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The tightly super 2-extra connectivity and 2-extra diagnosability of crossed cubes

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ABSTRACT. Connectivity and diagnosability play an important role in measuring the fault tolerance of interconnection networks. As a topology structure of interconnection networks, the n-dimensional crossed cube CQ_n has many good properties. In this paper, we study the 2-extra connectivity and 2-extra diagnosability of CQ_n. We prove that CQ_n is tightly $(3n-5)$ super 2-extra connected for $n \geq 5$, and the 2-extra diagnosability of CQ_n is $3n-3$ under the PMC model $(n \geq 5)$ and MM* model $(n \geq 6)$.

1 Introduction

With the rapid development of large multiprocessor systems, processors fault tolerance and identification play an important role in the reliability of computers. In order to maintain the reliability of large multiprocessor systems, the fault processors in such systems should be found and replaced by fault-free processors in time. The process of identifying the faulty processors is called the diagnosis of the system. A system is said to be t-diagnosable if all faulty processors can be identified without replacement, provided that the number of faults presented does not exceed t. The diagnosability $t(G)$ of a system G is the maximum value of t such that G is t-diagnosable [4, 7, 9]. For a t-diagnosable system, Dahbura and Masson [4] proposed an algorithm with time complex $O(n^{2.5})$, which can effectively identify the set of faulty processors.

Many system-level diagnosis models have been proposed to identify the faulty processors in the literature [1, 2, 6, 8, 10, 11, 14, 17, 18]. Among the proposed models, one major approach is the Preparata, Metze, and

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Chien's (PMC) model [14]. Another important model, called the comparison diagnosis model (MM model), was proposed by Maeng and Malek [11]. Sengupta and Dahbura [16] proposed a special case of the MM model, called the MM* model, in which each node must test its any pair of adjacent nodes. The PMC model and MM* model have been widely used in the pioneering work. In 2012, Peng et al. [13] proposed a measure for fault diagnosis of systems, namely, the g-good-neighbor diagnosability (which is also called the g-good-neighbor conditional diagnosability), which requires that every fault-free node contains at least g fault-free neighbors. In [13], they studied the g-good-neighbor diagnosability of the n-dimensional hypercube under the PMC model. In 2016, Wang and Han [19] studied the g-good-neighbor diagnosability of the n-dimensional hypercube under the MM* model. In 2016, Xu et al. [24] studied the g-good-neighbor diagnosability of complete cubic networks under the PMC model and MM* model. In 2016, Ren and Wang [15] gave some properties of the g -good-neighbor diagnosability of a multiprocessor system. Yuan et al. [25, 26] studied that the g-good-neighbor diagnosability of the k-ary n-cube ($k \geq 3$) under the PMC model and MM* model. The Cayley graph $C\Gamma_n$ generated by the transposition tree Γ_n has recently received considerable attention. In [20, 21], Wang et al. studied the g-good-neighbor diagnosability of $C\Gamma_n$ under the PMC model and MM* model for $g = 1, 2$. The n-dimensional bubble-sort star graph BS_n has many good properties. In 2016, Wang et al. [23] studied the 2-good-neighbor connectivity and 2-good-neighbor diagnosability of BS_n. In 2015, Zhang et al. [27] proposed a new measure for fault diagnosis of the system, namely, the g-extra diagnosability, which restrains that every fault-free component has at least $(g + 1)$ fault-free nodes. In [27], they studied the g-extra diagnosability of the n-dimensional hypercube under the PMC model and MM* model. In 2016, Wang et al. [22] studied the 2-extra diagnosability of BS_n under the PMC model and MM* model.

As a topology structure of interconnection networks, the n-dimensional crossed cube CQ_n has many good properties. In this paper, we study the connectivity and 2-extra diagnosability of CQ_n. It is proved that (1) CQ_n is tightly n super connected for $n \geq 2$; (2) CQ_n is tightly $(3n - 5)$ super 2-extra connected for $n \geq 5$; (3) the 2-extra diagnosability of CQ_n is $3n - 3$ under the PMC model for $n \geq 5$; (4) the 2-extra diagnosability of CQ_n is $3n - 3$ under the MM* model for $n \geq 6$.

2 Preliminaries

2.1 Notations

A multiprocessor system is represented by an undirected simple graph $G = (V, E)$, where the set of vertices $V(G)$ represents processors and the set of edges $E(G)$ represents communication links between processors. The degree $d_G(v)$ of a vertex v in G is the number of edges of G incident with v. Suppose that V' is a nonempty vertex subset of V. The subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V' is called the subgraph of G induced by V' and is denoted by $G[V']$; we say that $G[V']$ is an induced subgraph of G. For any vertex v, we define the neighborhood $N_G(v)$ of v in G to be the set of vertices adjacent to v. Let $S \subseteq V(G)$. We use $N_G(S)$ to denote the set $\cup_{v \in S} N_G(v) \setminus S$. A path in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. We use $P = v_1 v_2 \cdots v_n$ to denote a path that begins with v_1 and ends with v_n. The connectivity $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left when G is complete. We say that an isolated vertex is trivial. Let F_1 and F_2 be two distinct subsets of V, and let the symmetric difference $F_1 \triangle F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. Let $B_1, \ldots, B_k$ $(k \geq 2)$ be the components of $G - F_1$. If $|V(B_1)| \leq \cdots \leq |V(B_k)|$ $(k \geq 2)$, then B_k is called the maximum component of $G - F_1$.

Definition 2.1. *Let $G = (V, E)$ ba an undirected simple graph. A fault set $F \subseteq V$ is called a g-good-neighbor faulty set if $|N(v) \cap (V \backslash F)| \geq g$ for every vertex v in $V \backslash F$.*

A g-good-neighbor cut of a connected graph G is a g-good-neighbor faulty set F such that $G - F$ is disconnected. The minimum cardinality of g-good-neighbor cuts is said to be the g-good-neighbor connectivity of G, denoted by $\kappa^{(g)}(G)$. A connected graph G is said to be g-good-neighbor connected if G has a g-good-neighbor cut.

Definition 2.2. *Let $G = (V, E)$ ba an undirected simple graph. A fault set $F \subseteq V$ is called a g-extra faulty set if every component of $G - F$ has $(g + 1)$ vertices.*

A g-extra cut of a connected graph G is a g-extra faulty set F such that $G - F$ is disconnected. The minimum cardinality of g-extra cuts is said to be the g-extra connectivity of G, denoted by $\tilde{\kappa}^{(g)}(G)$. For graph-theoretical terminology and notation not defined here we follow [3].

2.2 The crossed cube CQ_n

Definition 2.3. *Let $R = \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. Two digit binary strings $u = u_1 u_0$ and $v = v_1 v_0$ are pair related, denoted as $u \sim v$, if and only if $(u, v) \in R$.*

Definition 2.4. *The vertex set of a crossed cube CQ_n is $\{v_{n-1} v_{n-2} \cdots v_0 : 0 \leq i \leq n-1, v_i \in \{0, 1\}\}$. Two vertices $u = u_{n-1} u_{n-2} \cdots u_0$ and $v = v_{n-1} v_{n-2} \cdots v_0$ are adjacent if and only if one of the following conditions is satisfied.*

1. There exists an integer l $(1 \leq l \leq n-1)$ such that

(1) $u_{n-1} u_{n-2} \cdots u_l = v_{n-1} v_{n-2} \cdots v_l$;

(2) $u_{l-1} \neq v_{l-1}$;

(3) if l is even, $u_{l-2} = v_{l-2}$;

(4) $u_{2i+1} u_{2i} \sim v_{2i+1} v_{2i}$, for $0 \leq i < \lfloor \frac{l-1}{2} \rfloor$.

2.

(1)$u_{n-1} \neq v_{n-1}$;

(2)if n is even, $u_{n-2} = v_{n-2}$;

(3) $u_{2i+1} u_{2i} \sim v_{2i+1} v_{2i}$ for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.

