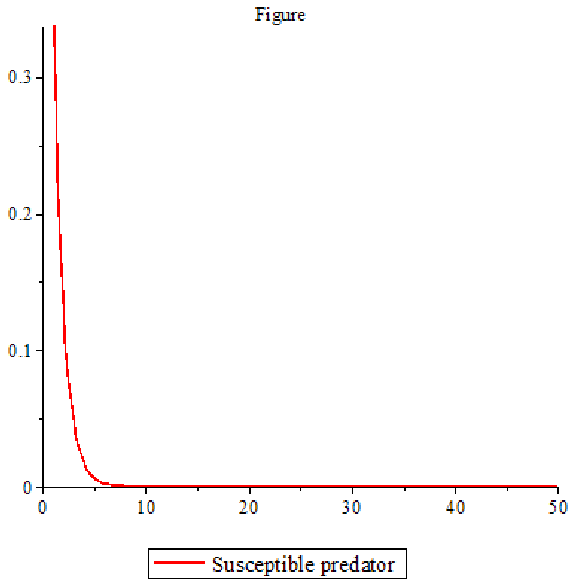

Figure 3: The susceptible predator population.

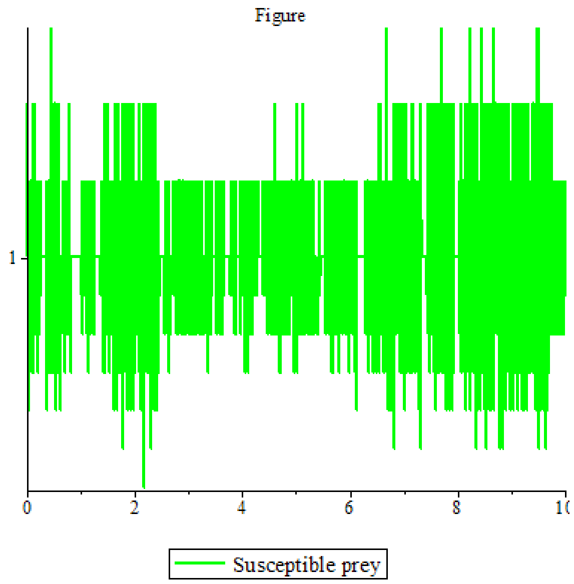

Figure 4: The susceptible prey population.

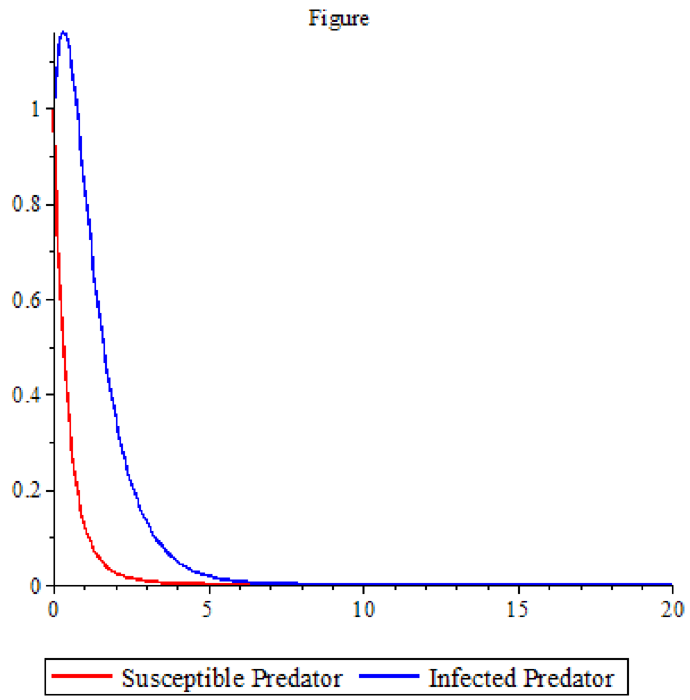

Figure 5: The susceptible prey population.

