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Dynamics of a rational difference equation $x_{n+1} = ax_n + \dfrac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}}$

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ABSTRACT. A nonlinear rational difference equation $x_{n+1} = ax_n + \dfrac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}}$, $\quad n = 0, 1, ...,$ of higher order is considered to apprehend the dynamics viz. the invariant intervals, periodic solutions, the character of semi-cycles and global asymptotic stability. Here all the parameters a, α, β and A, B and the initial conditions x_k, ..., x_1, x_0 are positive real numbers $k = \{1, 2, 3, ...\}$. It is shown that the equilibrium point is globally asymptotically stable under the condition $\beta \leq A$, and the unique positive solution is also globally asymptotically stable under the conditions $\beta \leq A \leq \beta$. Finally, we study the global stability of this equation through numerically solved examples and confirm our theoretical discussion through it. We also have considered parameters as real numbers (positive and negative) just to comprehend additional dynamics.

1 Introduction

We attempt to investigate the global stability character and the periodicity of the solutions of the following rational higher order difference equation

$$x_{n+1} = ax_n + \frac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}}, \quad n = 0, 1, ..., \tag{1.1}$$

where the parameters a, α, β and A, B and the initial conditions $x_k, ..., x_1, x_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$ is a positive integer, and the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are non-negative real numbers.

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Various biological systems naturally leads to their study by means of discrete variables. Appropriate examples include ecological dynamics and medicine and etc. Some fundamental models of biological phenomena, including harvesting of fish, a single species population model, ventilation volume and blood CO2 levels, the production of red blood cells, a simple epidemics model, and a model of waves of disease that can be analyzed by difference equations are shown in [1]. Newly, there has been interest in so-called dynamical diseases, which correspond to physiological disorders for which a generally stable control system becomes unstable. One of the first papers on this subject was that of Mackey and Glass [2]. In which they investigated a first-order difference-delay equation that models the concentration of blood-level CO2. They also discussed models of a second class of diseases associated with the production of red cells, white cells, and platelets in the bone marrow. The dynamical characteristics of population system have been modeled, among others by differential equations in the case of species with overlapping generations and by difference equations in the case of species with non-overlapping generations. In process, one can develop a discrete model directly from observations and experiments. Periodically, for numerical purposes, one wants to propose a finite-difference scheme to numerically solved a given differential equation model, especially when the differential equation cannot be solved explicitly. For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points [3]. But unless we can explicitly solve both equations, it is impossible to satisfy this requirements. Most of the time, it is fascinating that a differential equation, when extracted from a difference equation, marmalade the dynamical features of the corresponding continuous-time model such as equilibria, their local and global stability characteristics, and bifurcation behaviors. If alike discrete models can be derived from continuous time models, and it will preserve the considered realities, such discrete-time models can be called 'dynamically consistent' with the continuous-time models.

The study of oscillatory and asymptotic stability properties of solution behavior of difference equations is extremely advantageous in the behavior of various biological system and other applications. This is because difference equations are relevant models for expressing situations where the variable is assumed to take only a discrete set of values and they appear frequently in the formulation and analysis of discrete time systems, in the study of biological systems, the study of deterministic chaos, the numerical integration of differential equations by finite difference schemes and so on. Difference equations are good models for describing situations where population growth is not continuous but seasonal with overlapping generations. For example, the difference equation

$$x_{n+1} = x_n e^{\left[r(1-\frac{y_n}{k})\right]}$$

has been expressed to model different animal populations.

The generalized Beverton–Holt stock recruitment model has been investigated in [10,11]

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1 + cx_{n-1} + dx_n}.$$

Several other researchers have studied the behavior of the solution of difference equations, for example, in [15] E.M. Elsayed investigated the solution of the following non-linear difference equation.

$$x_{n+1} = ax_n + \frac{bx_n^2}{cx_n + dx_{n-1}^2}.$$

Elabbasy et al. [16] studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation.

$$x_{n+1} = \frac{\alpha x_{n-l} + \beta x_{n-k}}{Ax_{n-l} + Bx_{n-k}}.$$

Keratas et al. [20]gave the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

Elabbasy et al. [17] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} + cx_{n-q}}.$$

Yalçınkaya et al. [18] has studied the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Saleh et. al. [19]study the solution of difference

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}.$$

Elsayed et al. [22] studied the global behavior of rational recursive sequence

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-k} + cx_{n-s}}{d + ex_{n-t}}$$

As a matter of fact, numerous papers negotiate with the problem of solving nonlinear difference equations in any way possible, see, for instance [7]–[15]. The long-term behavior and solutions of rational difference equations of order greater than one has been extensively studied during the last decade. For example, various results about periodicity, boundedness, stability, and closed form solution of the second-order rational difference equations, see [5–9, 21–29]. Other related work on rational difference equations see in refs. [30–44]

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let $F : I^{k+1} \to I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

(Equilibrium Point) A point $\overline{x} \in I$ is called an equilibrium point of Eq.(1.2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(1.2), or equivalently, $\overline{x}$ is a fixed point of F.

(Periodicity) A Sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

(Fibonacci Sequence). The sequence $\{F_m\}_{m=1}^{\infty} = \{1, 2, 3, 5, 8, 13, ...\}$ *i.e.* $F_m = F_{m-1} + F_{m-2} \geq 0, F_{-2} = 0, F_{-1} = 1$ *is called Fibonacci Sequence.*

(Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.2).

