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On a Conjecture on Unimodal Sequences

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ABSTRACT. Wang and Yeh proved that if $P(x)$ is a polynomial with non- negative and nondecreasing coefficients, then $P(x+d)$ is unimodal for any $d > 0$. A mode of a unimodal polynomial $f(x) = a_0 + a_1x + \cdots + a_mx^m$ is an index k such that a_k is the maximum coefficient. Suppose that $M_*(P, d)$ is the smallest mode of $P(x+d)$, and $M^*(P, d)$ the greatest mode. Wang and Yeh conjectured that if $d_2 > d_1 > 0$, then $M_*(P, d_1) \geq M_*(P, d_2)$ and $M^*(P, d_1) \geq M^*(P, d_2)$. This conjecture has already been proved in [7] but we give a different proof of this conjecture. We also show that if $\{d_j : \ 0 \leq j \leq m\}$ is a unimodal sequence, then there is a polynomial $p(x) = \sum_{i=0}^{m} a_ix^i$ with nonnegative and non- decreasing coefficients such that $p(x+n) = \sum_{j=0}^{m} d_jx^j$, where n is a positive integer number. Furthermore, we define the almost unimodal sequences and prove that under some conditions the polynomial $p(x^k + d)$ for any positive real number d and integer number $k \geq 2$, is almost unimodal.

1 Introduction

A finite sequence of real numbers $\{d_0, d_1, \cdots, d_m\}$ is said to be unimodal if there exists an index $i (0 \leq i \leq m)$, called the mode of the sequence, such that $d_0 \leq d_1 \leq \cdots \leq d_i \geq d_{i+1} \geq \cdots \geq d_m$. A polynomial is said to be unimodal if its sequence of coefficients is unimodal. Unimodal polynomials arise often in Combinatorics, Geometry and Algebra. The reader is referred to [3] and [7] for surveys of the diverse techniques employed to prove that specific families of polynomials are unimodal. Let $M_*(p, d)$ and $M^*(p, d)$ be the smallest and the greatest mode of $P(x+d)$, respectively. Put $\overline{m(d)} = \lceil \frac{m-d}{d+1} \rceil$ and $\underline{m(d)} = \lfloor \frac{m}{d+1} \rfloor$, where $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the

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least integer $\geq x$ and the greatest integer $\leq x$ respectively. When there is no danger of confusion, we simply write $\overline{m}$ instead of $\overline{m(d)}$. Throughout this paper, let m be a positive integer and d a positive real number. We denote the set of monic polynomials of degree m with nonnegative and non-decreasing coefficients by $p_{\uparrow}^{m}$. We recall few basic results concerning the unimodality.

Theorem 1.1. ([3]). *If p is a polynomial with positive non-decreasing coefficients, then $p(x+1)$ is unimodal.*

Theorem 1.2. ([4]). *Let $0 \leq a_0 \leq a_1 \leq \cdots \leq a_m$ be a sequence of real numbers and consider the polynomial*

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m.$$

Then the polynomial $p(x+n)$ is unimodal with mode $\lfloor \frac{m}{n+1} \rfloor$ where $n \in N$.

Theorem 1.3. ([6]). *Let $p(x)$ be a polynomial of degree m with nonnegative coefficients. Suppose that $p(x)$ is non-decreasing and d is a positive real number. Then $p(x+n)$ is unimodal.*

We now state the following conjecture.

Conjecture 1.1. suppose that $p \in p_{\uparrow}^{m}$ and $0 < d_1 < d_2$. Then $M_*(P, d_1) \geq M_*(P, d_2)$ and $M^*(P, d_1) \geq M^*(P, d_2)$.

In this paper, we solve the above conjecture. We also characterize the unimodal sequences $\{d_j\}$ that appear in [5] and discuss the behavior of the coefficients of $p(x+n)$ for a unimodal polynomial $p(x)$. Moreover, we prove that under same conditions of [5], the polynomial $p(x^k + d)$ is almost unimodal, when k is an integer number ≥ 2.

2 Proof of conjecture 1.1

In this section we are going to prove the conjecture 1.1.