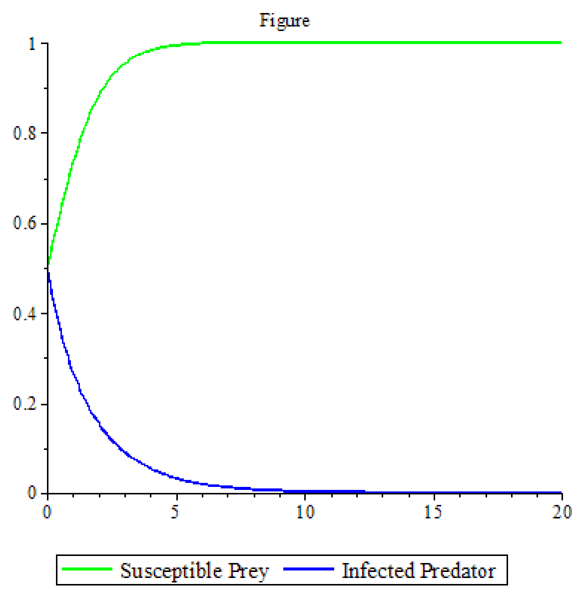

Figure 6: Interaction of the susceptible prey and infected predator population.

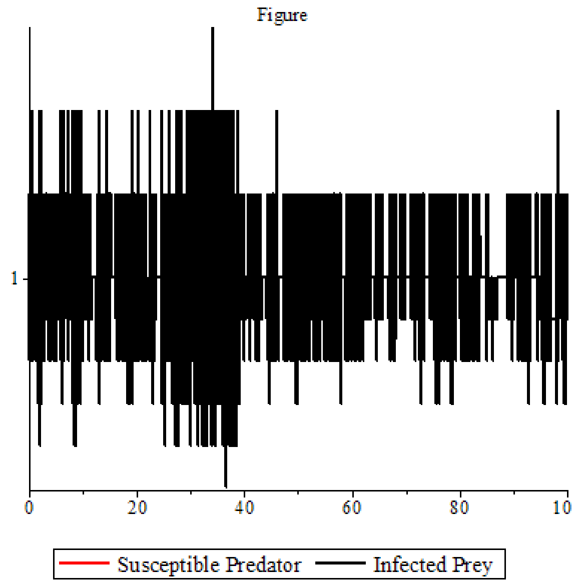

Figure 7: Interaction of the susceptible predator and infected prey population.

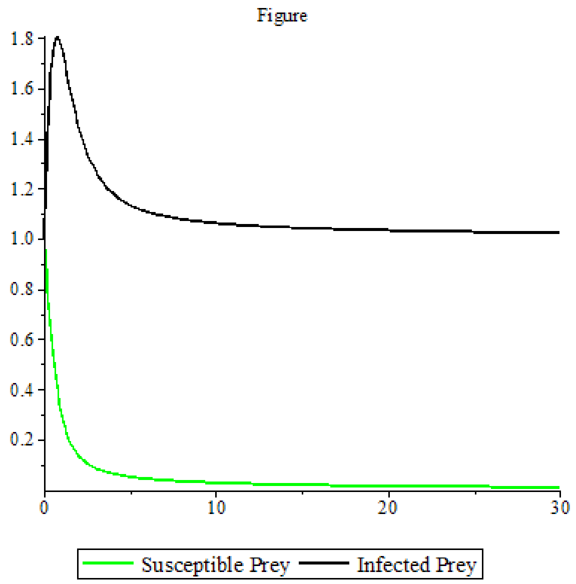

Figure 8: Interaction of the susceptible prey and infected prey population.

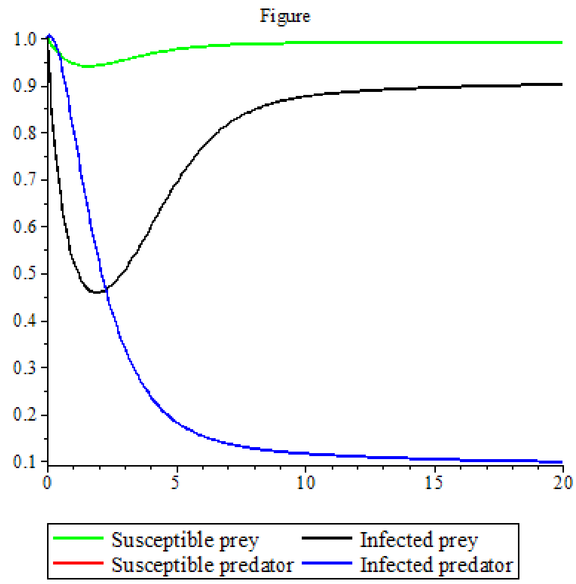

Figure 9: If we take the parameters of the system as ρ_2 Then the initial conditions satisfied $W(0) = 1, X(0) = 1, Y(0) = 1, Z(0) = 1$ is interaction of the both prey predator population for $\alpha, \beta = 0$.