(v) The equilibrium point $\overline{x}$ of Eq.(1.2) is unstable if $\overline{x}$ is not locally stable.

(vi) The linearized equation of Eq.(1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Theorem A [27] Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1, \tag{1.4}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

The following theorem will be useful for the proof of our results in this paper.

Theorem B [28]: Let $[l, m]$ be an interval of real numbers and assume that $f : [l, m]^2 \to [l, m]$,is a continuous function and consider the following equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n = 0, 1, ..., \tag{1.5}$$

satisfying the following conditions :

(a) $f(x, y)$ is non-decreasing in $x \in [l, m]$ for each fixed $y \in [\alpha, \beta]$.and $g(x, y)$ is non-increasing in $y \in [l, m]$ for each fixed $x \in [l, m]$

(b) If $(m, M) \in [l, m] \times [l, m]$ is a solution of the system

$$M = g(M, m) \qquad and \qquad m = g(m, M),$$

then $m = M$,

then Eq. (5) has a unique equilibrium $\overline{x} \in [l, m]$ and every solution of Eq.(5) converges to $\overline{x}$.

2 Equilibrium points of Eq.(1.1)

In this section we shall study the equilibrium points of Eq.(1.1) and their computational existence. The equilibrium points of Eq.(1.1) are the solutions of the equation

$$\overline{x} = a\overline{x} + \frac{\alpha + \beta\overline{x}}{A + B\overline{x}}$$

or,

$$B(1-a)\overline{x}^2 + (A - Aa - \beta)\overline{x} - \alpha = 0$$

The solution of the Eq.(1) are $\frac{\sqrt{(-aA+A-\beta)^2+4\alpha(B-aB)}-aA+A-\beta}{2(aB-B)}$ and $\frac{\sqrt{(-aA+A-\beta)^2+4\alpha(B-aB)}+aA-A+\beta}{2(B-aB)}$. Then the only positive equilibrium point of Eq.(1)for all the positive parameters a, α, β and A, B is given by

$$\overline{x} = \frac{(\beta - A + Aa) + \sqrt{(\beta - A + Aa)^2 + 4\alpha B(1-a)}}{2B(1-a)}$$

It is noted that there does not exist any positive parameters (a, α, β, A, B) such that both the equilibrium points are positive. We found a set of 5000 positive parameters (a, α, β, A, B) such that the equilibrium point $\frac{\sqrt{(-aA+A-\beta)^2+4\alpha(B-aB)}+aA-A+\beta}{2(B-aB)}$ is positive. The parameter spaces (a, A, B) and (α, β) are represented in three and two dimensional coordinate system respectively in the Fig. 1.

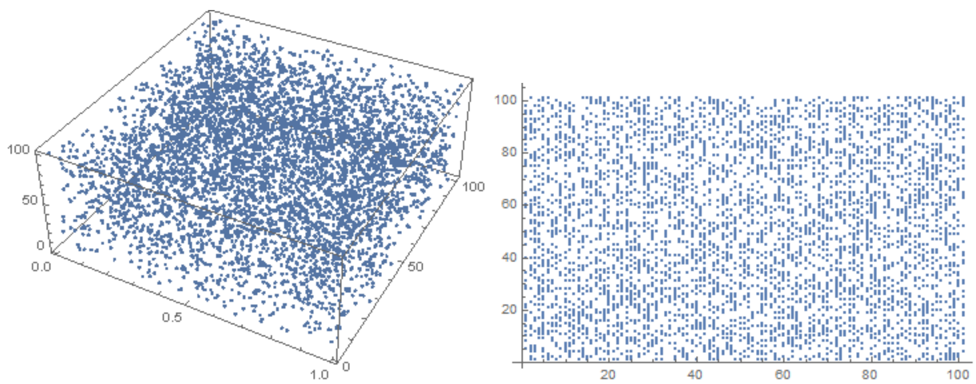

Figure 1: Positive Parameter space (a, A, B) and (α, β)(left, right)

It is worth noting that there are real (positive and negative) parameters (a, A, B) and (α, β) such that both the equilibrium points are positive. Such parameter spaces (a, A, B) and (α, β) are represented in the Fig. 2.

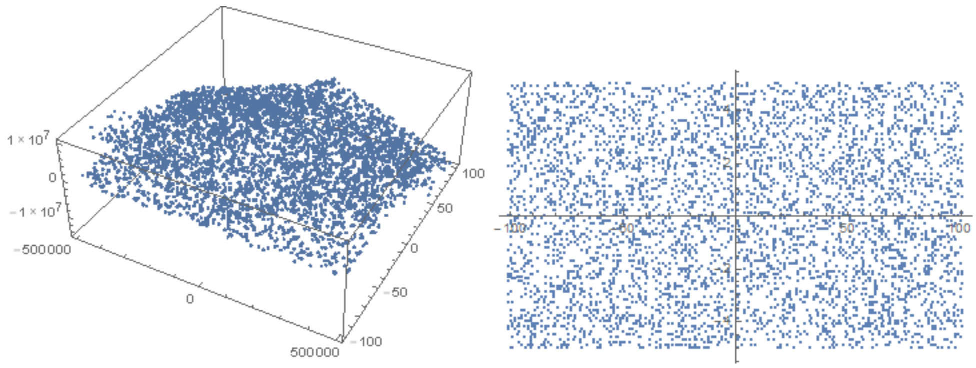

Figure 2: Parameter space (a, A, B) and (α, β)(left, right) such that both the equilibrium points are positive

There are real parameters (a, A, B) and (α, β) such that the equilibrium point $\frac{\sqrt{(-aA+A-\beta)^2+4\alpha(B-aB)}-aA+A-\beta}{2(aB-B)}$ is positive. Such kind of parameters (a, α, β, A, B) are plotted in the Fig. 3.