To prove this conjecture, let $p(x) = \sum_{i=0}^{m} a_i x^i$ and $p(x+d_1) = \sum_{j=0}^{m} b_j x^j$ where $b_j = \sum_{i=j}^{m} a_i d_1^{i-j} \binom{i}{j}$, $p(x+d_2) = \sum_{k=0}^{m} c_k x^k$ where $c_k = \sum_{i=k}^{m} a_i d_2^{i-k} \binom{i}{k}$. Let $M_*(p, d_2) = t$. Then by the definition, we can write $c_t > c_{t-1}$. We need to show that $M_*(p, d_1) \geq t$ otherwise $b_t \geq b_{t-1}$.

For $1 \leq t \leq \overline{m}$ let $r = \lceil (d_1+1)t \rceil$, then $t \leq r \leq m$. We have

$$\begin{aligned} t d_1^t (b_t - b_{t-1}) &= \sum_{i=t-1}^{m} a_i d_1^i \binom{i}{t-1} [(i+1) - (d_1+1)t] \\ &= \sum_{i=r}^{m} a_i d_1^i \binom{i}{t-1} [(i+1) - (d_1+1)t] \\ &- \sum_{i=t-1}^{r-1} a_i d_1^i \binom{i}{t-1} [(d_1+1)t - (i+1)] \end{aligned} \tag{1}$$

On the other hand if $i \leq r-1$ then

$$\begin{aligned} & i \leq r-1 = \lceil (d_1+1)t \rceil - 2 < (d_1+1)t - 1 \qquad \text{(by } x \leq \lceil x \rceil < x+1) \\ \Longrightarrow \quad & i+1 < (d_1+1)t. \end{aligned}$$

Since $d_1 < d_2$ implies $i+1 < (d_1+1)t < (d_2+1)t$, thus

$$(d_1+1)t - (i+1) < (d_2+1)t - (i+1). \tag{2}$$

By 1 and 2, we have

$$\begin{aligned} td_1^t(b_t - b_{t-1}) \geq & \sum_{i=r}^{m} a_i d_1^i \binom{i}{t-1} [(i+1) - (d_1+1)t] \\ & - \sum_{i=t-1}^{r-1} a_i d_2^i \binom{i}{t-1} [(d_2+1)t - (i+1)] \end{aligned} \tag{3}$$

And also $c_t - c_{t-1} > 0$ implies that

$$\begin{aligned} td_2^t(c_t - c_{t-1}) = & \sum_{i=t-1}^{m} a_i d_2^i \binom{i}{t-1} [(i+1) - (d_2+1)t] \\ = & \sum_{i=r}^{m} a_i d_2^i \binom{i}{t-1} [(i+1) - (d_2+1)t] \\ & - \sum_{i=t-1}^{r-1} a_i d_2^i \binom{i}{t-1} [(d_2+1)t - (i+1)] > 0 \end{aligned}$$

$$\Rightarrow \sum_{i=t-1}^{r-1} a_i d_2^i \binom{i}{t-1} [(d_2+1)t - (i+1)] > -\sum_{i=r}^{m} a_i d_2^i \binom{i}{t-1} [(i+1) - (d_2+1)t] \tag{4}$$

Thus from 3 and 4 we have

$$\begin{aligned} td_1^t(b_t - b_{t-1}) > & \sum_{i=r}^{m} a_i d_1^i \binom{i}{t-1} [(i+1) - (d_1+1)t] \\ & - \sum_{i=r}^{m} a_i d_2^i \binom{i}{t-1} [(i+1) - (d_2+1)t] \\ = & \sum_{i=r}^{m} a_i \binom{i}{t-1} [d_1^i((i+1) - (d_1+1)t) - d_2^i((i+1)) - (d_2+1)t)] \\ = & \sum_{i=r}^{m} a_i \binom{i}{t-1} [(i+1-t)(d_1^i - d_2^i) + t(d_2^{i+1} - d_1^{i+1})]. \end{aligned} \tag{5}$$