Let $n \geq 2$. We define two graphs CQ_n^0 and CQ_n^1 as follows. If $u = u_{n-2} u_{n-3} \cdots u_0 \in V(CQ_{n-1})$, then $u^0 = 0 u_{n-2} u_{n-3} \cdots u_0 \in V(CQ_n^0)$ and $u^1 = 1 u_{n-2} u_{n-3} \cdots u_0 \in V(CQ_n^1)$. If $uv \in E(CQ_{n-1})$, then $u^0 v^0 \in E(CQ_n^0)$ and $u^1 v^1 \in E(CQ_n^1)$. Then $CQ_n^0 \cong CQ_{n-1}$ and $CQ_n^1 \cong CQ_{n-1}$. Define the edges between the vertices of CQ_n^0 and CQ_n^1 according to the following rules.

The vertex $u = 0 u_{n-2} u_{n-3} \cdots u_0 \in V(CQ_n^0)$ and the vertex $v = 1 v_{n-2} v_{n-3} \cdots v_0 \in V(CQ_n^1)$ are adjacent if and only if

1. $u_{n-2} = v_{n-2}$ if n is even;

2. $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$, for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.

The edges between the vertices of CQ_n^0 and CQ_n^1 are said to be cross edges.

Proposition 2.1. *All cross edges of CQ_n is a perfect matching.*

Proof. By definition of CQ_2, all cross edges is a perfect matching. When $n \geq 3$, we define a mapping from $V(CQ_n^0)$ to $V(CQ_n^1)$ given by

$$\varphi : 0u_{n-2}u_{n-3} \cdots u_0 \to 1v_{n-2}v_{n-3} \cdots v_0$$

and satisfies the following two conditions:

1. $u_{n-2} = v_{n-2}$ if n is even;

2. $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$ for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.

We will prove that φ is bijective. Let $v = 1v_{n-2}v_{n-3} \cdots v_0$ be an arbitrary vertex in $V(CQ_n^1)$. We construct a string $u = 0u_{n-2}u_{n-3} \cdots u_0$. If n is even, then let $u_{n-2} = v_{n-2}$ and $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$ for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$. If n is odd, then let $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$ for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$. By the definition of R, u is a digit binary string. By the definition of $V(CQ_n^0)$, $u \in V(CQ_n^0)$. By the definition of φ, $\varphi(u) = v$. So φ is a surjection. Let $u = 0u_{n-2}u_{n-3} \cdots u_0$ and $u' = 0u'_{n-2}u'_{n-3} \cdots u'_0$ are two distinct vertices in $V(CQ_n^0)$. Since $u \neq u'$, there is a j $(0 \leq j \leq n-2)$ such that $u_j \neq u'_j$. Suppose that n is even. Then $u_{n-2} = v_{n-2}$ and $u_j \in \{u_{2i+1}, u_{2i} : (u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R, 0 \leq i < \lfloor \frac{n-1}{2} \rfloor\}$. without loss of generality, we suppose that $u_j = u_{2i+1}$. If $u_j = 0$, then $u'_j = 1$, $\varphi(u_j) = 0$, and $\varphi(u'_j) = 1$ by $R = \{(00,00), (10,10), (01,11), (11,01)\}$. If $u_j = 1$, then $u'_j = 0$, $\varphi(u_j) = 1$, and $\varphi(u'_j) = 0$ by $R = \{(00,00), (10,10), (01,11), (11,01)\}$. Therefore, $\varphi(u_j) \neq \varphi(u'_j)$ and hence $\varphi(u) \neq \varphi(u')$. Suppose that n is odd. Then $u_j \in \{u_{2i+1}, u_{2i} : (u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R, 0 \leq i < \lfloor \frac{n-1}{2} \rfloor\}$. Similarly, we have $\varphi(u) \neq \varphi(u')$. Therefore, So φ is an injective.

By the definition of the crossed cube CQ_n, $u\varphi(u)$ is an edge in CQ_n. By φ is bijective, $\{u\varphi(u) : u \in V(CQ_n^0)\}$ is a perfect matching. □

By Proposition 2.1, CQ_n can be recursively defined as follows.

Definition 2.5. *Define that $CQ_1 \cong K_2$. For $n \geq 2$, CQ_n is obtained by CQ_n^0 and CQ_n^1, and a perfect matching between the vertices of CQ_n^0 and CQ_n^1 according to the following rules (see Fig.1):*
The vertex $u = 0u_{n-2}u_{n-3} \cdots u_0 \in V(CQ_n^0)$ and the vertex $v = 1v_{n-2}v_{n-3} \cdots v_0 \in V(CQ_n^1)$ are adjacent in CQ_n if and only if

1. $u_{n-2} = v_{n-2}$ if n is even;

2. $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$, for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.

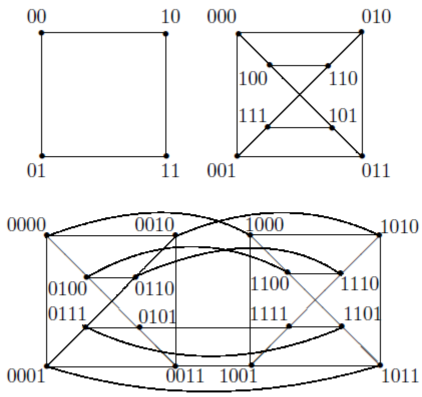

Fig.1. CQ_2,CQ_3, and CQ_4.

2.3 The PMC Model and the MM* Model

In the PMC model, a processor (vertex) can perform tests on its neighbors. For two adjacent vertices u and v in $V(G)$, the ordered pair (u, v) represents the test performed by u on v. If the vertex u is fault-free, the testing result is reliable. The outcome of a test (u, v) is 1 (resp. 0) if v is faulty (resp. fault-free). If the vertex u is faulty, the testing result is unreliable, and the outcome may be 0 or 1 whether v is faulty or fault-free.

A test assignment T for a system G can be modeled as a directed graph $T = (V(G), L)$, where $(u, v) \in L$ indicates that u and v are adjacent in G. The result of all tests in T is called a syndrome. Briefly, a syndrome σ is a function L from $\{0, 1\}$. For a given syndrome σ, a subset $F \subseteq V(G)$ is consistent with σ if the syndrome σ can be produced from the situation that, for any $(u, v) \in L$ such that $u \in V \setminus F$, $\sigma(u, v) = 1$ if and only if $v \in F$. Since the testing result by a faulty processor is unreliable, a given faulty set F may produce many different syndromes. On the other hand, different fault sets may produce the same syndrome. Let $\sigma(F)$ denote all syndromes which F is consistent with.

In the MM* model, a processor w sends the same task to some pair of distinct neighbors, u and v, and then compares their responses to diagnose a system G. The comparison scheme of a system $G = (V(G), E(G))$ is modeled as a multigraph, denoted by $M = (V(G), L)$, where L is the labeled-edge set. A labeled edge $(u, v)_w \in L$ represents a comparison in which two vertices u and v are compared by a vertex w, which implies $uw, vw \in E(G)$. The collection of all comparison results in $M = (V(G), L)$ is called the syndrome, denoted by $\sigma*$, of the diagnosis. If the comparison $(u, v)_w$ disagrees, then $\sigma * ((u, v)_w) = 1$. Otherwise, $\sigma * ((u, v)_w) = 0$. Hence, a syndrome is a function from L to $\{0, 1\}$.

In a system, two distinct vertex sets F_1 and F_2 are indistinguishable if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$; otherwise, F_1 and F_2 are said to be distinguishable. If $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$, we say (F_1, F_2) is an indistinguishable pair; else, (F_1, F_2) is a distinguishable pair.

In a system $G = (V, E)$, a faulty set $F \subseteq V$ is called a g-extra faulty set if every component of $G - F$ has more than g nodes. G is g-extra t-diagnosable if and only if for each pair of distinct faulty g-extra vertex subsets $F_1, F_2 \subseteq V(G)$ such that $|F_i| \leq t$, F_1 and F_2 are distinguishable. The g-extra diagnosability of G, denoted by $\tilde{t}_g(G)$, is the maximum value of t such that G is g-extra t-diagnosable.

3 The connectivity of the crossed cube CQ_n

Lemma 3.1. *([5])* $\kappa(CQ_n) = n$ *for* $n \geq 1$.

Lemma 3.2. *([12])* $\kappa^{(1)}(CQ_n) = 2n - 2$ *for* $n \geq 3$.