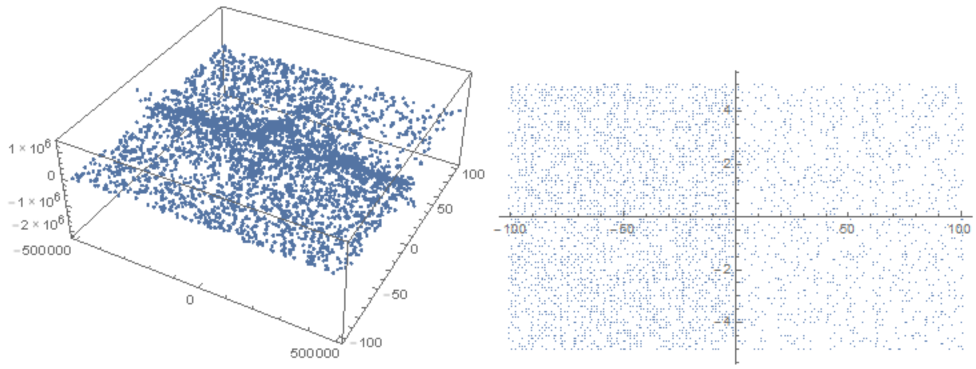

Figure 3: Parameter space (a, A, B) and (α, β)(left, right) such that equilibrium point $\frac{\sqrt{(-aA+A-\beta)^2+4\alpha(B-aB)}-aA+A-\beta}{2(aB-B)}$ is positive

There are real parameters (a, A, B) and (α, β) such that the equilibrium point $\frac{\sqrt{(-aA+A-\beta)^2+4\alpha(B-aB)}+aA-A+\beta}{2(B-aB)}$ is positive. Such kind of parameters are plotted in the Fig. 4.

Present computational plots of parameters ensure the existence of both the equilibriums (positive) for real (positive and negative) parameters. Therefore it would certainly be interesting enough to explore the local stability of the equilibriums for real parameters instead just positive parameters.

To find the linearization for our problem, consider the function of two variables . . .

$$f(u, v) = au + \frac{\alpha + \beta v}{A + Bv}$$

Now,

$$\frac{\partial f(u, v)}{\partial u} = a \quad and \quad \frac{\partial f(u, v)}{\partial v} = \frac{\beta A - \alpha B}{(A + Bv)^2}$$

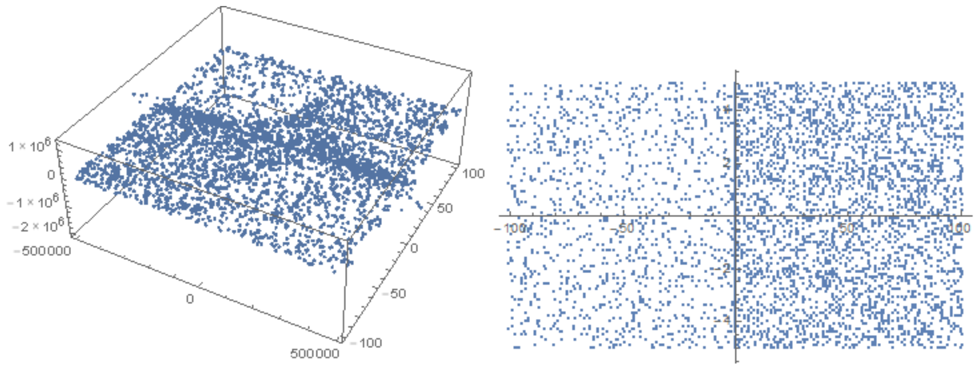

Figure 4: Parameter space (a, A, B) and (α, β)(left, right) such that equilibrium point $\frac{\sqrt{(-aA+A-\beta)^2+4\alpha(B-aB)}+aA-A+\beta}{2(B-aB)}$ is positive

Hence, for $\overline{x} = \frac{(\beta-A+Aa)+\sqrt{(\beta-A+Aa)^2+4\alpha B(1-a)}}{2B(1-a)}$,

$$f_u(\overline{x}, \overline{x}) = a = p,$$

and

$$f_v(\overline{x}, \overline{x}) = \frac{(\beta A - \alpha B)}{(A + Bv)^2} = q.$$

So, the linearized equation about the $\overline{x} = \frac{(\beta-A+Aa)+\sqrt{(\beta-A+Aa)^2+4\alpha B(1-a)}}{2B(1-a)}$

$$y_{n+1} = (a)\, y_n + \left(\frac{(\beta A - \alpha B)}{(A + Bv)^2}\right) y_{n-k} \tag{2.1}$$

and its characteristic equation is

$$\lambda^{k+1} - (a)\, \lambda^k - \left(\frac{(\beta A - \alpha B)}{(A + Bv)^2}\right) = 0. \tag{2.2}$$

3 Local Asymptotic Stability of Eq.(1.1)

3.1 Local stability about $\overline{x} = \frac{(\beta-A+Aa)+\sqrt{(\beta-A+Aa)^2+4\alpha B(1-a)}}{2B(1-a)}$

Theorem 3.1. *The equilibrium $\overline{x} = \frac{(\beta-A+Aa)+\sqrt{(\beta-A+Aa)^2+4\alpha B(1-a)}}{2B(1-a)}$ of the difference equation Eq.(1.1) is locally asymptotically stable iff.*

$$(A + B\overline{x})^2 > \frac{|\beta A - \alpha B|}{(1 - a)}, \qquad a < 1. \tag{3.1}$$