Now by $i \geq r$ we have

$$i \geq r = \lceil (d_1+1)t \rceil - 1 \geq (d_1+1)t - 1$$

$$\Rightarrow \quad i + 1 - t \geq td_1.$$

Thus

$$\begin{aligned} (i+1-t)(d_1^i - d_2^i) + t(d_2^{i+1} - d_1^{i+1}) \geq & td_1(d_1^i - d_2^i) + t(d_2^{i+1} - d_1^{i+1}) \\ = & td_1^{i+1} - td_1 d_2^i + td_2^{i+1} - td_1^{i+1} \\ = & t(d_2^{i+1} - d_1 d_2^i) > 0. \qquad (\text{by } 0 < d_1 < d_2) \end{aligned}$$

We conclude that every term in the sum 5 are positive. So $b_t - b_{t-1} > 0$ Which implies that $M_*(p, d_1) \geq t$ otherwise $M_*(p, d_1) \geq M_*(p, d_2)$.

Now assume that $M^*(p, d_1) = t$. Then by definition, we can write $b_t > b_{t+1}$. Therefore, it is enough to prove

$M^*(p, d_2) \leq t$. Otherwise $c_t \geq c_{t+1}$. For $1 \leq t \leq \overline{m}$ let $r = \lceil (d_1+1)t \rceil - 1$, then $t \leq r \leq m$. We have

$$\begin{aligned}
-(t+1)d_2^{t+1}(c_{t+1}-c_t) &= -\sum_{i=t}^{m} a_i d_2^i \binom{i}{t}[(i+1)-(d_2+1)(t+1)] \\
&= -\sum_{i=r}^{m} a_i d_2^i \binom{i}{t}[(i+1)-(d_2+1)(t+1)] \\
&\quad + \sum_{i=t}^{r-1} a_i d_2^i \binom{i}{t}[(d_2+1)(t+1)-(i+1)] \\
&> -\sum_{i=r}^{m} a_i d_2^i \binom{i}{t}[(i+1)-(d_2+1)(t+1)] \\
&\quad + \sum_{i=t}^{r-1} a_i d_1^i \binom{i}{t}[(d_1+1)(t+1)-(i+1)] \qquad (6)
\end{aligned}$$

On the other hand $b_{t+1} - b_t < 0$ implies that

$$\begin{aligned}
(t+1)d_1^{t+1}(b_{t+1}-b_t) &= \sum_{i=r}^{m} a_i d_1^i \binom{i}{t}[(i+1)-(d_1+1)(t+1)] \\
&\quad - \sum_{i=t}^{r-1} a_i d_1^i \binom{i}{t}[(d_1+1)(t+1)-(i+1)] < 0
\end{aligned}$$

$$\Rightarrow \sum_{i=t}^{r-1} a_i d_1^i \binom{i}{t}[(d_1+1)(t+1)-(i+1)] > \sum_{i=r}^{m} a_i d_1^i \binom{i}{t}[(i+1)-(d_1+1)(t+1)] \qquad (7)$$

Thus from 6 and 7 we have

$$\begin{aligned}
-(t+1)d_2^{t+1}(c_{t+1}-c_t) &> -\sum_{i=r}^{m} a_i d_2^i \binom{i}{t}[(i+1)-(d_2+1)(t+1)] \\
&\quad + \sum_{i=r}^{m} a_i d_1^i \binom{i}{t}[(i+1)-(d_1+1)(t+1)] \\
&= \sum_{i=r}^{m} a_i \binom{i}{t}[(i-t)(d_1^i - d_2^i) + (t+1)(d_2^{i+1} - d_1^{i+1})] \qquad (8)
\end{aligned}$$

Now by $i \geq r$ we have $(i-t) \geq td_1 - 1$. Thus we get the following

$$\begin{aligned}
(i-t)(d_1^i - d_2^i) + (t+1)(d_2^{i+1} - d_1^{i+1}) &\geq (td_1 - 1)(d_1^i - d_2^i) + (t+1)(d_2^{i+1} - d_1^{i+1}) \\
&= td_1^{i+1} - td_1 d_2^i + (d_2^i - d_1^i) + td_2^{i+1} - td_1^{i+1} + (d_2^{i+1} - d_1^{i+1}) \\
&= (td_2^{i+1} - td_1 d_2^i) + (d_2^i - d_1^i) + (d_2^{i+1} - d_1^{i+1}) > 0.
\end{aligned}$$

So $d_1 < d_2$ implies that the above statement is positive. Then from 8 we get $c_t > c_{t+1}$, that implies $M^*(p, d_2) \leq t$. Otherwise $M^*(p, d_1) \geq M^*(p, d_2)$.