Lemma 3.3. *([12]) Any two vertices in crossed cube of CQ_n have at most two common neighbors for $n \geq 2$.*

Lemma 3.4. *Let F be the set of vertices in CQ_n $(n \geq 2)$ with $|F| = n$. If $CQ_n - F$ is disconnected, then $CQ_n - F$ has exactly two components, one of which is an isolated vertex.*

Proof. Let F be the set of vertices in CQ_n $(n \geq 2)$ with $|F| = n$. We will prove this lemma by induction on n. The lemma clearly holds for $n = 2$. Assume that the lemma holds for $n - 1$, i.e., $CQ_{n-1} - F$ has exactly two components, one of which is an isolated vertex if $CQ_{n-1} - F$ is disconnected. We consider n for $n \geq 3$. We can decompose CQ_n along dimension $n - 1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_{n-1}. Let $F_0 = F \cap V(CQ_n^0)$ and $F_1 = F \cap V(CQ_n^1)$ with $|F_0| \leq |F_1|$. Then $|F| = |F_1| + |F_2|$. We consider the following three cases.

Case 1. $|F_0| = 0$.

Since $|F_0| = 0$, $CQ_n^0 - F_0$ is connected. By Proposition2.1, $CQ_n[V(CQ_n^0 - F_0) \cup V(CQ_n^1 - F_1)] = CQ_n - F$ is connected, a contradiction to that F is a cut of $CQ_n - F$.

Case 2. $|F_0| \geq 2$.

Since $|F| = |F_0| + |F_1| = n$, we have $|F_1| \leq n - 2$. Note that $|F_0| \leq |F_1|$. So we have $|F_0| \leq |F_1| \leq n - 2$. By Lemma 3.1, Both $CQ_n^0 - F_0$ and $CQ_n^1 - F_1$ are connected. Since $2^{n-1} - n \geq 1$ $(n \geq 3)$, by Proposition 2.1, $CQ_n[V(CQ_n^0 - F_0) \cup V(CQ_n^1 - F_1)]$ is connected, a contradiction to that F is a cut of $CQ_n - F$.

Case 3. $|F_0| = 1$.

By Lemma 3.1, $CQ_n^0 - F_0$ is connected. Since $|F| = |F_0| + |F_1| = n$, we have $|F_1| = n - 1$. If $CQ_n^1 - F_1$ is connected, then it is similar to Case 2. If $CQ_n^1 - F_1$ is disconnected, by the inductive hypothesis, $CQ_n^1 - F_1$ has two components, one of which is an isolated vertex. Let the isolated vertex be u and let $CQ_n^1 - F_1 - u$ be B_1. When $n = 3$, $|V(B_1)| = 1$. Since $|F_0| = 1$, by Proposition 2.1, $CQ_n - F$ has exactly two components, one of which is an isolated vertex. When $n \geq 4$, $|V(B_1)| \geq 2$. Since $|V(B_1)| \geq 2$ and $|F_0| = 1$, by Proposition 2.1, $CQ_n[V(CQ_n^0 - F_0) \cup V(B_1)]$ is connected. If $N_{CQ_n}(u) \cap V(CQ_n^0) = F_0$, then $CQ_n - F$ has exactly two components, one of which is an isolated vertex. If $N_{CQ_n}(u) \cap V(CQ_n^0) \neq F_0$, then $CQ_n - F$ is connected, a contradiction. □

A connected graph G is super connected if every minimum cut F of G isolates one vertex. If, in addition, $G - F$ has two components, one of which is an isolated vertex, then G is tightly $|F|$ super connected.

Theorem 3.1. *The crossed cube CQ_n $(n \geq 2)$ is tightly n super connected.*

Lemma 3.5. *Let $F \subseteq V(CQ_n)$ $(n \geq 3)$ with $n \leq |F| \leq 2n - 3$. If $CQ_n - F$ is disconnected, then $CQ_n - F$ has exactly two components, one of which is an isolated vertex.*

Proof. Let $F \subseteq V(CQ_n)$ $(n \geq 3)$ with $n \leq |F| \leq 2n-3$. We will prove this lemma by induction on n. When $n = 3$, we can get $|F| = 3$. By Lemma 3.4, the result is true. Assume that the lemma holds for $n-1$ and $n \geq 4$, i.e., $CQ_{n-1} - F$ has exactly two components, one of which is an isolated vertex if $n-1 \leq |F| \leq 2n-5$. Now we consider n for $n \geq 4$. We can decompose CQ_n along dimension $n-1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_{n-1}. Let $F_0 = F \cap V(CQ_n^0)$ and $F_1 = F \cap V(CQ_n^1)$ with $|F_0| \leq |F_1|$. Since $|F_0| + |F_1| = |F| \leq 2n-3$, we can get $|F_0| \leq n-2$. By Lemma 3.4, $CQ_n^0 - F_0$ is connected. We consider the following three cases.

Case 1. $|F_1| \leq n-2$.

By Lemma 3.1, $CQ_n^1 - F_1$ is connected. Since $2^{n-1} - (2n-3) \geq 1$ $(n \geq 4)$, by Proposition 2.1, $CQ_n[V(CQ_n^0 - F_0) \cup V(CQ_n^1 - F_1)] = CQ_n - F$ is connected, a contradiction to that F is a cut of CQ_n.

Case 2. $n-1 \leq |F_1| \leq 2n-5$.

If $CQ_n^1 - F_1$ is connected, then it is similar to Case 1. We suppose that $CQ_n^1 - F_1$ is disconnected. By the inductive hypothesis, $CQ_n^1 - F_1$ has two components, one of which is an isolated vertex. Let the isolated vertex be u and let the nontrivial component of $CQ_n^1 - F_1$ be B_1. Since $2^{n-1} - (2n-3) - 1 \geq 1$ $(n \geq 4)$, by Proposition 2.1, $CQ_n[V(CQ_n^0 - F_0) \cup V(B_1)]$ is connected. If $N_{CQ_n}(u) \cap V(CQ_n^0) \subseteq F_0$, then $CQ_n - F$ has exactly two components, one of which is an isolated vertex. If $N_{CQ_n}(u) \cap V(CQ_n^0) \nsubseteq F_0$, then $CQ_n - F$ is connected, a contradiction to that F is a cut of CQ_n.

Case 3. $|F_1| \geq 2n-4$.

If $CQ_n^1 - F_1$ is connected, then it is similar to Case 1. Suppose that $CQ_n^1 - F_1$ is disconnected. Since $n \leq |F| \leq 2n-3$, we have $|F_0| \leq 1$. When $|F_0| = 0$, by Proposition 2.1, $CQ_n[V(CQ_n^0 - F_0) \cup V(CQ_n^1 - F_1)]$ is connected, a contradiction. When $|F_0| = 1$, by Proposition 2.1, there is a vertex $u \in V(CQ_n^1)$ such that $N_{CQ_n}(u) \cap V(CQ_n^0) = F_0$ in CQ_n. If u is the isolated vertex of $CQ_n^1 - F_1$, then $CQ_n - F$ has two components, one of which is an isolated vertex. If u is not the isolated vertex of $CQ_n^1 - F_1$, then $CQ_n - F$ is connected, a contradiction. □

Lemma 3.6. *Let $CQ_n (n \geq 5)$ be the crossed cube, and let $F \subseteq V(CQ_n)$ with $|F| = 2n-2$. Suppose that $CQ_n - F$ is disconnected. Then $CQ_n - F$ satisfies one of the following conditions:*

(1) $CQ_n - F$ has two components, one of which is a K_2;

(2) $CQ_n - F$ has two components, one of which is an isolated vertex;

(3) $CQ_n - F$ has three components, two of which are isolated vertices.

Proof. Let $F \subseteq V(CQ_n)$ with $|F| = 2n-2$. We can decompose CQ_n along dimension $n-1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_{n-1}. Let $F_0 = F \cap V(CQ_n^0)$ and $F_1 = F \cap V(CQ_n^1)$ with $|F_0| \leq |F_1|$. We consider the following two cases.

Case 1. $|F_0| = |F_1|$

Since $|F| = 2n-2$ and $|F_0| \leq |F_1|$, we have $|F_0| = |F_1| = n-1$.

Case 1.1. Both $CQ_n^0 - F_0$ and $CQ_n^1 - F_1$ are connected.

Since $2^{n-1} - (2n-2) \geq 1$ $(n \geq 5)$, by Proposition 2.1, $CQ_n - F$ is connected, a contradiction to that F is a cut of CQ_n.