Proof. Let $f : (0, \infty)^2 \to (0, \infty)$ be a continuous function defined by

$$f(u, v) = au + \frac{\alpha + \beta v}{A + Bv} \tag{3.2}$$

It follows from Theorem 1.1 that, Eq.(1.1) is asymptotically stable $\Leftrightarrow$

$$p < |1 - q| < 2.$$

□

Thus,

$$a + \left| \frac{(\beta A - \alpha B)}{(A + Bv)^2} \right| < 1,$$

also

$$\left| \frac{(\beta A - \alpha B)}{(A + Bv)^2} \right| < (1 - a), \qquad a < 1,$$

$$|\beta A - \alpha B| < (A + Bv)^2 (1 - a), \qquad a < 1,$$

or

$$\frac{|\beta A - \alpha B|}{(1 - a)} < (A + Bv)^2, \quad a < 1.$$

We have now found 5000 positive parameters (a, A, B, α, β) such that the positive equilibrium

$$\bar{x} = \frac{(\beta - A + Aa) + \sqrt{(\beta - A + Aa)^2 + 4\alpha B(1 - a)}}{2B(1 - a)}$$

is *locally asymptotically stable*. The parameters (a, A, B) & (A, B) are plotted in three and two dimensional coordinate system respectively in the following figure Fig. 5.

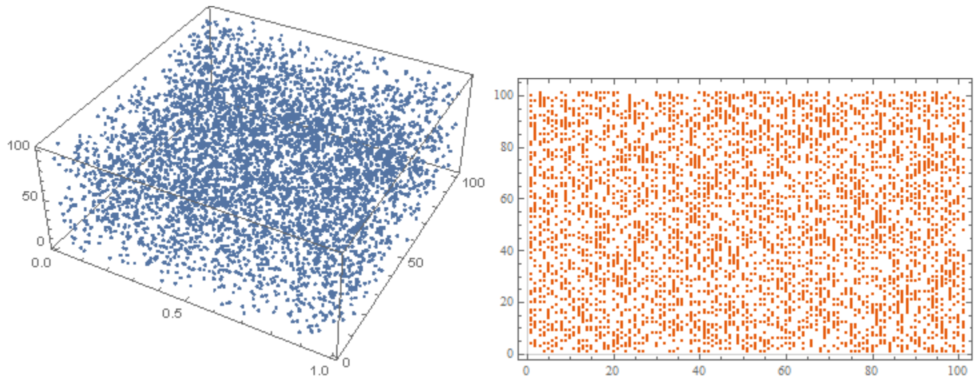

Figure 5: Positive Parameter space (a, A, B) and (α, β)(left, right) such that the positive equilibrium is locally asymptotically stable

We have also found 5000 real (positive and negative) parameters (a, A, B, α, β) such that the positive equilibrium is *locally asymptotically stable*. The parameters (a, A, B) & (A, B) are plotted in three and two dimensional coordinate system respectively in the following figure Fig. 6.

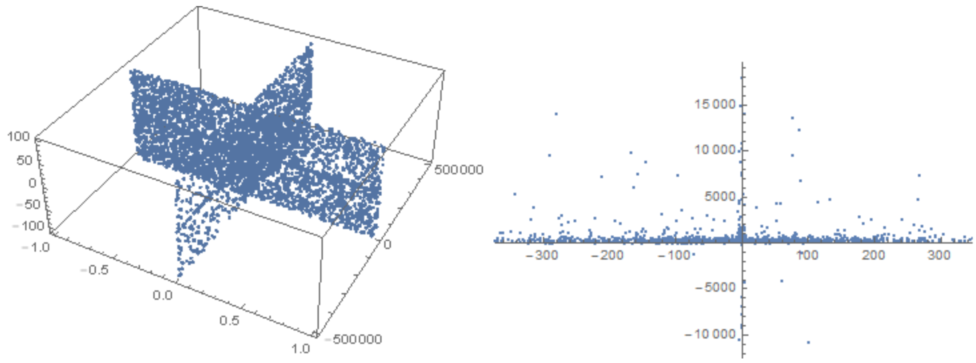

Figure 6: Real Parameter space (a, A, B) and (α, β)(left, right) such that the positive equilibrium is locally asymptotically stable.

We have also found 5000 real (positive and negative) parameters (a, A, B, α, β) such that the equilibrium point $\frac{\sqrt{(-aA+A-\beta)^2+4\alpha(B-aB)}-aA+A-\beta}{2(aB-B)}$ is *locally asymptotically stable*. The parameters (a, A, B) & (A, B) are plotted in three and two dimensional coordinate system respectively in the following figure Fig. 7.