3 The convers of the criterion for unimodality

In this section we discuss the following inverse question : Given a unimodal sequence $\{d_j : \ 0 \leq j \leq m\}$, is there a polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ with nonnegative and non-decreasing coefficients such that $p(x+n) = \sum_{j=0}^{m} d_j x^j$? We begin by expressing the conditions on $\{a_j\}$ that guaranteed unimodality of $p(x+n)$ in terms of the coefficients $\{d_j\}$. First we let $p(x+n) = \sum_{j=0}^{m} d_j x^j$. Then

$$p(x) = \sum_{j=0}^{m} d_j (x-n)^j = \sum_{j=0}^{m} d_j \sum_{k=0}^{j} \binom{j}{k} (-n)^{j-k} x^j$$
$$= \sum_{j=0}^{m} a_j x^j \tag{9}$$

And

$$a_j = \sum_{k=j}^{m} (-1)^{k-j} n^{k-j} d_k \binom{k}{j} \tag{10}$$

Now, we investigate the necessary and sufficient condition for a_i to be non-negative.

Lemma 3.1. *Let $0 \le j \le m$. Then the following holds*

$$a_j \ge 0 \iff d_j \ge \sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j} d_k \binom{k}{j} \tag{11}$$

Proof.

$$\begin{aligned} a_j \ge 0 &\iff \sum_{k=j}^{m} (-1)^{k-j} n^{k-j} d_k \binom{k}{j} \ge 0 \qquad \text{(by 10)} \\ &\iff \sum_{k=j+1}^{m} (-1)^{k-j} n^{k-j} d_k \binom{k}{j} + d_j \ge 0 \\ &\iff d_j \ge -\sum_{k=j+1}^{m} (-1)^{k-j} n^{k-j} d_k \binom{k}{j} \\ &\iff d_j \ge \sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j} d_k \binom{k}{j}. \end{aligned}$$

□

Another condition that is needed for unimodality of $p(x+n)$, is nondecreasing of the sequence of coefficients of $p(x)$, which we state in the following lemma.

Lemma 3.2. *Let $0 \le j \le m$. Then*

$$a_j \le a_{j+1} \iff d_j \le \sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j-1} d_k \binom{k+1}{j+1} \left(\frac{k+j(n-1)+n}{k+1} \right). \tag{12}$$

Proof.

$$\begin{aligned} a_{j+1} - a_j &= \sum_{k=j+1}^{m} (-1)^{k-j-1} n^{k-j-1} d_k \binom{k}{j+1} - \sum_{k=j}^{m} (-1)^{k-j} n^{k-j} d_k \binom{k}{j} \\ &= \sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j-1} d_k \binom{k}{j+1} + n \sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j-1} d_k \binom{k}{j} - d_j \\ &= \sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j-1} d_k \left[\binom{k}{j+1} + n \binom{k}{j} \right] - d_j \\ &= \sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j-1} d_k \binom{k+1}{j+1} \left(\frac{k+j(n-1)+n}{k+1} \right) - d_j. \end{aligned}$$

Thus we get

$$\begin{aligned} a_j \le a_{j+1} \quad &\Longleftrightarrow \quad \sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j-1} d_k \binom{k+1}{j+1} \left(\frac{k+j(n-1)+n}{k+1} \right) - d_j \ge 0 \\ &\Longleftrightarrow \quad d_j \le \sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j-1} d_k \binom{k+1}{j+1} \left(\frac{k+j(n-1)+n}{k+1} \right). \end{aligned}$$

□

Now, we combine the previous two lemmas to produce a criterion for unimodality.

Theorem 3.3. *Let $Q(x) = d_0 + d_1 x + \cdots + d_m x^m$ and assume the coefficients $\{d_j\}$ satisfy the inequalities*

$$\sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j} d_k \binom{k}{j} \le d_j \le \sum_{k=j+1}^{m} (-1)^{k-j+1} n^{k-j-1} d_k \binom{k+1}{j+1} \left(\frac{k+j(n-1)+n}{k+1} \right) \tag{13}$$

Then $Q(x)$ is a unimodal polynomial for which $P(x) := Q(x-n)$ has positive and non-decreasing coefficients.