Case 1.2. Only one of $CQ_n^0 - F_0$ and $CQ_n^1 - F_1$ is connected.

Without loss of generality, we assume that $CQ_n^0 - F_0$ is connected and $CQ_n^1 - F_1$ is disconnected. By Lemma 3.4, $CQ_n^1 - F_1$ has exactly two components, one is trivial and the other is nontrivial. Let the nontrivial component be B_1. Since $2^{n-1} - (2n-2) - 1 \geq 1 (n \geq 5)$, $CQ_n[V(B_1) \cup V(CQ_n^0 - F_0)]$ is connected by Proposition 2.1. Thus, $CQ_n - F$ satisfies one of the following conditions:

(1) $CQ_n - F$ is connected, a contradiction;

(2) $CQ_n - F$ has two components, one of which is an isolated vertex.

Case 1.3. Both $CQ_n^0 - F_0$ and $CQ_n^1 - F_1$ are disconnected.

By Theorem 3.1, both $CQ_n^0 - F_0$ and $CQ_n^1 - F_1$have exactly two components, one is trivial and the other is nontrivial. Let the two nontrivial components be B_0 and B_1, respectively. Since $2^{n-1} - (2n-2) - 2 \geq 1$ $(n \geq 5)$, $CQ_n[V(B_0) \cup V(B_1)]$ is connected by Proposition 2.1. Thus, $CQ_n - F$ satisfies one of the following conditions:

(1) $CQ_n - F$ is connected, a contradiction

(2) $CQ_n - F$ has two components, one of which is a K_2;

(3) $CQ_n - F$ has two components, one of which is an isolated vertex;

(4) $CQ_n - F$ has three components, two of which are isolated vertices.

Case 2. $|F_0| \neq |F_1|$.

In this case, $|F_0| < |F_1|$. Since $|F| = 2n-2$ and $|F| = |F_0| + |F_1|$, we have $|F_0| < n-1$ and $|F_1| > n-1$. By Lemma 3.4, $CQ_n^0 - F_0$ is connected.

Case 2.1. $n - 1 < |F_1| \leq 2n - 5$.

Suppose that $CQ_n^1 - F_1$ is connected. Since $2^{n-1} > 2n-2$ as $n \geq 5$, by Proposition 2.1, $CQ_n - F$ is connected, a contradiction to that F is a cut of CQ_n. Then $CQ_n^1 - F_1$ is disconnected. Since $n - 1 < |F_1| \leq 2n-5 = 2(n-1) - 3$, by Lemma 3.5, $CQ_n^1 - F_1$ have exactly two components, one is trivial and the other is nontrivial. This is similar to Case 1.2.

Case 2.2. $|F_1| \geq 2n - 4$.

If $CQ_n^1 - F_1$ is connected, then it is similar to Case 1.1. We suppose that $CQ_n^1 - F_1$ is disconnected. Since $|F| = 2n-2$, we have $|F_0| \leq 2$. When $|F_0| = 0$, by Proposition 2.1, $CQ_n[V(CQ_n^0 - F_0) \cup V(CQ_n^1 - F_1)]$ is connected, a contradiction. When $|F_0| = 1$, by Proposition 2.1, there is a vertex u in $CQ_n^1 - F_1$ such that $N_{CQ_n}(u) \cap V(CQ_n^0) = F_0$. If u is an isolated vertex of $CQ_n^1 - F_1$, then $CQ_n - F$ has two components, one of which is an isolated vertex. If u is not an isolated vertex of $CQ_n^1 - F_1$, then $CQ_n - F$ is connected, a contradiction. When $|F_0| = 2$, by Proposition 2.1, there are two vertices u and v in $CQ_n^1 - F_1$ such that $N_{CQ_n}(\{u, v\}) \cap V(CQ_n^0) = F_0$. If u and v are two isolated vertices of $CQ_n^1 - F_1$, then $CQ_n - F$ has three components, two of which are isolated vertices. If uv is an isolated edge of $CQ_n^1 - F_1$, then $CQ_n - F$ has two components, one of which is a K_2. If only u is an isolated vertex of $CQ_n^1 - F_1$, then $CQ_n - F$ has two components, one of which is an isolated vertex. □

Lemma 3.7. *([15]) Let G be a connected graph. Then* $\kappa^{(1)}(G) = \tilde{\kappa}^{(1)}(G)$.

Combining Lemmas 3.2 and 3.7, we have the following corollary.

Theorem 3.2. $\kappa^{(1)}(CQ_n) = \tilde{\kappa}^{(1)}(CQ_n) = 2n - 2$ *for* $n \geq 3$.

A connected graph G is super g-extra connected if every minimum g-extra cut F of G isolates one connected subgraph of order $g+1$. If, in addition, $G-F$ has two components, one of which is the connected subgraph of order $g+1$, then G is tightly $|F|$ super g-extra connected.

Combining Lemma 3.6 and Theorem 3.2, we have the following theorem.

Theorem 3.3. *For $n \geq 5$, the crossed cube CQ_n is tightly $(2n-2)$ super 1-extra connected.*

Lemma 3.8. *Let CQ_n be the crossed cube and let $A = \{0\cdots000, 0\cdots001,$ $0\cdots011\}$. If $n \geq 4$, $F_1 = N_{CQ_n}(A)$, $F_2 = A \cup N_{CQ_n}(A)$, then $|F_1| = 3n-5$, $|F_2| = 3n-2$, F_1 is a 2-extra cut of CQ_n, and $CQ_n - F_1$ has two components $CQ_n - F_2$ and $CQ_n[A]$.*

Proof. It is easy to see that $CQ_n[A]$ is a path of length 2. By the definition of the crossed cube, we can find that $0\cdots000$ and $0\cdots011$ have two common neighbors $0\cdots001 \in A$ and $0\cdots010 \notin A$. Thus, $|F_1| = |N_{CQ_n}(A)| = (n-1)+(n-1)+(n-2)-1 = 3n-5$ and $|F_2| = 3n-2$.

We prove that $CQ_n - F_2$ is connected by induction on n. When $n = 4$, it is easy to see that $CQ_4 - F_2$ is connected (See Fig.1). We assume that the result is true for $n-1$, i.e., $CQ_{n-1} - F_2$ is connected. Now we show that the result is also true for n ($n \geq 5$). We can decompose CQ_n along dimension $n-1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_{n-1}. Let $F_2^0 = F_2 \cap V(CQ_n^0)$ and $F_2^1 = F_2 \cap V(CQ_n^1)$. By the inductive hypothesis, $CQ_n^0 - F_2^0$ is connected. Note that $A \subseteq V(CQ_n^0)$ and $|A| = 3$. By Proposition 2.1, we have $|F_2^1| = |N_{CQ_n}(A) \cap V(CQ_n^1)| = 3$. By Lemma 3.4, $CQ_n^1 - F_2^1$ is connected. Since $2^{n-1} - |F_2| = 2^{n-1} - (3n-2) \geq 1$ ($n \geq 5$), by Proposition 2.1, $CQ_n - F_2$ is connected.

Note that $CQ_n - F_1$ has two components $CQ_n - F_2$ and $CQ_n[A]$ with $|A| = 3$. Since $|V(CQ_n - F_2)| = 2^n - (3n-2) \geq 3$ ($n \geq 4$), F_1 is a 2-extra cut of CQ_n. □

Lemma 3.9. *Let $F \subseteq V(CQ_n)$ ($n \geq 5$) with $2n-2 \leq |F| \leq 3n-6$. If $CQ_n - F$ is disconnected, then $CQ_n - F$ satisfies one of the following conditions:*

(1) $CQ_n - F$ has two components, one of which is a K_2;

(2) $CQ_n - F$ has two components, one of which is an isolated vertex;

(3) $CQ_n - F$ has three components, two of which are isolated vertices.