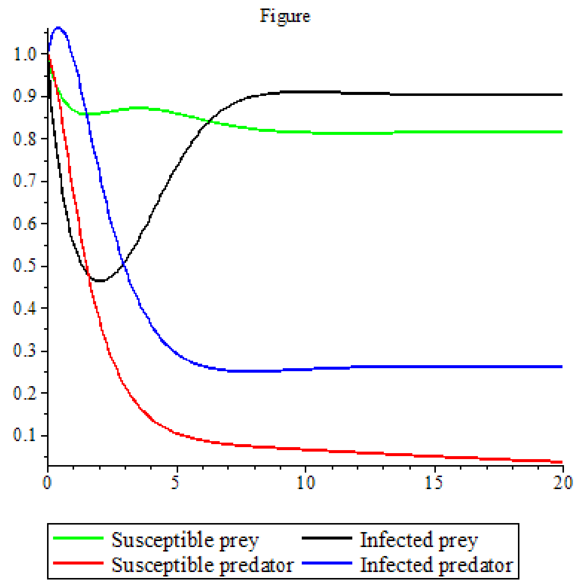

Figure 10: If we take the parameters of the system as ρ_2 Then the initial conditions satisfied $W(0) = 1, X(0) = 1, Y(0) = 1, Z(0) = 1$ is interaction of the both prey predator population for $\alpha, \beta = 0.2$

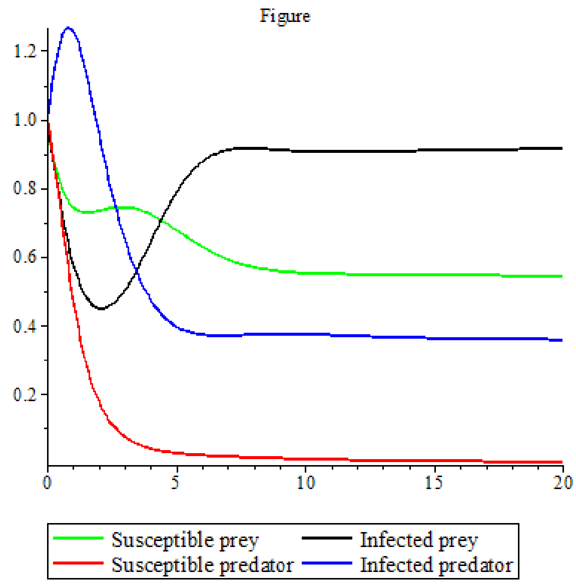

Figure 11: If we take the parameters of the system as ρ_2 Then the initial conditions satisfied $W(0) = 1, X(0) = 1, Y(0) = 1, Z(0) = 1$ is interaction of the both prey predator population for $\alpha, \beta = 0.5$

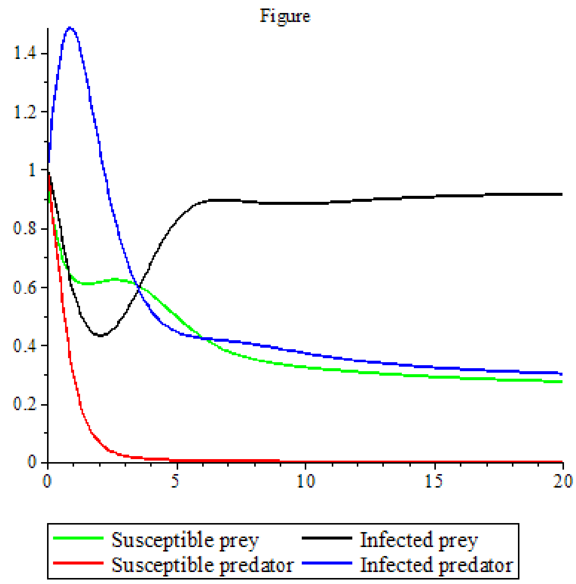

Figure 12: If we take the parameters of the system as ρ_2 Then the initial conditions satisfied $W(0) = 1, X(0) = 1, Y(0) = 1, Z(0) = 1$ is interaction of the both prey predator population for $\alpha, \beta = 0.8$.