Proof.

$$\begin{aligned} Q(x) &= \sum_{j=0}^{m} d_j x^j \\ Q(x-n) &= \sum_{j=0}^{m} d_j (x-n)^j \\ &= \sum_{j=0}^{m} d_j \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} n^{j-k} x^k \\ &= \sum_{j=0}^{m} \left(\sum_{k=j}^{m} d_j \binom{j}{k} (-1)^{k-j} n^{k-j} \right) x^j \end{aligned}$$

And by 10 we get $Q(x-n) = \sum_{j=0}^{m} a_j x^j$. Thus from 9we have $p(x) = Q(x-n)$. So inequalities 13and two lemmas 3.1 and 3.2 implies that $p(x)$ has nonnegative, nondecreasing coefficients.Therefore Theorem 1.2 yields the result. □

4 An extension of a criterion for almost unimodality

There exists a set of polynomials in which the sequences of their coefficients has several zeros. For instance, in $p(x) = \sum_{i=0}^{m} a_i x^i$, if x goes to the power of integer $k \ge 2$, then the sequence of coefficients of $p(x^k) = \sum_{i=0}^{m} a_i x^{ki}$ will be

$$a_0 00 \cdots 0 a_1 00 \cdots 0 a_2 \cdots$$

For such sequences, we will use the concept of almost unimodality. A finite sequence of real numbers $\{c_0, c_1, \cdots, c_m\}$ is called **almost nondecreasing** if it is nondecreasing excepting a subsequence which is zero. It is clear that, if the sequence $\{c_0, c_1, \cdots, c_m\}$ is nondecreasing, then it is almost nondecreasing. The converse is not true, as we can see from the following example. The sequence $\{0, 1, 0, 2, 0, 3, \cdots, 0, m\}$ is almost nondecreasing but it is not non-decreasing. A finite sequence of real numbers $\{d_0, d_1, \cdots, d_m\}$ is called **almost unimodal** if there exists an index

$i(0 \leq i \leq m)$, such that d_j almost increases up to $j = i$ and almost decreases from then on. Also, a polynomial is is said to be almost unimodal, if its sequence of coefficients is almost unimodal. A criterion for almost unimodality is appear in [1] as follow:

Theorem 4.1. *Let* $0 \leq a_0 \leq a_1 \leq \cdots \leq a_m$ *be a sequence of real numbers, n be a positive integer number and* $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ *be an arbitrary polynomial. Then for any integer* $k \geq 2$*, the polynomial* $p(x^k + n)$ *is almost unimodal.*

Now we would like to show that this criterion is true for real and positive number.

Theorem 4.2. *Let* $0 \leq a_0 \leq a_1 \leq \cdots \leq a_m$ *be a sequence of real numbers, and consider the polynomial* $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$*. Then for any integer* $k \geq 2$*, the polynomial* $p(x^k + d)$ *is almost unimodal.*

Proof. First note that if $Q(x) = \sum_{j=0}^{m} b_jx^j$ is a unimodal polynomial, then its sequence of coefficients is unimodal. Further $Q(x^k) = \sum_{j=0}^{m} b_jx^{kj}$, and sequence that arise with excepting a subsequence of coefficients of $Q(x^k)$ which is zero, is the sequence of coefficients of $Q(x)$. Since this sequence is unimodal, then according to the definition, $Q(x^k)$ is almost unimodal. Applying Theorem 1.3, we conclude that $p(x+d)$ is unimodal and now using the remark above it follow that $p(x^k + d)$ is almost unimodal. □

5 Remark

Our results can be restated in terms of sequences instead of polynomials. In this paper we peresented a conjecture on comparison of modes of two polynomials. It often occurs that unimodality of a sequence is known, but to find out the exact number and location of modes of the sequence is a much more difficult task. For example, it is well known that, for each positive integer n, the Stirling number of the second kind $S(n,k)$ is unimodal in k with at most two modes K_n, $K_n + 1$. However it is very difficult to determine whether the mode of $S(n,k)$ is unique or not. See [2],[6] for the related results.

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