Proof. Let $F \subseteq V(CQ_n)$ ($n \geq 5$) with $2n-2 \leq |F| \leq 3n-6$. We prove this lemma by induction on n. When $n = 5$, we have $8 \leq |F| \leq 9$. If $|F| = 8$, then $CQ_5 - F$ satisfies one of the conditions (1)-(3) by Lemma 3.6. When $|F| = 9$, we can decompose CQ_5 along dimension 4 into CQ_5^0 and CQ_5^1. Then both CQ_5^0 and CQ_5^1 are isomorphic to CQ_4. Let $F_0 = F \cap V(CQ_5^0)$ and $F_1 = F \cap V(CQ_5^1)$ with $|F_0| \leq |F_1|$. Since $|F_0| + |F_1| = |F| = 9$, we have $|F_0| \leq 4$. If $|F_0| = 4$, then $|F_1| = 5$. By Theorem 3.1 , $CQ_5^0 - F_0$ is connected or has two components, one of which is an isolated vertex. By Lemma 3.5, $CQ_5^1 - F_1$ is connected or has two components, one of which is an isolated vertex. Thus, $CQ_5 - F$ satisfies one of the conditions (1)-(3). If $|F_0| = 3$, then $|F_1| = 6$. By Lemma 3.1, $CQ_5^0 - F_0$ is connected. Suppose that $CQ_5^1 - F_1$ is connected. Since $2^4 - 9 \geq 1$, $CQ_5 - F$ is connected, a contradiction. Suppose that $CQ_5^1 - F_1$ is disconnected. Let the components of $CQ_5^1 - F_1$ be $C_1, C_2, \ldots, C_k$ ($k \geq 2$). Since $|F_0| = 3$, $k \leq 3$ holds. When $|V(C_i)| \geq 4$ for $i \in \{1, \ldots, k\}$, we have $CQ_5[V(CQ_5^0 - F_0) \cup V(C_i)]$ is connected. When $|V(C_i)| = 3$ for $i \in \{1, \ldots, k\}$, C_i is a K_3 or a path of length 2. Since CQ_5 has no triangle, $CQ_5^1 - F_1$ has no a component of K_3. Suppose that C_i is a path of length 2. Let the path be $P = uvw$. Since C_i is a component of $CQ_5^1 - F_1$, we have $N_{CQ_5^1}(V(C_i)) \subseteq F_1$. Then we have $N_{CQ_5^1}(\{u,v,w\}) \subseteq F_1$. By Lemma 3.3, $|N_{CQ_5^1}(\{u,v,w\})| = |N_{CQ_5^1}(\{u\}) \setminus \{v\}| + |N_{CQ_5^1}(\{v\} \setminus \{u,w\})| + |N_{CQ_5^1}(\{w\}) \setminus \{v\}| - |N_{CQ_5^1}(\{u\}) \cap N_{CQ_5^1}(\{w\})| \geq 3+2+3-1 = 7$. Since $|N_{CQ_5^1}(\{u,v,w\})| \geq 7 > 6 = |F_1|$, we have $N_{CQ_5^1}(\{u,v,w\}) \nsubseteq F_1$, a contradiction. So $CQ_5^1 - F_1$ has no a component of order 3. When $|V(C_i)| \leq 2$ for $i \in \{1,\ldots,k-1\}$, $CQ_5[V(CQ_5^0 - F_0) \cup V(C_i)]$ is connected or $N_{CQ_5}(V(C_i)) \cap V(CQ_5^0) \subseteq F_0$. Thus, $CQ_5 - F$ satisfies one of the conditions (1)-(3). If $|F_0| \leq 2$, by Proposition 2.1, then $CQ_5 - F$ satisfies one of the conditions (1)-(3). Therefore, the lemma is true for CQ_5.

We assume that the result is true for $n-1$, i.e., $CQ_{n-1} - F$ satisfies one of the conditions (1)-(3) if $|F| \leq 3(n-1) - 6 = 3n-9$, then. Now we prove that the result is true for n $(n \geq 6)$. We can decompose CQ_n along dimension $n-1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_{n-1}. Let $F_0 = F \cap V(CQ_n^0)$ and $F_1 = F \cap V(CQ_n^1)$ with $|F_0| \leq |F_1|$.

Case 1. $|F_1| \leq n-2$.

Since $|F_0| \leq |F_1|$, we have $|F_0| \leq n-2$. By Lemma 3.1, $CQ_n^0 - F_0$ and both $CQ_n^1 - F_1$ are connected. Since $2^{n-1} - (3n-6) \geq 1 (n \geq 6)$, by Proposition 2.1, $CQ_n - F$ is connected, a contradiction.

Case 2. $n-1 \leq |F_1| \leq 2n-5$.

By Lemma 3.5, $CQ_n^1 - F_1$ is connected or has exactly two components, one of which is an isolated vertex. Since $|F_0| \leq |F_1|$, we have $|F_0| \leq \lfloor \frac{3n}{2} - 3 \rfloor$. Since $n \geq 6$, $|F_0| \leq \lfloor \frac{3n}{2} - 3 \rfloor \leq 2n-5$ holds. Therefore, by Lemma 3.5, $CQ_n^0 - F_0$ is connected or has exactly two components, one of which is an isolated vertex. Since $2^{n-1} > 3n-6+2 = 3n-4$ as $n \geq 6$, by Proposition 2.1, then $CQ_n - F$ satisfies one of the conditions (1)-(3).

Case 3. $|F_1| = 2n-4$.

Since $|F| \leq 3n-6$, we have $|F_0| = |F| - |F_1| \leq (3n-6) - (2n-4) = n-2$. By Lemma 3.1, $CQ_n^0 - F_0$ is connected. Note that $|F_1| = 2n-4 = 2(n-1)-2$. By Lemma 3.6, $CQ_n^1 - F_1$ satisfies one of the conditions (1)-(3). Let the maximum component of $CQ_n^1 - F_1$ be B_1. Since $2^{n-1} - (3n-6) - 2 \geq 1$ $(n \geq 6)$, by Proposition 2.1, $CQ_n[V(B_1) \cup V(CQ_n^0 - F_0)]$ is connected. $CQ_n - F$ satisfies one of the conditions (1)-(3) in this case.

Case 4. $2n-3 \leq |F_1| \leq 3n-9$.

Since $|F| \leq 3n-6$, we have $|F_0| = |F| - |F_1| \leq (3n-6) - (2n-3) = n-3$. By Lemma 3.1, $CQ_n^0 - F_0$ is connected. Note that $|F_1| \leq 3n-9 = 3(n-1)-6$. By the inductive hypothesis, $CQ_n^1 - F_1$ satisfies one of the conditions (1)-(3). This is similar to Case 3.

Case 5. $3n-8 \leq |F_1| \leq 3n-6$.

Since $|F| \leq 3n-6$, we have $|F_0| = |F| - |F_1| \leq (3n-6) - (3n-8) = 2$. By Lemma 3.1, $CQ_n^0 - F_0$ is connected. Suppose that $CQ_n^1 - F_1$ is connected. Since $2^{n-1} - (3n-6) \geq 1$ $(n \geq 6)$, by Proposition 2.1, $CQ_n - F$ is connected, a contradiction. Suppose that $CQ_n^1 - F_1$ is disconnected. When $|F_0| = 0$, by Proposition 2.1, $CQ_n[V(CQ_n^0 - F_0) \cup V(CQ_n^1 - F_1)]$ is connected, a contradiction. When $|F_0| = 1$, by Proposition 2.1, there is a u in $CQ_n^1 - F_1$ such that $N_{CQ_n}(u) \cap V(CQ_n^0) = F_0$. If u is an isolated vertex of $CQ_n^1 - F_1$, then $CQ_n - F$ has two components, one of which is isolated vertex. This situation satisfies the condition (2). If u is not isolated vertex of $CQ_n^1 - F_1$, then $CQ_n - F$ is connected, a contradiction. When $|F_0| = 2$, by Proposition 2.1, there are two vertices u and v in $CQ_n^1 - F_1$ such that $N_{CQ_n}(\{u,v\}) \cap V(CQ_n^0) = F_0$. If u and v are two isolated vertices of $CQ_n^1 - F_1$, then $CQ_n - F$ has three components, two of which are isolated vertices. This situation satisfies the condition (3). If uv is an isolated edge of $CQ_n^1 - F_1$, then $CQ_n - F$ has two components, one of which is a K_2. This situation satisfies the condition (1). $\square$

By Lemma 3.9, we have the following corollary.

