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Minimal Embeddings of Latin Tableaux in Latin Squares

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ABSTRACT. Minimal embeddings of Latin Tableaux LT in strictly larger Latin Squares LS are established. Embeddings of $LT(N)$ into $LS(N+2)$ are studied in particular: it is proven they exist if and only if $N+2$ is prime. For all other N, there exists a Latin Tableau $LT(N)$ with a minimal embedding in a larger Latin Squares $LS(N+k)$ for all $1 < k \leq N$. For odd $k > 3$ the proof depends on the validity of the Wide Partition Conjecture for Young Diagrams.

1 Introduction

A *Young diagram* is a partition of a natural number visualized as set of cells consisting of left-aligned rows whose length corresponds to the numbers in the partition: we take the rows in non-increasing length order. E.g. the Young diagram corresponding to the partition $[7,4,4,1]$ of 16 looks like Figure 1.

The *conjugate* $conj(\lambda)$ of a Young diagram λ is obtained by interchanging the rows with the columns, or reflecting it around the main diagonal. E.g. $conj([7,4,4,1]) = [4,3,3,3,1,1,1]$.

We are here only concerned with self-adjoint Young diagrams, i.e. $\lambda = conj(\lambda)$: such diagrams are symmetric around the main anti-diagonal. We will most often leave out the adjective *self-adjoint*, but it is always assumed from now on.

[1] names a Young diagram with a solution *Latin* and its shape a *Latin Tableau*. A solution consists in filling out all cells of each row with all different numbers from 1 to the length of that row, and similarly for the columns:

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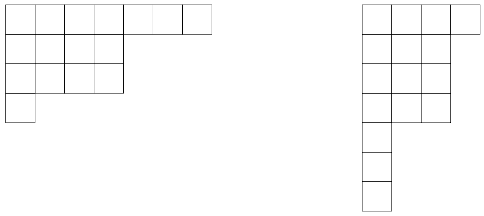


Figure 1: Visualization of the Young diagram [7,4,4,1] and its conjugate

a Latin Square is then just a square Latin Tableau. Some Young diagrams have no solution, others have many, but a handy characterization of solvable Young diagrams seems not to exist. However, [1] makes the following conjecture that applies to Young diagrams and Latin Tableaux:

The Wide Partition Conjecture for free matroids: A partition is Latin if and only if it is wide.

[1] contains a proof of the only-if part of WPC. Wideness is defined in a number of steps.

Definition 1.1. *A partition* μ dominates *a partition* ν*, denoted as* $\mu \geq \nu$ *if*
$\forall j : \sum_{i=1}^{j} \mu_i \geq \sum_{i=1}^{j} \nu_i$ *where any non-existent* μ_i *or* ν_i *are taken to be zero.*

Definition 1.2. *A* lower subpartition *of a partition* $\mu = [\mu_1, \mu_2, ...\mu_l]$ *is any partition* $[\mu_k, \mu_{k+1}, ...\mu_l]$ *for* $k \in 1..l$.

We start counting lower subpartitions form 0, so $[\mu_k, \mu_{k+1}, ...\mu_l]$ is the $(k-1)^{th}$ lower subpartitions of $\mu = [\mu_1, \mu_2, ...\mu_l]$

Definition 1.3. *A partition* μ *is* wide *if for every lower subpartition* λ *of* μ, $\lambda \geq conj(\lambda)$.

The definition above is not the original one, but the one proven to be equivalent in Proposition 3 of [1].

We use the following notation:

- a partition is denoted by a non-increasing sequence of strictly positive numbers, like $[7, 4, 4, 4, 2]$; we also use a shorthand for repeated entries in a partition: $[7, 4(3), 2]$ denotes the same partition
- $YD(N)$ denotes the set of all partitions whose first element equals N (and since self-adjoint, have exactly N elements); $YD[5, 3, 3, 1, 1]$ denotes the (not solvable) YD with partition $[5, 3, 3, 1, 1]$
- $LT(N)$ denotes the elements of $YD(N)$ that have a solution: the particular solution will very often be immaterial for our purposes, although sometimes we need to establish a concrete one; $LT[5, 5, 4, 3, 2]$ denotes the (solvable) YD with partition $[5, 5, 4, 3, 2]$
- $LS(N)$ is the Latin Square of size N, or otherwise denoted $LT[N(N)]$.