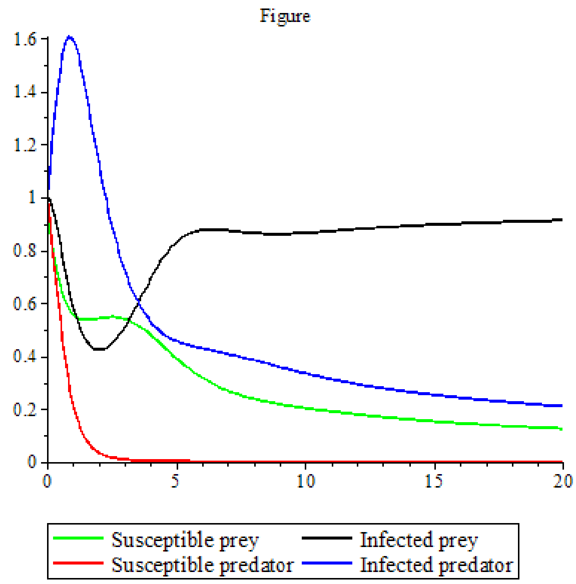

Figure 13: If we take the parameters of the system as ρ_2 Then the initial conditions satisfied $W(0) = 1, X(0) = 1, Y(0) = 1, Z(0) = 1$ is interaction of the both prey predator population for $\alpha, \beta = 1$

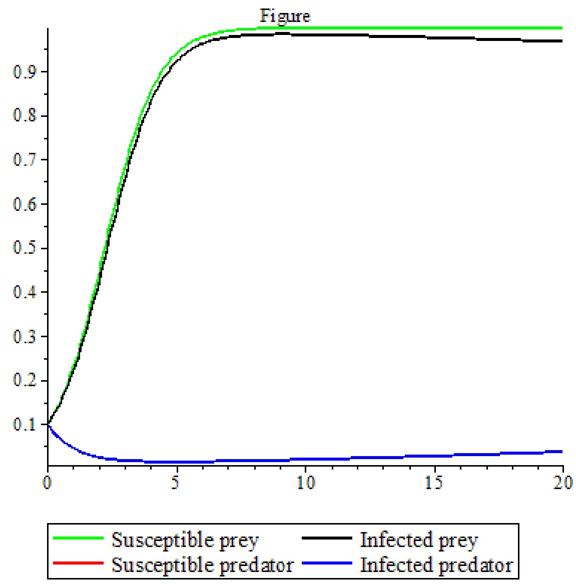

Figure 14: If we take the parameters of the system as ρ_2 Then the initial conditions satisfied $W(0) = 0.1, X(0) = 0.1, Y(0) = 0.1, Z(0) = 0.1$ is interaction of the both prey predator population for $\alpha, \beta = 0$

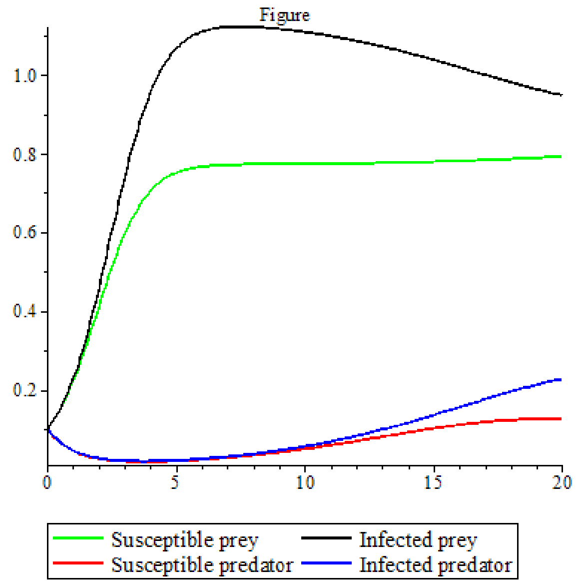

Figure 15: If we take the parameters of the system as ρ_2 Then the initial conditions satisfied $W(0) = 0.1, X(0) = 0.1, Y(0) = 0.1, Z(0) = 0.1$ is interaction of the both prey predator population for $\alpha, \beta = 0.2$