Corollary 3.1. *Let $n \geq 5$. If F is a 2-extra cut of CQ_n, then $|F| \geq 3n - 5$.*

Theorem 3.4. *Let CQ_n be the crossed cube. Then $\tilde{\kappa}^{(2)}(CQ_n) = 3n - 5 (n \geq 5)$.*

Proof. Let A be defined as Lemma 3.8. By Lemma 3.8, $N_{CQ_n}(A)$ is a 2-extra cut of CQ_n and $|N_{CQ_n}(A)| = 3n - 5$. By the definition of the 2-extra connectivity, we have $\tilde{\kappa}^{(2)}(CQ_n) \leq 3n - 5$. Let F be a 2-extra cut of CQ_n. By Corollary 3.1, $|F| \geq 3n - 5$. So we can get $\tilde{\kappa}^{(2)}(CQ_n) \geq 3n - 5$. Thus, we have $\tilde{\kappa}^{(2)}(CQ_n) = 3n - 5$. $\square$

Theorem 3.5. *For $n \geq 5$, the crossed cube CQ_n is tightly $(3n - 5)$ super 2-extra connected.*

Proof. Now we consider CQ_n for any minimum 2-extra cut $F \subseteq V(CQ_n)$. By Theorem 3.4, $|F| = 3n - 5$. We can decompose CQ_n along dimension $n - 1$ into CQ_n^0 and CQ_n^1. Then both CQ_n^0 and CQ_n^1 are isomorphic to CQ_{n-1}. Let $F_0 = F \cap V(CQ_n^0)$ and $F_1 = F \cap V(CQ_n^1)$ with $|F_0| \leq |F_1|$. We consider the following two cases.

Case 1. $|F_0| \leq 2$.

By Lemma 3.1, $CQ_n^0 - F_0$ is connected. By Proposition 2.1, $CQ_n - F$ satisfies one of the following conditions: (1) $CQ_n - F$ is connected; (2) $CQ_n - F$ has two components, one of which is a K_2; (3) $CQ_n - F$ has two components, one of which is an isolated vertex; (4) $CQ_n - F$ has three components, two of which are isolated vertices. These are a contradiction to that F is a 2-extra cut of CQ_n.

Case 2. $|F_0| = 3$.

By Lemma 3.1, $CQ_n^0 - F_0$ is connected. By Proposition 2.1, there are three vertices u, v, w in $CQ_n^1 - F_1$ such that $N_{CQ_n}(\{u, v, w\}) = F_0$. If $CQ_n[\{u, v, w\}]$ is a component of order 3 in $CQ_n^1 - F_1$, then $CQ_n - F$ has two components, one of which is a subgraph of order 3. Thus, F is a minimum 2-extra cut of CQ_n. Otherwise, F is not a 2-extra cut of CQ_n.

Case 3. $4 \leq |F_0| \leq n - 2$.

By Lemma 3.1, $CQ_n^0 - F_0$ is connected. Since $|F| = 3n - 5$, we can get $|F_1| \leq (3n - 5) - 4 = 3n - 9$. By Lemma 3.9, $CQ_n^1 - F_1$ satisfies one of the following conditions: (1) $CQ_n^1 - F_1$ is connected; (2) $CQ_n^1 - F_1$ has two components, one of which is a K_2; (3) $CQ_n^1 - F_1$ has two components, one of which is an isolated vertex; (4) $CQ_n^1 - F_1$ has three components, two of which are isolated vertices. When $CQ_n^1 - F_1$ satisfies one of the conditions (2)-(4), then $CQ_n^1 - F_1$ is disconnected. Let B_1 be the maximum component of $CQ_n^1 - F_1$. Since $2^{n-1} - (3n - 5) - 2 \geq 1$ $(n \geq 5)$, by Proposition 2.1, $CQ_n[V(B_1) \cup V(CQ_n^0 - F_0)]$ is connected. Therefore, $|V(CQ_n - F - (V(B_1) - V(CQ_n^0)| \leq 2$. This is a contradiction to that F is a 2-extra cut of CQ_n. When $CQ_n^1 - F_1$ satisfies the condition (1), then $CQ_n^1 - F_1$ is connected. Since $2^{n-1} - (3n - 5) \geq 1$ $(n \geq 5)$, $CQ_n - F$ is connected. Thus, F is not a 2-extra cut of CQ_n, a contradiction.

Case 4. $|F_0| = n - 1$.

By Theorem 3.1, $CQ_n^0 - F_0$ is connected or has two components, one of which is an isolated vertex. Since $|F| = 3n - 5$, we can get $|F_1| = 2n - 4$. By Lemma 3.6, $CQ_n^1 - F_1$ is connected or satisfies one of the following conditions: (1) $CQ_n^1 - F_1$ has two components, one of which is a K_2; (2) $CQ_n^1 - F_1$ has two components, one of which is an isolated vertex; (3) $CQ_n^1 - F_1$ has three components, two of which are isolated vertices. Suppose that $CQ_n^0 - F_0$ is connected, then it is similar to Case 3. Suppose that $CQ_n^0 - F_0$ is disconnected. Let B_0 be the maximum component of $CQ_n^0 - F_0$.

Case 4.1. $CQ_n^1 - F_1$ is connected.

Since $2^{n-1} - (3n-5) - 1 \geq 1$ $(n \geq 5)$, by Proposition 2.1, $CQ_n[V(B_0) \cup V(CQ_n^1 - F_1)]$ is connected. We have $|V(CQ_n - F - (V(B_1) - V(CQ_n^0)| \leq 1$. This is a contradiction to that F is a 2-extra cut of CQ_n.

Case 4.2. $CQ_n^1 - F_1$ is disconnected.

Let B_1 be the maximum component of $CQ_n^1 - F_1$. Since $2^{n-1} - (3n-5) - 3 \geq 1$ $(n \geq 5)$, by Proposition 2.1, $CQ_n[V(B_0) \cup V(B_1)]$ is connected.

Case 4.2.1. $CQ_n^1 - F_1$ satisfies condition (1).

In this case, $CQ_n^1 - F_1$ has two components, one of which is a K_2. Note that $CQ_n - F$ satisfies one of the following conditions: (a) $CQ_n - F$ is connected; (b) $CQ_n - F$ has three components, two of which is an isolated vertex and a K_2; (c) $CQ_n - F$ has two components, one of which is a K_2; (d) $CQ_n - F$ has two components, one of which is an isolated vertex; (e) $CQ_n - F$ has two components, one of which is a path of length 2. If $CQ_n - F$ satisfies one of the conditions (a)-(d), then it is a contradiction to that F is a 2-extra cut of CQ_n. If $CQ_n - F$ satisfies the condition (e), then F is a 2-extra cut of CQ_n.

Case 4.2.2. $CQ_n^1 - F_1$ satisfies condition (2).

In this case, $CQ_n^1 - F_1$ has two components, one of which is an isolated vertex. We have $|V(CQ_n - F - (V(B_0) \cup V(B_1))| \leq 2$. This is a contradiction to that F is a 2-extra cut of CQ_n.

Case 4.2.3. $CQ_n^1 - F_1$ satisfies condition (3).

In this case, $CQ_n^1 - F_1$ has three components, two of which are isolated vertices. Note that $CQ_n - F$ satisfies one of the following conditions: (a) $CQ_n - F$ is connected; (b) $CQ_n - F$ has four components, three of which are isolated vertices; (c) $CQ_n - F$ has three components, two of which are an isolated vertex and a K_2; (d) $CQ_n - F$ has three components, two of which are isolated vertices; (e) $CQ_n - F$ has two components, one of which is an isolated vertex; (f) $CQ_n - F$ has two components, one of which is a K_2. This is a contradiction to that F is a 2-extra cut of CQ_n.

Case 5. $n \leq |F_0| \leq \lfloor \frac{3n-5}{2} \rfloor$.

Since $n \leq |F_0| \leq \lfloor \frac{3n-5}{2} \rfloor \leq 2n-5$ $(n \geq 5)$, By Lemma 3.5, $CQ_n^0 - F_0$ is connected or has two components, one of which is an isolated vertex. Note that $|F| = 3n-5$. We have $n \leq |F_1| \leq 2n-5$. By Lemma 3.5, $CQ_n^1 - F_1$ is connected or has two components, one of which is an isolated vertex. By Proposition 2.1, $CQ_n - F$ satisfies one of the following conditions: (a) $CQ_n - F$ is connected; (b) $CQ_n - F$ has three components, two of which are isolated vertices; (c) $CQ_n - F$ has two components, one of which is an isolated vertex; (d) $CQ_n - F$ has two components, one of which is a K_2. This is a contradiction to that F is a 2-extra cut of CQ_n. $\square$

4 The 2-extra diagnosaility of the crossed cube CQ_n under the PMC model

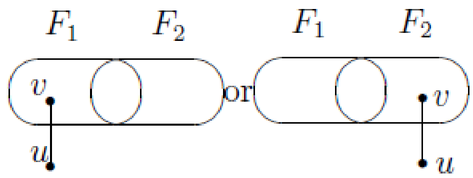

Fig.2. Illustration of a distinguishable pair (F_1, F_2) under the PMC model.