An LT is an incomplete Latin Square: see for instance [4] for a theorem proving that every incomplete Latin Square of size N can be embedded in a Latin Square of size $2N$ and that this bound is tight for some incomplete Latin Squares. This motivated us to study the (minimal) embedding of LTs in an LS.

2 Embedding an LT in an LS: the principles

Figure 2 shows an $LT[5,4,4,4,1]$ embedded in an $LS(8)$. Clearly, any solution of the $LT[5,4,4,4,1]$ can be lembedded in the same way in an $LS(8)$, and the part of the $LS(8)$ not belonging to the $LT[5,4,4,4,1]$ is in fact an $LT[8,8,8,7,4,4,4,3]$ with a suitable renumbering of the entries.

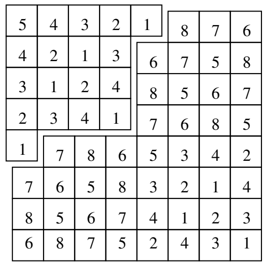


Figure 2: [5,4,4,4,1] and its 3-complement [8,8,8,7,4,4,4,3]

We name that $LT[8,8,8,7,4,4,4,3]$ the 3-complement of the $LT[5,4,4,4,1]$. More generally

Definition 2.1. *The k-complement of a self-adjoint partition* $\lambda = [\lambda_N, \lambda_{N-1}, ...\lambda_2, \lambda_1]$ *(in which* $\lambda_N = N$*) is the sequence obtained by pointwise subtracting* $[\underbrace{0,0,...,0}_{k\ times}, \lambda_1, \lambda_2, ..., \lambda_{N-1}, \lambda_N]$ *from* $[\underbrace{N+k, N+k, ..., N+k}_{N+k\ times}]$

For a partition $\lambda = [\lambda_N, \lambda_{N-1}, ...\lambda_2, \lambda_1]$ (with $\lambda_N = N$), the following two statements are clearly equivalent:

- $LT[\lambda]$ can be embedded in $LS(N+k)$
- the k-complement of $LT[\lambda]$ is in $LT(N+k)$

Assuming the WPC, one can prove that adding a top row and a left column to the partition of an LT results in an LT partition, so we can conclude:

if the $LT[\lambda]$ can be embedded in $LS(N+k)$ for some $k > 0$, it can be embedded in $LS(N+k'), \forall k' \geq k$

The condition $k > 0$ is necessary, as one can check that the 0-complement of $[4,4,2,2]$ is $LT[2,2]$ and has a solution, while the 1-complement of $[4,4,2,2]$ is $YD[5,3,3,1,1]$ with no solution.

It thus makes sense to study the smallest $k > 0$ for which the k-complement of particular LT has a solution: we name this k the *gap* of the LT.

In Section 3, we show

1. there exists exactly one $LT[N]$ with gap equal to 1 - see Section 3.1
2. there exists an $LT[N]$ with gap equal to 2, if and only if $(N+2)$ is composite - see Section 3.2
3. gap 3 exists for all N - see Section 3.3
4. for all N all odd gaps exist - see Section 3.4

For each case, we give explicit constructions, in particular, in case (2), we establish all $LT[N]$ with gap equal to 2: it turns out that there are as many as there are divisors of $(N+2)$.

Only (4) depends on the Wide Partition Conjecture. Moreover, empirically, we also found that for $3 \leq k \leq N \leq 50$, there exists an $LT[N]$ with gap equal to k, but we failed to find a nice characterization.

In Section 3.5 we mention some additional results without proofs or constructions.

3 Embedding a Latin Tableau in a Latin Square: the gaps

3.1 Latin Tableaux with gap 1

Theorem 3.1. *An $LT(N)$ with an $LS(N+1)$-embedding equals $LT([N, N-1, N-2, ..., 2, 1])$.*

Proof. To be solvable, both the $LT(N)$ and its 1-complement must contain the cells in their anti-diagonal (see Lemma 1 of [1]) and the result follows. □

3.2 Latin Tableaux with gap 2

Theorem 3.2. *An $LT[N]$ has gap 2 if and only if $(N+2)$ is composite.*

The *only if* part is proven in Section 3.2.1, the *if* part in Section 3.2.2.