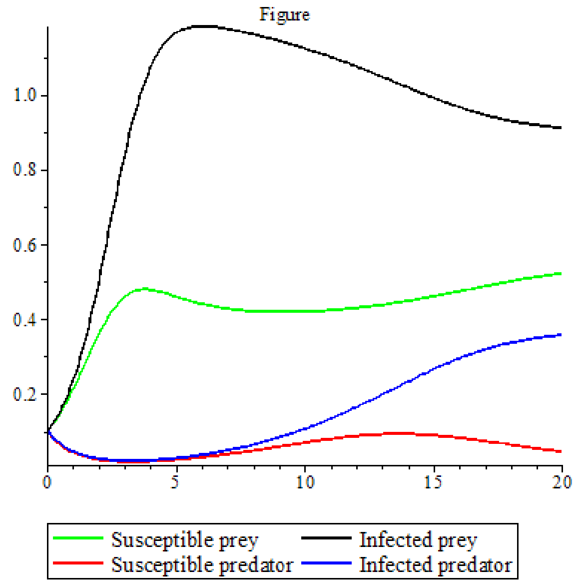

Figure 16: If we take the parameters of the system as ρ_2 Then the initial conditions satisfied $W(0) = 0.1, X(0) = 0.1, Y(0) = 0.1, Z(0) = 0.1$ is interaction of the both prey predator population for $\alpha, \beta = 0.5$

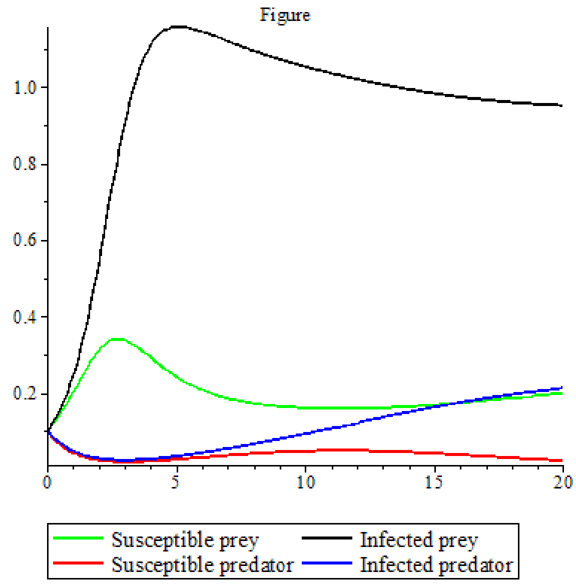

Figure 17: If we take the parameters of the system as ρ_2 Then the initial conditions satisfied $W(0) = 0.1, X(0) = 0.1, Y(0) = 0.1, Z(0) = 0.1$ is interaction of the both prey predator population for $\alpha, \beta = 0.8$

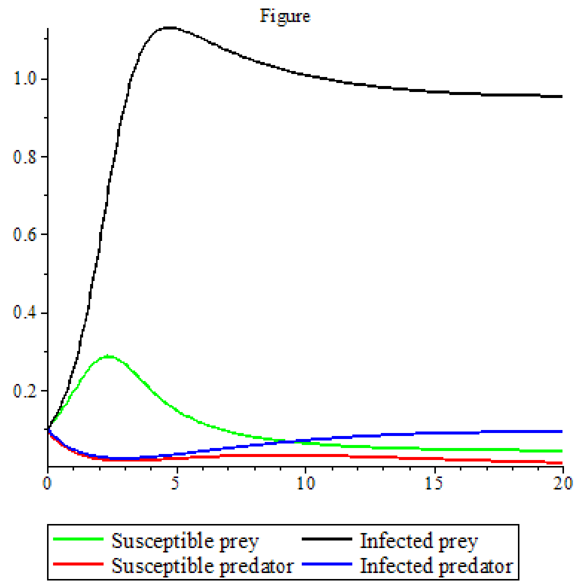

Figure 18: If we take the parameters of the system as ρ_2 Then the initial conditions satisfied $W(0) = 0.1, X(0) = 0.1, Y(0) = 0.1, Z(0) = 0.1$ is interaction of the both prey predator population for $\alpha, \beta = 1$