Theorem 4.1. *([25]) A system $G = (V, E)$ is g-extra t-diagnosable under the PMC model if and only if there is an edge $uv \in E$ with $u \in V \in \setminus(F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of g-extra faulty subsets F_1 and F_2 of $V(CQ_n)$ with $|F_1| \leq t$ and $|F_2| \leq t$ (See Fig.2.).*

Lemma 4.1. *Let $n \geq 4$. Then the 2-extra diagnosability of the crossed cube CQ_n under the PMC model is less than or equal to $3n - 3$, i.e., $\tilde{t_2}(CQ_n) \leq 3n - 3$.*

Proof. Let A be defined in Lemma 3.8, $F_1 = N_{CQ_n}(A)$ and $F_2 = A \cup N_{CQ_n}(A)$. By Lemma 3.8, $|F_1| = 3n - 5$, $|F_2| = 3n - 2$, F_1 is a 2-extra cut of CQ_n, and $CQ_n - F_1$ has two components $CQ_n - F_2$ and $CQ_n[A]$. Thus, F_1 and F_2 are both 2-extra faulty sets of CQ_n with $|F_1| = 3n - 5$ and $|F_2| = 3n - 2$. Since $A = F_1 \triangle F_2$ and $N_{CQ_n}(A) = F_1 \subset F_2$, there is no edge of CQ_n between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 4.1, CQ_n is not 2-extra $(3n - 2)$-diagnosable under PMC model. By the definition of 2-extra diagnosability, we can deduce that the 2-extra diagnosability of CQ_n is less than $3n - 2$, i.e., $\tilde{t_2}(CQ_n) \leq 3n - 3$. □

Lemma 4.2. *Let $n \geq 5$. Then the 2-extra diagnosability of the crossed cube CQ_n under the PMC model is more than or equal to $3n - 3$, i.e., $\tilde{t_2}(CQ_n) \geq 3n - 3$.*

Proof. By the definition of 2-extra diagnosability, it is sufficient to show that CQ_n is 2-extra $(3n - 3)$-diagnosable. By Theorem 4.1, we need to prove that there is an edge $uv \in E$ with $u \in V(CQ_n) \setminus (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of 2-extra faulty subsets F_1 and F_2 of $V(CQ_n)$ with $|F_1| \leq 3n - 3$ and $|F_2| \leq 3n - 3$.

Suppose, on the contrary, that there are two distinct 2-extra faulty subsets F_1 and F_2 of $V(CQ_n)$ with $|F_1| \leq 3n - 3$ and $|F_2| \leq 3n - 3$, but there is no edge between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Suppose that $V(CQ_n) = F_1 \cup F_2$. $2^n = |V(CQ_n)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(3n - 3) = 6n - 6$, a contradiction to $n \geq 5$. Therefore, $V(CQ_n) \neq F_1 \cup F_2$. Since there is no edge between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $CQ_n - F_1$ has two parts $CQ_n \setminus (F_1 \cup F_2)$ and $CQ_n[F_2 \setminus F_1]$. Note that F_1 is a 2-extra faulty set. Thus, every component G_i of $CQ_n \setminus (F_1 \cup F_2)$ has $|V(G_i)| \geq 3$ and every component B_i of $CQ_n[F_2 \setminus F_1]$ has $|V(B_i)| \geq 3$. If $F_1 \setminus F_2 = \emptyset$, then $F_1 \cap F_2 = F_1$ is a 2-extra faulty set. If $F_1 \setminus F_2 \neq \emptyset$, similarly, every component C_i of $CQ_n[F_1 \setminus F_2]$ has $|V(C_i)| \geq 3$. Therefore, $F_1 \cap F_2$ is also a 2-extra faulty set. Since there is no edge between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 2-extra cut of CQ_n. By Theorem 3.4, $|F_1 \cap F_2| \geq 3n - 5$. Thus,

$|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 3 + 3n - 5 = 3n - 2$. This is a contradiction to that $|F_2| \leq 3n - 3$. Therefore, CQ_n is 2-extra $(3n-3)$-diagnosable, i.e., $\tilde{t_2}(CQ_n) \geq 3n - 3$. □

Combining Lemmas 4.1 and 4.2, we have the following theorem.

Theorem 4.2. *Let $n \geq 5$. Then the 2-extra diagnosability of the crossed cube CQ_n under the PMC model is $3n-3$, i.e., $\tilde{t_2}(CQ_n) = 3n - 3$.*

5 The 2-extra diagnosaility of the crossed cube CQ_n under the MM* model

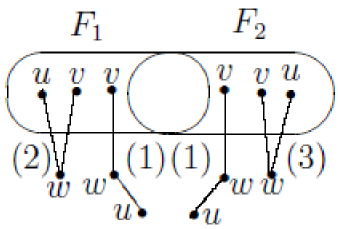

Fig.3. Illustration of a distinguishable pair (F_1, F_2) under the MM* model.

Theorem 5.1. *([25]) A system $G = (V, E)$ is g-extra t-diagnosable under the MM* model if and only if each distinct pair of g-extra faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ satisfies one of the following conditions:*

(1) There exist two vertices $u, w \in V(G) \setminus (F_1 \cup F_2)$ and there exists a vertex $v \in F_1 \triangle F_2$ such that $uw, vw \in E(G)$.

(2) There exist two vertices $u, v \in F_1 \setminus F_2$ and there exists a vertex $w \in V(G) \setminus (F_1 \cup F_2)$ such that $uw, vw \in E(G)$.

(3) There exist two vertices $u, v \in F_2 \setminus F_1$ and there exists a vertex $w \in V(G) \setminus (F_1 \cup F_2)$ such that $uw, vw \in E(G)$ (See Fig.3.).

Lemma 5.1. *Let $n \geq 4$. Then the 2-extra diagnosability of the crossed cube CQ_n under the MM* model is less than or equal to $3n-3$, i.e., $\tilde{t_2}(CQ_n) \leq 3n - 3$.*

Proof. Let A be defined in Lemma 3.8, $F_1 = N_{CQ_n}(A)$, and $F_2 = A \cup N_{CQ_n}(A)$. By Lemma 3.8, $|F_1| = 3n - 5$, $|F_2| = 3n - 2$, F_1 is a 2-extra cut of CQ_n, and $CQ_n - F_1$ has two components $CQ_n - F_2$ and $CQ_n[A]$. Thus, F_1 and F_2 are both 2-extra faulty sets of CQ_n with $|F_1| = 3n - 5$ and $|F_2| = 3n - 2$. Since $A = F_1 \triangle F_2$ and $N_{CQ_n}(A) = F_1 \subset F_2$, there is no edge of CQ_n between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 5.1, CQ_n is not 2-extra $(3n-2)$-diagnosable under MM* model. By the definition of 2-extra diagnosability, we can deduce that the 2-extra diagnosability of CQ_n is less than $3n-2$, i.e., $\tilde{t_2}(CQ_n) \leq 3n - 3$. □

A component of a graph G is odd according as it has an odd number of vertices. We denote by $o(G)$ the number of odd component of G.

Lemma 5.2. *([3]) A graph $G = (V, E)$ has a perfect matching if and only if $o(G - S) \leq |S|$ for all $S \subseteq V$.*

Lemma 5.3. *Let $n \geq 6$. Then the 2-extra diagnosability of the crossed cube CQ_n under the MM* model is more than or equal to $3n - 3$, i.e., $\tilde{t_2}(CQ_n) \geq 3n - 3$.*

Proof. By the definition of 2-extra diagnosability, it is sufficient to show that CQ_n is 2-extra $(3n - 3)$-diagnosable. On the contrary, there are two distinct 2-extra faulty subsets F_1 and F_2 of CQ_n with $|F_1| \leq 3n - 3$ and $|F_2| \leq 3n - 3$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 5.1. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similar to the discussion on $V(CQ_n) = F_1 \cup F_2$ in Lemma 4.2, we can deduce $V(CQ_n) \neq F_1 \cup F_2$.