3.2.1 If the gap equals 2, $(N+2)$ is composite

Figure 3 shows how a candidate $LT[9]$ and its two-complement (an $LT[11]$) could look: a priori, each of the grey cells could be part of the $LT[9]$ (if the corresponding $\delta_i = 1$) or its two-complement (if the $\delta_i = 0$). Since the diagrams are self-adjoint, $\delta_i = \delta_j$ for $i + j = N + 2$, or $[\delta_2, \delta_3, ..., \delta_N] = [\delta_N, \delta_{N-1}, ..., \delta_2]$.

Moreover, not all δ's can be equal to zero, neither to one: in the first case, the one-complement would have a solution, so we would not have established an LT with a gap equal to two; in the second case, one can check that the two-complement has no solution or alternatively that it is not wide.

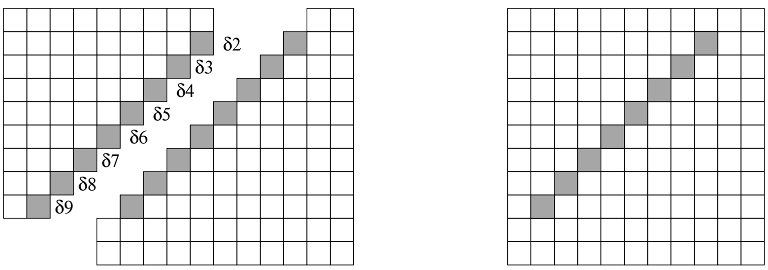


Figure 3: An $LT[9]$, its 2-complement, and how they fit together

Generalizing from Figure 3, we see that the partition of the $LT[N]$ and its two-complement are

$$[N, N-1+\delta_2, N-2+\delta_3, ..., 1+\delta_N] \text{ and } [N+2, N+2, N+1-\delta_2, N-\delta_3, ..., 3-\delta_N, 2].$$

It will be more convenient to represent these sequences as the pointwise sum, or difference, of two other sequences as follows;

$[N, N-1+\delta_2, N-2+\delta_3, ..., N-i-1+\delta_i, ...1+\delta_N] = [N, N-1, N-2, ..., 1] + [0, \delta_2, \delta_3, ..., \delta_N]$

and

$[N+2, N+2, N+1-\delta_2, N-\delta_3, ..., 3-\delta_N, 2] = [N+2, N+2, N+1, N, ..., 3, 2] - [0, 0, \delta_2, \delta_3, ..., \delta_N, 0]$

We refer to the first summand - the one without the δs - the *fixed part*.

If in such a sum or difference, one list is shorter than the other, then think about the shorter one as if extended (at the end) with enough zeros.

If both diagrams are solvable, their partitions must be wide. We express the wideness of the sequences as a number of steps in which it is checked that the next lower subpartition dominates its conjugate. For the sequence of the $LT[N]$ this is summarized in Table 1.

step			
0	$[N, N-1, N-2, ..., 1] + [0, \delta_2, \delta_3, ..., \delta_N]$	$\geq$	$[N, N-1, N-2, ..., 1] + [0, \delta_2, \delta_3, ..., \delta_N]$
1	$[N-1, N-2, ..., 1] + [\delta_2, \delta_3, ..., \delta_N]$	$\geq$	$[N-1, N-2, N-3, ..., 1] + [0, \delta_2, \delta_3, ..., \delta_N]$
2	$[N-2, N-3, ..., 1] + [\delta_3, \delta_4, ..., \delta_N]$	$\geq$	$[N-2, N-3, ..., 1] + [0, \delta_2, \delta_3, ..., \delta_{N-1}]$
...	...		...
n	$[N-n, N-n-1, ..., 1] + [\delta_{n+1}, \delta_{n+2}, ..., \delta_N]$	$\geq$	$[N-n, N-n-1, ..., 1] + [0, \delta_2, \delta_3, ..., \delta_{N-n+1}]$
...	...		...
N-1	$[1] + [\delta_N]$	$\geq$	$[1] + [0, \delta_2]$

Table 1: Dominance checking to be done for the $LT[N]$

In the left column of the table, at each step, one element is chopped off from both lists in the sum. In the right column, 1 is subtracted from each element in the fixed part. The fixed parts in the left and right column are equal at each step, so they can be ignored while checking dominance.