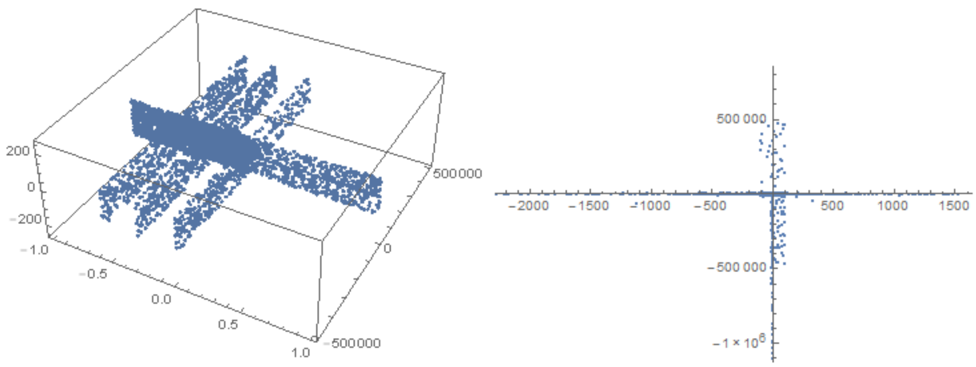

Figure 7: Real Parameter space (a, A, B) and (α, β)(left, right) such that the equilibrium $\frac{\sqrt{(-aA+A-\beta)^2+4\alpha(B-aB)}-aA+A-\beta}{2(aB-B)}$ is locally asymptotically stable

4 Global Stability of Eq.(1.1)

In this section we will study the global stability character of the solutions of Eq.(1.1).

Lemma 4.1. *For any values of the quotient $\frac{\alpha}{A}$ and $\frac{\beta}{B}$ the function $f(u, v)$ defined by Eq.(3.2) has monotonicity behavior in its two arguments.*

Proof. The proof follows by easy computations and is omitted. $\square$

Theorem 4.1. *The equilibrium point $\overline{x}$ of Eq.(1.1) is a global attractor $\Leftrightarrow$ one of the following statements holds*

$$i)\ \beta A \geq \alpha B \text{ and } \beta > A(1-a), \quad a < 1. \tag{5.1}$$

$$ii) \quad \beta A \leq \alpha B \text{ and } a < 1. \tag{5.2}$$

Proof. Let $l,\ m$ are real numbers and assume that $f : [l, m]^2 \to [l, m]$ be a function defined by

$$f(u, v) = au + \frac{\alpha + \beta v}{A + Bv}$$

$$\frac{\partial f(u, v)}{\partial u} = a \quad and \quad \frac{\partial f(u, v)}{\partial v} = \frac{\beta A - \alpha B}{(A + Bv)^2}$$

Now, two cases must be considered.

Case (1): Suppose that (i) is true, then we can easily see that the function is increasing in u, v. □

Let x be a solution of the equation $x = g(x, x)$. Then from Eq.(1.1), we can write

$$x = ax + \frac{\alpha + \beta x}{A + Bx}$$

or

$$x(1 - a) = \frac{\alpha + \beta x}{A + Bx},$$

then the equation

$$B(1 - a)x^2 + \{A(1 - a) - \beta\}x - \alpha = 0,$$

has a unique positive solution when $\beta > A(1 - a), \quad a < 1$ which is,

$$x = \frac{(\beta - A(1 - a)) + \sqrt{(\beta - A(1 - a))^2 + 4\alpha B(1 - a)}}{2B(1 - a)}$$

By using Theorem B, it follows that $\overline{x}$ is a global attractor of Eq.(1.1) and then the proof is completed.

Case (2): Suppose that (5.2) is true, let $l,\ m$ are real numbers and assume that $f : [l, m]^2 \to [l, m]$ be a function defined by $f(u, v) = au + \frac{\alpha+\beta v}{A+Bv}$, then we can easily see that the function is $f(u, v)$ is increasing in u and decreasing in v.

Let (m, M) be a solution of the system $M = f(M, m)$ and $m = f(m, M)$.Then from Eq.(1.1), we see that

Then from Eq.(1.1) we see that

$$m = am + \frac{\alpha + \beta M}{A + BM} \; and \;\; M = aM + \frac{\alpha + \beta m}{A + Bm}$$

by subtracting we get

$$M(1 - a) = \frac{\alpha + \beta m}{A + Bm}, \;\; m(1 - a) = \frac{\alpha + \beta M}{A + BM}$$

and

$$A(1 - a)M + B(1 - a)Mm = \alpha + \beta m, \;\; A(1 - a)m + B(1 - a)mM = \alpha + \beta M.$$

Subtracting we obtain

$$(M - m)\{A(1 - a)(M + m) + \beta\} = 0,$$

under the condition $a < 1$, we conclude that,

$$m = M.$$

It follows by Theorem C that $\overline{x}$ is a global attractor of Eq.(1.1) and then the proof is completed.

5 Boundedness of Solution of Eq.(1.1)

In this section we will study the boundedness of solutions of Eq.(1.1).

Theorem 5.1. *Every solution of Eq.(1.1) is boumded and persist if* $a < 1$.

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (1.1). It follows from Eq. (1.1) that

$$x_{n+1} = ax_n + \frac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}} = ax_n + \frac{\alpha}{A + Bx_{n-k}} + \frac{\beta x_{n-k}}{A + Bx_{n-k}}$$

Then

$$x_{n+1} \leq ax_n + \frac{\alpha}{A} + \frac{\beta x_{n-k}}{Bx_{n-k}} = ax_n + \frac{\alpha}{A} + \frac{\beta}{B} \quad for\ all \quad n \geq 1.$$

By using comparison, the right hand side can be written as follows

$$y_{n+1} = ay_n + \frac{\alpha}{A} + \frac{\beta}{B}.$$

So, we can write

$$y_n = a^n y_0 + \text{constant},$$

and this equation is locally asymptotically stable because $a < 1$, and converges to the equillibrium point $\overline{y} = \frac{\alpha B + \beta A}{AB(1-a)}$

Therefore

$$\lim_{n\to\infty} \sup x_n \leq \frac{\alpha B + \beta A}{AB(1-a)}$$

Hence the solution is bounded. □

Theorem 5.2. *Every solution of Eq.(1.1) is bounded is unbounded if* $a > 1$.