Claim 1. $CQ_n - (F_1 \cup F_2)$ has no isolated vertex.

We suppose, on the contrary, that $CQ_n - (F_1 \cup F_2)$ has at least one isolated vertex w. Since F_1 is one 2-extra faulty set, there is a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Note that the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 5.1. By the condition (3) of Theorem 5.1, there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Thus, there is just a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. If $F_1 \setminus F_2 = \emptyset$, then $F_1 \subseteq F_2$. Since F_2 is a 2-extra faulty set, every component G_i of $CQ_n - (F_1 \cup F_2) = CQ_n - F_2$ has $|V(G_i)| \geq 3$. So $CQ_n - (F_1 \cup F_2)$ has no isolated vertex. It is contradict with the hypothesis. Thus, $F_1 \setminus F_2 \neq \emptyset$. Similarly, we can deduce that there is just a vertex $v \in F_1 \setminus F_2$ such that v is adjacent to w. Let $W \subseteq V(CQ_n) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $CQ_n[V(CQ_n) \setminus (F_1 \cup F_2)]$, and let H be the induced subgraph by the vertex set $V(CQ_n) \setminus (F_1 \cup F_2 \cup W)$. Then for any vertex $w \in W$, we can get that w has $(n-2)$ neighbors in $F_1 \cap F_2$. By Lemma 5.2 and Proposition 2.1, $|W| \leq o(CQ_n - (F_1 \cup F_2)) \leq |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq 2(3n-3) - (n-2) = 5n - 4$. We assume that $V(H) = \emptyset$. Then $2^n = |V(CQ_n)| = |F_1 \cup F_2| + |W| = |F_1| + |F_2| - |F_1 \cap F_2| + |W| \leq 2(3n-3) - (n-2) + (5n-4) = 10n - 8$, a contradiction to that $n \geq 6$. Therefore, $V(H) \neq \emptyset$.

Since the vertex set pair (F_1, F_2) is not satisfied with the condition (1) of Theorem 5.1 and any vertex of $V(H)$ is not isolated in H, we induce that there is no edge between $V(H)$ and $F_1 \triangle F_2$. If $F_1 \cap F_2 = \emptyset$, then $F_1 \triangle F_2 = F_1 \cup F_2$. Since there is no edge between $V(H)$ and $F_1 \triangle F_2$, CQ_n is disconnected, a contradiction to that CQ_n is connected. Thus, $F_1 \cap F_2 \neq \emptyset$. Note that $CQ_n - (F_1 \cap F_2)$ has two parts: H and $CQ_n[(F_1 \setminus F_2) \cup (F_2 \setminus F_1) \cup W]$. Thus, $F_1 \cap F_2$ is a vertex cut of CQ_n. Since F_1 is a 2-extra faulty set of CQ_n, we have that every component H_i of H has $|V(H_i)| \geq 3$ and every component B_i^2 of $CQ_n[(F_2 \setminus F_1) \cup W]$ has $|V(B_i^2)| \geq 3$. Similarly, every component B_i^1 of $CQ_n[(F_1 \setminus F_2) \cup W]$ has $|V(B_i^1)| \geq 3$. Note that any vertex $w \in W$ has two neighbors u and v such that $u \in V(B_i^2)$ and $v \in V(B_i^1)$. Thus, w is belongs to a component $CQ_n[V(B_i^1) \cup V(B_i^2)]$ of $CQ_n[(F_1 \setminus F_2) \cup (F_2 \setminus F_1) \cup W]$ with $|V(B_i^1) \cup V(B_i^2)| \geq 3$. Let $v_2 \in F_2 \setminus F_1$. Then v_2 is belongs to a component B_i^2 of $CQ_n[(F_2 \setminus F_1) \cup W]$, where $|V(B_i^2)| \geq 3$. Note that B_i^2 is belongs to a component $\mathbb{B}_i^2$ of $CQ_n[(F_2 \setminus F_1) \cup W \cup (F_1 \setminus F_2)]$. Since $|V(B_i^2)| \geq 3$, $|V(\mathbb{B}_i^2)| \geq 3$. Similarly, $v_1 \in F_1 \setminus F_2$. Therefore, every component B_i of $CQ_n[(F_1 \setminus F_2) \cup (F_2 \setminus F_1) \cup W]$ has $|V(B_i)| \geq 3$. By the definition of 2-extra cut, $F_1 \cap F_2$ is a 2-extra cut of CQ_n. By Theorem 3.4, $|F_1 \cap F_2| \geq 3n - 5$. Note that $|V(B_i^2)| \geq 3$. We can get $|F_2 \setminus F_1| \geq 2$. Since $|F_1 \cap F_2| = |F_2| - |F_2 \setminus F_1| \leq (3n-3) - 2 = 3n - 5$, we have $|F_1 \cap F_2| = 3n - 5$. Note that $|F_2| \leq 3n - 3$. We can get $|F_2 \setminus F_1| = 2$ and $|F_2| = 3n - 3$. Similarly, we have $|F_1 \setminus F_2| = 2$ and $|F_1| = 3n - 3$. By Theorem 3.5, the crossed cube CQ_n is tightly $(3n - 5)$ super 2-extra connected, i.e., $CQ_n - (F_1 \cap F_2)$ has two components, one of which is a subgraph of order 3. Thus, $CQ_n - (F_1 \cap F_2)$ has two components, one of which is $CQ_n[(F_1 \setminus F_2) \cup (F_2 \setminus F_1) \cup W]$ and the other one is H with $|V(H)| = 3$. Note that $|W| \leq 5n-4$. $2^n = |V(CQ_n)| = |F_1 \setminus F_2| + |F_2 \setminus F_1| + |F_1 \cap F_2| + |W| + |V(H)| \leq 2+2+(3n-5)+(5n-4)+3 = 8n-2$, a contradiction to $n \geq 6$. The proof of Claim 1 is complete.

Let $u \in V(CQ_n) \setminus (F_1 \cup F_2)$. By Claim 1, u has at least one neighbor in $CQ_n - F_1 - F_2$. Since (F_1, F_2) is not satisfied with any one condition in Theorem 5.1, u has no neighbor in $F_1 \triangle F_2$. By the arbitrariness of u, there is no edge between $V(CQ_n) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since F_1 and F_2 are two 2-extra faulty set, every component H_i of $CQ_n - (F_1 \cup F_2)$ has $|V(H_i)| \geq 3$, every component B_i of $CQ_n([F_2 \setminus F_1])$ has $|V(B_i)| \geq 3$, and every component C_i of $CQ_n([F_1 \setminus F_2])$ has $|V(C_i)| \geq 3$ when $F_1 \setminus F_2 \neq \emptyset$. Thus, $F_1 \cap F_2$ is also a 2-extra faulty set. Since there is no edge between $V(CQ_n \setminus (F_1 \cup F_2))$ and $F_1 \triangle F_2$, we have $F_1 \cap F_2$ is a 2-extra cut of CQ_n. By Theorem 3.4, we have $|F_1 \cap F_2| \geq 3n-5$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 3 + (3n-5) = 3n-2$, which contradicts $|F_2| \leq 3n-3$. Therefore, CQ_n is 2-extra $(3n-3)$-diagnosable, i.e., $\tilde{t}_2(CQ_n) \geq 3n-3$. The proof is complete.

Combining Lemmas 5.1 and 5.3, we can get the following theorem.

Theorem 5.2. *Let $n \geq 6$. Then the 2-extra diagnosability of the crossed cube CQ_n under the MM* model is $3n-3$, i.e., $\tilde{t}_2(CQ_n) = 3n-3$.*

$\square$

6 Conclusions

We prove that the crossed cube CQ_n is tightly $(3n-5)$ super 2-extra connected for $n \geq 5$, and the 2-extra diagnosability of CQ_n is $3n-3$ under the PMC model $(n \geq 5)$ and MM* model $(n \geq 6)$. On the basis of this study, the researchers can continue to study the g-extra connectivity and diagnosability of networks.

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