In step 2, expressing dominance results in the inequalities:

- $\delta_3 \geq 0$
- $\delta_3 + \delta_4 \geq \delta_2$
- $\delta_3 + \delta_4 + \delta_5 \geq \delta_2 + \delta_3$, or simplified: $\delta_4 + \delta_5 \geq \delta_2$
- ...

Clearly, the dominance condition at step 2 in Table 1 can be summarized as

$$\delta_{i+1} + \delta_{i+2} \geq \delta_2 \; for \; i = 2...(N-2) \quad (1)$$

Similarly, one can check that the dominance relation in step n in Table 1 implies

$$\delta_{i+1} + \delta_{i+2} + ... + \delta_{i+n} \geq \delta_2 + \delta_3 + ... + \delta_n \; for \; i = n...(N-n) \quad (2)$$

Equation 2 generalizes Equation 1.

Analogous to Table 1, we get Table 2 for the two-complement:

step		
0	$[N+2, N+2, N+1, N, ..., 3, 2] - [0, 0, \delta_2, \delta_3, ..., \delta_N] \geq$	$[N+2, N+2, N+1, N, ..., 3, 2] - [0, 0, \delta_2, \delta_3, ..., \delta_N]$
1	$[N+2, N+1, N, ..., 3, 2] - [0, \delta_2, \delta_3, ..., \delta_N] \geq$	$[N+1, N+1, N, N-1, ..., 2, 1] - [0, 0, \delta_2, \delta_3, ..., \delta_N]$
2	$[N+1, N, ..., 3, 2] - [\delta_2, \delta_3, ..., \delta_N] \geq$	$[N, N, N-1, N-2, ..., 1] - [0, 0, \delta_2, \delta_3, ..., \delta_N]$
3	$[N, N-1, ..., 3, 2] - [\delta_3, ..., \delta_N] \geq$	$[N-1, N-1, N-2, N-3, ..., 1] - [0, 0, \delta_2, \delta_3, ..., \delta_{N-1}]$
...	$... \geq$	...
n	$[N+3-n, N+2-n, ..., 3, 2] - [\delta_n, \delta_{n+1}, ..., \delta_N] \geq$	$[N+2-n, N+2-n, ..., 1] - [0, 0, \delta_2, \delta_3, ..., \delta_{N+2-n}]$
...	$... \geq$	...
N+1	$[2] - [] \geq$	$[1, 1] - []$

Table 2: Dominance checking to be done for the 2-complement of the $LT[N]$

From step 2 on, the sum of a prefix of the fixed part at the left is 1 more than the corresponding prefix sum of the fixed part at the right, so we readily get the inequalities for step 2 as:

- $1 - \delta_2 \geq 0$ or $1 \geq \delta_2$
- $1 - \delta_2 - \delta_3 \geq 0$ or $1 \geq \delta_2 + \delta_3$
- $1 - \delta_2 - \delta_3 - \delta_4 \geq -\delta_2$ or $1 \geq \delta_3 + \delta_4$
- $1 - \delta_2 - \delta_3 - \delta_4 - \delta_5 \geq -\delta_2 - \delta_3$ or $1 \geq \delta_4 + \delta_5$
- ...

The interesting part of the pattern here is

$$1 \geq \delta_{i+1} + \delta_{i+2} \; for \; i = 2...N-2 \tag{3}$$

Similarly, one can check that step n in Table 2 implies

$$1 + \delta_2 + \delta_3 + ... + \delta_{n-1} \geq \delta_{i+1} + \delta_{i+2} + ... + \delta_{i+n} \; for \; i = n..(N-n) \tag{4}$$

Putting Equations 2 and 4 together, we obtain for $i = n..(N-n)$

$$1 + \delta_2 + \delta_3 + ... + \delta_{n-1} \geq \delta_{i+1} + \delta_{i+2} + ... + \delta_{i+n} \geq \delta_2 + \delta_3 + ... + \delta_n \tag{5}$$