Proof. Let $\{x_n\}_{n=-1}^{\infty}$be a solution of Eq.(1.1). Then from Eq.(1.1) we see that

$$x_{n+1} = ax_n + \frac{\alpha + \beta x_{n-k}}{A + Bx_{n-k}} > ax_n \quad for\ all \quad n \geq 1.$$

the right hand side can be written as follows

$$y_{n+1} = ay_n \quad \Rightarrow \quad y_n = a^n y_0,$$

and this equation is unstable because $a > 1$, and $\lim_{n\to\infty} y_n = \infty$. Then by using ratio test $\{x_n\}_{n=-1}^{\infty}$is unbounded from above.

□

6 Existence of Periodic Solution

In this section, we are looking for the positive prime period two solution of Eq.(1.1). Let us assume the two cycle period of Eq.(1.1) will be in the form

$$...p,\ q,\ p,\ q...$$

Case (1): if k is odd then,

$$x_{n+1} = x_{n-k} \tag{4.1}$$

we get,

$$\begin{aligned} p &= aq + \frac{\alpha + \beta p}{A + Bp} \\ q &= ap + \frac{\alpha + \beta q}{A + Bq} \end{aligned}$$

This transform to

$$\begin{aligned} Ap + Bp^2 &= aAq + aBpq + \alpha + \beta p \quad \text{and} \\ Aq + Bq^2 &= aAp + aBpq + \alpha + \beta q. \end{aligned}$$

by subtracting (4.5) from (4.4) we get,

$$A(p-q) + B(p^2 - q^2) = -aA(p-q) + \beta(p-q)$$

Then since $p \neq q$, it follows that

$$(p+q) = \frac{\beta - aA - A}{B} \tag{4.4}$$

again by adding (4.5) from (4.4) we get,

$$\begin{aligned} A(p+q) + B(p^2+q^2) &= aA(p+q) + 2aBpq + 2\alpha + \beta(p+q), \\ B(p^2+q^2) &= (aA - A + \beta)(p+q) + 2aBpq + 2\alpha \end{aligned}$$

By using (4.4), (4.5) and the relation

$$p^2 + q^2 = (p+q)^2 - 2pq, \qquad p, q \in R,$$

we get

$$B((p+q)^2 - 2pq) = (aA - A + \beta)(p+q) + 2aBpq + 2\alpha$$

$$2(1+a)Bpq = -2aA(p+q) - 2\alpha$$

So,

$$pq = \frac{-(\beta - aA - A)aA - \alpha B}{(1+a)B^2}. \tag{4.5}$$

Now, it is obvious from Eq.(4.4) and Eq.(4.5) that, p and q are two distinct real roots of the quadratic equation

$$t^2 - \left(\frac{\beta - aA - A}{B}\right)t - \left(\frac{\alpha B + (\beta - aA - A)aA}{(1+a)B^2}\right) = 0,$$

or,

$$Bt^2 - (\beta - aA - A)t - \left(\frac{\alpha B + (\beta - aA - A)aA}{(1+a)B}\right) = 0,$$

Thus,

$$[\beta - aA - A]^2 + \frac{4\left[\alpha B + (\beta - aA - A)aA\right]}{(1+a)} > 0, \tag{4.6}$$

or

$$[\beta - aA - A]^2(1+a) + 4\left[\alpha B + (\beta - aA - A)aA\right] > 0.$$

Now suppose,

$$p = \frac{\beta - aA - A + \delta}{2B}$$

and

$$q = \frac{\beta - aA - A - \delta}{2B}$$

where $\delta = \sqrt{[\beta - aA - A]^2 + \frac{4[\alpha B + (\beta - aA - A)aA]}{(1+a)}}$
from (4.6) we get that $\delta^2 > 0$, therefore p and q are distinct real numbers, set

$$x_{-1} = p \quad \text{and} \quad x_0 = q.$$

It follows from Eq.(1.1) that

$$\begin{aligned} x_1 &= aq + \frac{\alpha + \beta p}{A + Bp} \\ &= a\left(\frac{\beta - aA - A - \delta}{2B}\right) + \frac{\alpha + \beta\left(\frac{\beta - aA - A - \delta}{2B}\right)}{A + B\left(\frac{\beta - aA - A - \delta}{2B}\right)} \end{aligned}$$

dividing numerator and denominator by $2(A + aB)$ we get

$$x_1 = a\left(\frac{\beta - aA - A + \delta}{2B}\right) + \frac{2B\alpha + \beta(\beta - aA - A + \delta)}{2BA + B(\beta - aA - A + \delta)}$$

multiply denominator and numerator of right hand side by $2BA + B(\beta - aA - A - \delta)$ and by computation we get

$$x_1 = p$$

Similarly

$$x_2 = q$$

Then by induction we get

$$x_{2n} = q \quad \text{and} \quad x_{2n+1} = p \qquad \text{for all} \quad n \geq -1.$$

Thus Eq.(1.1) has prime period two solution.