Since some δ_j must be equal to one - otherwise, the LT would be $LT[N, N-1, N-2, ..., 1]$ whose gap equals 1, not 2 - let n be the smallest j for which $\delta_j = 1$, i.e. $\delta_n = 1$ and $\forall i < n : \delta_i = 0$. Now Equation 5 reduces to $\delta_{i+1} + \delta_{i+2} + ... + \delta_{i+n} = 1$. This means that in each stretch of n consecutive δ's, there is exactly one δ equal to 1 and it must be at locations that are a multiple of n. I.e. $\delta_{i \times n} = 1$ for all possible i, and $\delta_j = 0$ for j not a multiple of n. Using the self-adjointness of the $LT[N]$, we obtain that $(N+2)$ must be a multiple of n, and we conclude:

> If a self-adjoint $YD[N]$ and its two-complement are both wide, then $(N+2)$ is not prime. Moreover, for each divisor d of $(N+2)$ there is exactly one such $YD[N]$.

Since wideness is a prerequisite to being solvable, we conclude

Lemma 3.3. *If an $LT[N]$ has gap 2, $(N+2)$ is composite.*

3.2.2 For composite $(N+2)$, there is an $LT[N]$ with gap equal to two

For each proper divisor d of $(N+2)$, we construct a solved $LT[N]$ with its embedding in $LS(N+2)$. Furthermore, we show that it has no $LS(N+1)$.

Let $+\!\!+$ symbolize list concatenation. For given N and d, let $\lambda_{N,d} = [N, N-1, N-2, ..., 2, 1] +\!\!+ [\delta_2, \delta_3, ..., \delta_{N-1}, \delta_N]$ with $\delta_j = 1$ if and only if j is a multiple of d. Considering $YD[\lambda_{N,d}]$, it is easy to check that it is self-conjugate if and only if d divides $(N+2)$.

Lemma 3.4. *If d divides $(N+2)$, $YD[\lambda_{N,d}]$ and its 2-complement are solvable.*

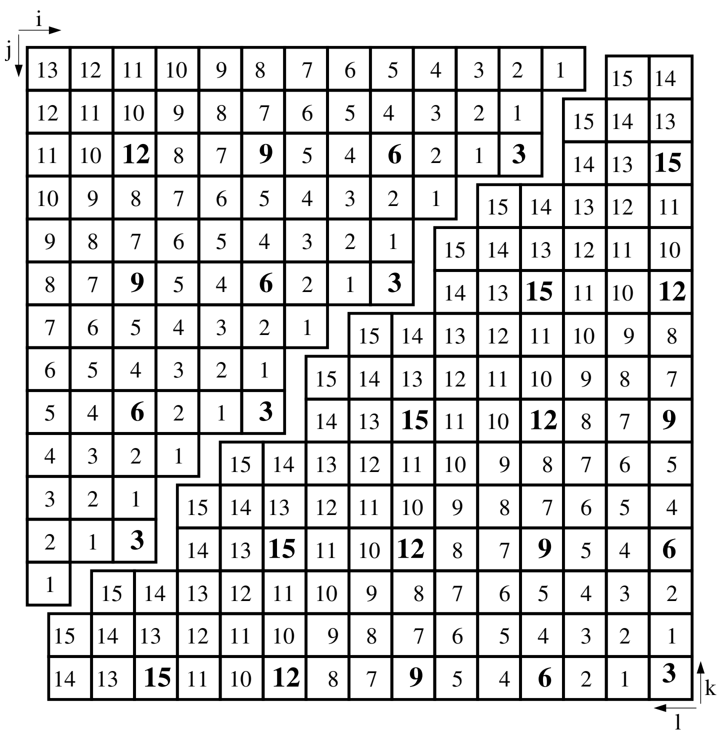


Figure 4: A solution for $YD[\lambda_{13,3}]$ and its 2-complement

Proof. As an example, Figure 4 shows a solution for $YD[\lambda_{13,3}]$ and for its 2-complement. These solutions were systematically constructed starting from filling out diagonals with a particular number and then introducing multiples of 3 at appropriate places. For the general description of such solutions for an $YD[\lambda_{N,d}]$ with d a divisor of $(N+2)$, we number the cells of $YD[\lambda_{N,d}]$ with (i,j) starting at $(1,1)$ as in the figure, similarly for its two-complement with (k,l) also starting at $(1,1)$. The solution for $YD[\lambda_{N,d}]$ is

$cell(i \times d, j \times d) = N + 2 - i - j + d$ for $1 \leq i, j \leq \frac{N}{d}$

$cell(i, j) = N + 2 - i - j$ for the other relevant indices

and for its two-complement it is

$cell(k, l) = l + k - 1$ for all relevant (k, l) except for

$cell(1 + k \times d, 1 + l \times d) = k + l - 2 + d$ for relevant (k, l)

□

Together, Lemmas 3.3 and 3.4 imply Theorem 3.2

3.3 Latin Tableaux with gap 3

In this section, we establish constructively $LT[N]$ with gap equal to 3. We consider odd N and even N separately.

Odd N Figure 5 shows an $LT[N]$ for $N = 5, 7, 9$ with their 0-complement filled out with three new values $N+1, N+2$ and $N+3$, symbolized as a, b, c so that the similarity between the examples is more apparent. The cell with the question mark can be filled out with any non-conflicting value: a is always appropriate.

5	4	3	2	1
4	1	2	3	a
3	2	4	1	b
2	3	1	?	c
1	a	b	c	5

7	6	5	4	3	2	1
6	4	3	5	2	1	a
5	3	1	2	4	a	b
4	5	2	3	1	b	c
3	2	4	1	?	c	7
2	1	a	b	c	7	6
1	a	b	c	7	6	5

9	8	7	6	5	4	3	2	1
8	7	5	4	6	3	2	1	a
7	5	6	3	4	2	1	a	b
6	4	3	1	2	5	a	b	c
5	6	4	2	3	1	b	c	9
4	3	2	5	1	?	c	9	8
3	2	1	a	b	c	9	8	7
2	1	a	b	c	9	8	7	6
1	a	b	c	9	8	7	6	5

Figure 5: Three $LT[N]$ having an $LS(N+3)$-embedding for $N = 5, 7, 9$

The idea is as follows: given a solution of the particular $LT[N]$, the added cells form a Latin Rectangle of order N by N based on the numbers $1..(N+3)$ and in which every symbol occurs at least $N-3$ times. Theorem 2 in [5)] guarantees that this Latin Rectangle can be extended to an $LS(N+3)$, meaning that the gap of this $LT[N]$ is 3 or less. The gap is certainly not 1 (by Theorem 3.1), so we still need prove that the gap is not 2:

The tableau is $LT[2n+1, 2n, ..., n+2, n+2, n+1, n-1, n-2, ...2, 1]$ for some n. Its 2-complement is $LT[2n+3(2), 2n+2, ..., n+4, n+2, n+1, n+1, ..., 3, 2]$. Take the lower subpartition that starts at $(n+2)$, i.e. $[n+2, n+1, n+1, ..., 3, 2]$. Its conjugate starts with $[n+2, n+2, ...]$ so this subpartition does not dominate its conjugate and the 2-complement is not wide, so it cannot have a solution.

The above construction can be generalized to any odd $N \geq 5$.