Case (2): if k is even then

$$\begin{aligned} p &= aq + \frac{\alpha + \beta q}{A + Bq} \\ q &= ap + \frac{\alpha + \beta p}{A + Bp} \end{aligned}$$

This transforms to

$$\begin{aligned} Ap + Bpq &= aAq + aBq^2 + \alpha + \beta q \\ Aq + Bpq &= aAp + aBp^2 + \alpha + \beta p \end{aligned}$$

by subtracting (4.8) from (4.7) we get,

$$A(p-q) = -aA(p-q) - aB(p^2-q^2) - \beta(p-q)$$

Then since $p \neq q$, it follows that

$$(p+q) = -\left(\frac{\beta + aA + A}{aB}\right) \tag{4.9}$$

again by adding (4.7) from (4.8) we get,

$$\begin{aligned} A(p+q) + 2Bpq &= aA(p+q) + aB(p^2+q^2) + 2\alpha + \beta(p+q), \\ aB(p^2+q^2) &= (A - aA - \beta)(p+q) + 2aBpq - 2\alpha. \end{aligned}$$

by using (4.7), (4.8) and the relation

$$p^2 + q^2 = (p+q)^2 - 2pq, \qquad p, q \in R,$$

we get

$$aB((p+q)^2 - 2pq) = (A - aA - \beta)(p+q) + 2aBpq - 2\alpha$$

$$4aBpq = 2\alpha - 2A.$$

So,

$$pq = \frac{\alpha - A}{2aB}. \tag{4.10}$$

Now, it is obvious from Eq.(4.4) and Eq.(4.5) that, p and q are two distint real roots of the quadratic equation

$$t^2 + \left(\frac{\beta + aA + A}{aB}\right) t - \left(\frac{\alpha - A}{2aB}\right) = 0,$$

or,

$$2aBt^2 + 2(\beta + aA + A)t - (\alpha - A) = 0,$$

Thus,

$$4(\beta + aA + A)^2 + 8aB(\alpha - A) > 0. \tag{4.11}$$

Now suppose,

$$p = \frac{-2(\beta + aA + A) + \delta}{4aB}$$

and

$$q = \frac{-2(\beta + aA + A) - \delta}{4aB}$$

where $\delta = \sqrt{4(\beta + aA + A)^2 + 8aB(\alpha - A)}$
from (4.6) we get that $\delta^2 > 0$, therefore p and q are distinct real numbers, set

$$x_{-1} = p \quad \text{and} \quad x_0 = q.$$

It follows from Eq.(1.1) that

$$\begin{aligned} x_1 &= aq + \frac{\alpha + \beta q}{A + Bq} \\ &= a\left(\frac{-2(\beta + aA + A) - \delta}{4aB}\right) + \frac{\alpha + \beta\left(\frac{-2(\beta+aA+A)-\delta}{4aB}\right)}{A + B\left(\frac{-2(\beta+aA+A)-\delta}{4aB}\right)} \end{aligned}$$

so,

$$x_1 = -a\left(\frac{(\beta + aA + A) - \delta}{2aB}\right) + \frac{4aB\alpha + \beta(-2(\beta + aA + A) - \delta)}{4aBA + B(-2(\beta + aA + A) - \delta)}$$

multiply denominator and numerator of right hand side by $4aBA + B(-2(\beta + aA + A) + \delta)$ and by computation we get

$$x_1 = p$$

Similarly

$$x_2 = q$$

Then by induction we get

$$x_{2n} = q \quad \text{and} \quad x_{2n+1} = p \quad \text{for all } n \geq -1.$$

Thus Eq.(1.1) has prime period two solution.

Here we shall explore real parameters (a, α, β, A, B) and k such that the solutions of the rational difference equation Eq.(1.1) are periodic, which is listed in the Table 1.

It is noted that, in all the cases, the periodic solution exist if the parameter $|a| < 1$.

7 Existence of Chaotic Solution

Here computationally we wish to explore solution of the rational difference equation of chaotic nature. We assume here the parameters are real numbers. This assumption is based on the impression that there does not exist such dynamics for any strictly positive real parameters of the system. Here we gathered a set of examples as stated in the following Table 2.

In all the above set of examples of chaotic cases, without loss of generality, the parameters A, B, α, β are taken from the interval $[-100, 100]$ and a is taken to be absolutely less than 1. Also k is taken from the interval $[1, 50]$. In every case, ten different initial values are taken and ran the trajectory for the specific set parameters.

8 Existence of Fractal-like Solution

Computationally we explore solutions of the rational difference equation of self-similar (fractal-like) nature. We assume here as usual the parameters are real numbers. Here we gathered a set of examples as stated in the following Table 2.

In all the above set of examples of fractal-like(self-similar), without loss of generality, the parameters A, B, α, β are taken from the interval $[-100, 100]$ and a is taken to be absolutely less than 1. Also k is taken from the interval $[1, 50]$. In every case, ten different initial values are taken and ran the trajectory for the specific set parameters.