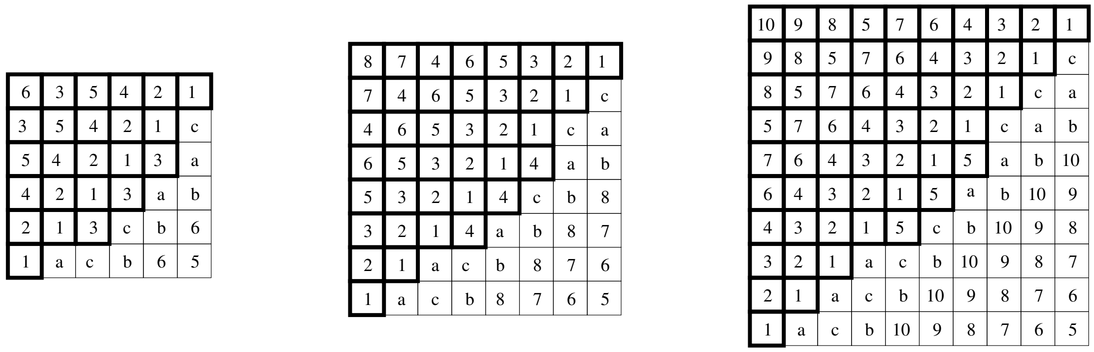


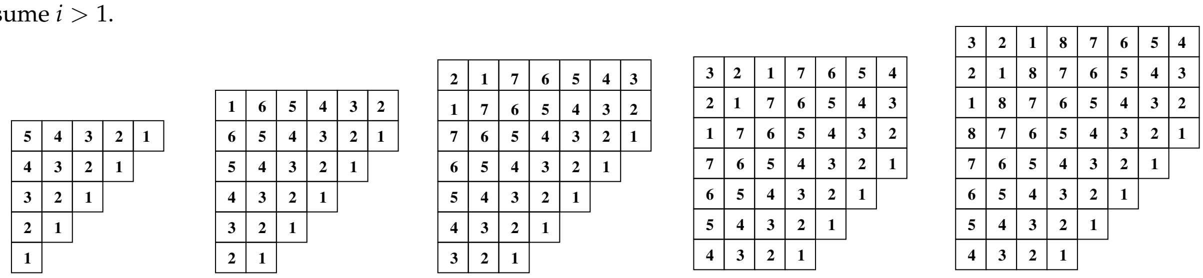
Figure 6: Three $LT[N]$ having an $LS(N+3)$-embedding for $N = 6, 8, 10$

Even N The reasoning is the similar to the one for odd N: Figure 6 can be generalized to all even $N \geq 4$.

We still need to prove that the gap is not 2. As in the odd case, it is enough to prove that the 2-complement is not wide. Let $N = 2n$. The shown LT has sequence $[2n, 2n-1, ..., n+2, n+2, n+1, n, n-2, n-3, ..., 2, 1]$. Its 2-complement has sequence $[2n+2(2), 2n+1, 2n, ..., n+4, n+2, n+1, n, n, n-1, n-2, ..., 3, 2]$. Take the lower subpartition that starts at $(n+1)$, i.e. $[n+1, n, n, n-1, n-2, ..., 2]$. Its conjugate starts as $[n+1, n+1, ...]$ so that lower subpartition is not wide, and therefore the 2-complement has no solution.

3.4 All odd gaps

We denote $LT[N(i), N-1, N-2, ..., i]$ by $DLT(N, i)$. $DLT(N, 1)$ has a 1-gap, and is not interesting, so we will assume $i > 1$.



DLT(5,1) **DLT(6,2)** **DLT(7,3)** **DLT(7,4)** **DLT(8,4)**

There is more than one way to show how the construction of a solution can be generalized to any $DLT(N, i)$. One can start from the unique solution of $DLT(N-i+1, 1)$ and add top rows and left columns, completing the solution gradually. Numbering the rows from N [the topmost] to 1, a general description of this solution is:

- row $r \in 1..N-i+1$ has length L_r and contents $L_r, L_r-1, ..., 1$
- any higher row is obtained from the one just below it by a circular right shift

An alternative description would consider the anti-diagonals.

Theorem 3.5. *For $k \leq N$, the k-complement of $DLT(N, i)$ is wide if and only if $2i-1 \leq k$.*

Proof. We first prove that the condition $2i - 1 \leq k$ is necessary. The k-complement of $DLT(N, i)$ has the sequence

$[(N + k)(k), N + k - i, N + k - i - 1, ..., k + 1, k(i)]$

The following expresses that the k^{th} lower subpartition is wide:

$[N + k - i, N + k - i - 1, ..., k + 1, k(i)] \geq [N(k), N - i, N - i - 1, ..., 1]$

Taking the sum of the first k elements of each sequence, gives

$\sum_{j=0}^{k-1}(N + k - i - j) \geq Nk$ or $\sum_{j=0}^{k-1} k = k^2 \geq \sum_{j=0}^{k-1}(i + j) = ik + k(k - 1)/2$ resulting in $k \geq 2i - 1$.

We now prove the condition $2i - 1 \leq k$ is sufficient to be wide. It is actually enough to prove this for $k = 2i - 1$, so we must prove that the sequence

$[(N + k)(k), N + k - i, N + k - i - 1, ..., k + 1, k(i)]$ is wide for $2 \leq i \leq (N + 1)/2$

or, replacing $N + k$ above by M, that $[M(2i - 1), M - i, M - i - 1, ..., 2i, (2i - 1)(i)]$ is wide. We consider three cases for the r^{th} lower subpartition:

1. $\mathbf{0 \leq r \leq 2i - 1}$

   We need to prove that

   $\lambda = [M(2i - 1 - r), M - i, M - i - 1, ..., 2i, (2i - 1)(i)] \geq$
   $[(M - r)(2i - 1), M - i - r, ..., 2i - r, (2i - 1 - r)(i)] = \mu$

   We split both sequences in 4 parts: $\lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ and $\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4$ as follows:

$\lambda_1 =$	$\lambda_2 =$	$\lambda_3 =$	$\lambda_4 =$
[M(2i-1-r)]	[M-i, M-i-1,...,M-i-r+1]	[M-i-r, ..., 2i]	[(2i-1)(i)]
$\mu_1 =$	$\mu_2 =$	$\mu_3 =$	$\mu_4 =$
[(M-r)(2i-1-r)]	[(M-r)(r)]	[M-i-r, ..., 2i]	[2i-1, 2i-2, ...2i-1-r(i)]

   The table makes it easy to check that the partial prefix sums fulfill the required condition.

2. $\mathbf{2i - 1 < r < M - i}$

   We need to check that $\lambda = [M + i - r - 1, M + i - r - 2, ..., 2i, (2i - 1)(i)]$ dominates $\mu = [(M - r)(2i - 1), M - i - r, ..., 1]$. We split the lists in 3 parts as follows:

$\lambda_1 =$	$\lambda_2 =$	$\lambda_3 =$
[M+i-r-1, M+i-r-2,..., M-i-r+1]	[M-i-r, M-i-r-1,...,2i]	[(2i-1)(i)]
$\mu_1 =$	$\mu_2 =$	$\mu_3 =$
[(M-r)(2i-1)]	[M-i-r, M-i-r-1,...,2i]	[2i-1, 2i-2, ..., 1]

   Once more, the table makes it easy to check that the partial prefix sums fulfill the required condition, especially since λ_1 and μ_1 have the same length, $sum(\lambda_1) = sum(\mu_1)$ and $\lambda_2 = \mu_2$.

3. $\mathbf{M - i \leq r}$

   We now need to check that $[(2i - 1)(M - r)]$ dominates $[(M - r)(2i - 1)]$. This follows from the fact that $(2i - 1) \geq (M - r)$ (since $i > 1$)

□

Since we have not constructed a solution for the $(2i - 1)$-complement of the $DLT(N, i)$, we can conclude that under the assumption of the WPC, all odd gaps occur for all N.

3.5 More LT-families with their gaps

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For other families of Latin Tableaux, we have found their gaps mostly by explicit constructions:

- $LT[N(i), N-1(N-i-1), i]$ has gaps N, $N-1$ and $N-2$
- $LT[2n(n), n(n)]$ has gap equal to n
- $LT[2n(n), (n+1), n(n-1)]$ has gap $n+1$
- $LT[N(i), (N-b)(N-i-b), i(b)]$ with $b \leq min(\frac{N}{2}, i)$ shows (amongst others) all gaps in $[\frac{N}{2}...N]$

The latter case was established only empirically and for $N < 50$: we used B-Prolog [6]. More details can be found in the technical report [3].

4 Conclusion

Our initial interest in Latin Tableaux was related to redundant disequalities in the CSP formulation of $LT(N)$: [2] discusses this issue for the Latin Square problem. We strayed into the study of embeddings of an $LT(N)$ in a strictly larger Latin Square $LS(M)$. The gap of a particular $LT(N)$ characterizes the smallest $M > N$ for which an $LT(N)$ can be embedded in an $LS(M)$. The gap depends on the shape of the particular $LT[N]$. We have established - and partly proven - the result that for all N, all gaps occur, except when $(N+2)$ is prime, in which case there is no $LT[N]$ with gap equal to 2. The relation with redundant disequalities needs to be investigated.

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