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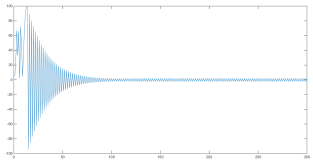
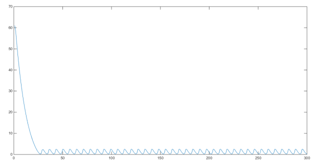
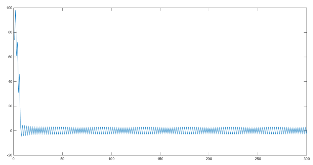
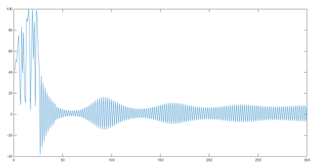
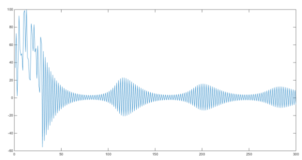
Parameters $(a, \alpha, \beta, A, B, k)$	**Period**	**Trajectory**
$(-0.9500, 0, 66, 14, -28, 13)$	2	
$(0.92, 0, 48, 26, -37, 1)$	7	
$(-0.97, 16, 87, -21, 15, 6)$	4	
$(-0.91, -44, 31, 58, 60, 25)$	2	
$(-0.93, 90, 53, -14, 50, 28)$	2	

Table 1: Periodic solutions for different parameters

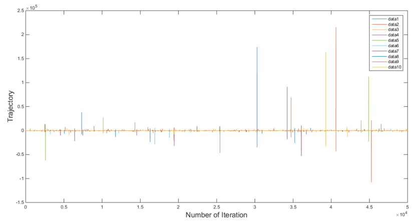
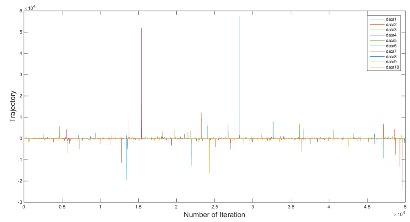
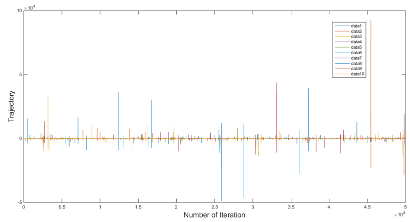
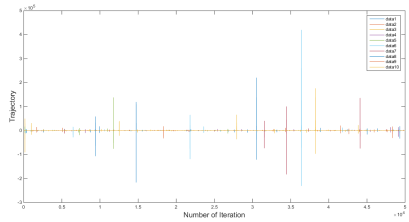
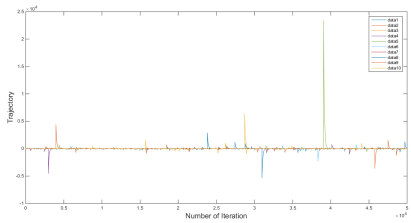
Parameters $(a, \alpha, \beta, A, B, k)$	**Nature and Lyapunav Exponent**	**Trajectory**
$(-0.2, 40, -60, -43, -66, 34)$	0.6784, hence chaotic	
$(0.17, -16, -82, 45, -44, 2)$	0.2361, hence chaotic	
$(-0.25, -56, -56, 72, 17, 27)$	0.5764, hence chaotic	
$(-0.55, -46, 35, 41, 46, 24)$	0.9854, hence chaotic	
$(0.99, -57, -79, 44, 5, 6)$	0.7653, hence chaotic	

Table 2: Chaotic solutions for different parameters

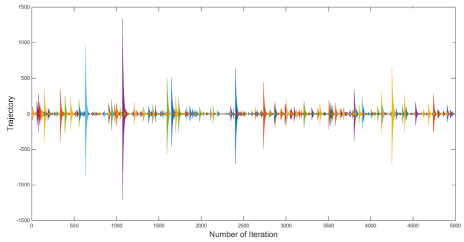
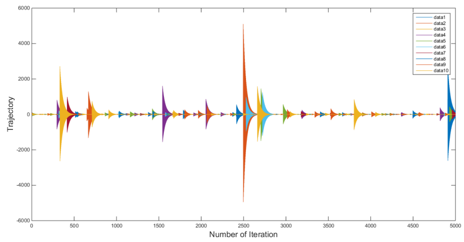
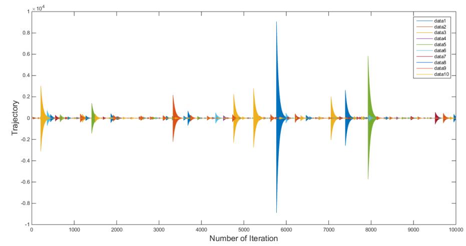
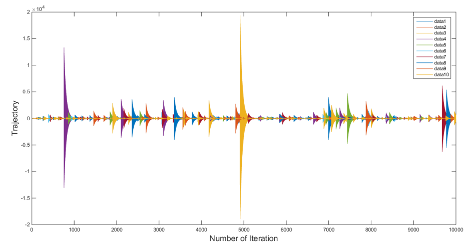
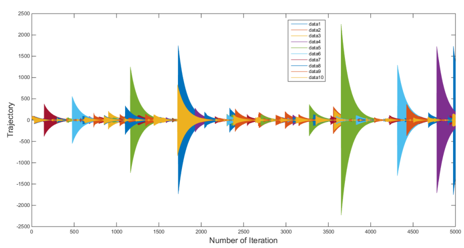
Parameters $(a, \alpha, \beta, A, B, k)$	**Box-Counting Dimension**	**Trajectory**
$(-0.91, -2, -62, 23, -28, 7)$	1.94823	
$(-0.97, 97, -67, -42, -14, 6)$	1.94850	
$(-0.98, 10, -76, -95, 70, 30)$	1.94783	
$(-0.98, -31, 7, 80, 22, 32)$	1.98410	
$(-0.99, 28, 58, -97, 87, 27)$	1.89751	

Table 3: Fractal-like solutions for different parameters