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Dynamics on Unique Numerators for Generating the Cantor Set

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Abstract. This work provides a fomular used to generate a Cantor set of any segment. This is called the Lower Bound Numerator Generator($LBNG$) In this new approach, you can construct a specified segment of Cantor set without having to first construct the segment before. This approach guarantees accuracy, effectiveness and effciency and speed. The construction of the Cantor set using the other approaches is rigorous, time demanding and complicated compared to this approach.

1 Introduction

The Cantor set was discovered by Henry John Stephen Smith [3] The Cantor set is a famous set first introduced by German mathematician Georg Cantor in 1883[6]. The Cantor set is a very popular concept which plays a very important role in many branches of mathematics[5].The Cantor set has a lot of deep and wonderful properties. People like Baire also have contributed to this concept. Cantor set is very important and integral in some branches in Mathematics including set theory , chaotic dynamical system and fractal geometry[5]. The Cantor set is the prototype of a fractal[5]. Cantor set is one of the basis forms of fractals. The most popular form for obtaining the set is by deleting or removing the middle thirds of a line segment $[0, 1]$

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2 Preliminaries

Definition 2.1. Let S be a subset of $\mathbb{R}$. If there exist a real number m such that $m \geq s$ for all $s \in S$, then m is called an upper bound for S, hence S is bounded above[7].
If $m \leq s$, for all $s \in S$,then m is a lower bound for s which implies S is bounded below. Any set that has both an upper bound and lower bound is said to be bounded.
Definition 2.2. A set S is compact if and only if every open cover contains a finite sub cover[7]. Heine Borel theorem on compact sets describes the set as a close and bounded set.
Definition 2.3. A subset A of a topological space X is closed if the set $X - A$ is open[1]. In terms of limit points , if a set is closed, then the set must contains all of its limit points.
Definition 2.4. A subset A of a topological space X is called dense if the closure of A is the same as the set X.

3 Methodology and Results

Over the years, different approaches and methods have been invented and used in the construction of the Cantor set. Some are popular whiles others are lesser known. Two different approaches are illustrated in this chapter for comparative reasons such that we can compare and conclude if any particular approach is by a distance better than the other. The conditions that will be key in describing which approach is the best includes : the amount of work done in the process , the speed and ease of construction , the margin of errors or how easily errors can be avoided, convenience etc.

3.1 General Approach and Construction of Cantor Set

This is the most common and possibly basic approach used in the construction of Cantor sets.
We begin with the closed real interval C_o, where the nth set formula is given as:

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$$

Where $n = 1, 2, 3, \ldots$ and $C_o = [0, 1]$

When n=1

$$\begin{aligned} C_1 &= \frac{C_0}{3} \cup \left(\frac{2}{3} + \frac{C_0}{3}\right) \\ C_1 &= \left(\frac{0}{3}, \frac{1}{3}\right) \cup \left(\frac{2}{3} + \left[\frac{0}{3}, \frac{1}{3}\right]\right) \end{aligned}$$

Open middle third $I_1 = (\frac{1}{3}, \frac{2}{3})$

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

When n=2

$$\begin{aligned} C_2 &= \frac{C_1}{3} \cup \left(\frac{2}{3} + \frac{C_1}{3} \right) \\ C_2 &= \frac{[0,\frac{1}{3}] \cup [\frac{2}{3},1]}{3} \cup \left(\frac{2}{3} + \frac{[0,\frac{1}{3}] \cup [\frac{2}{3},1]}{3} \right) \end{aligned}$$

Open middle third $I_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

When n=3

$$\begin{aligned} C_3 &= \frac{C_2}{3} \cup \left(\frac{2}{3} + \frac{C_2}{3} \right) \\ C_3 &= \frac{[0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]}{3} \cup \left(\frac{2}{3} + \frac{[0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]}{3} \right) \end{aligned}$$

Open middle third $I_3 = (\frac{1}{27}, \frac{2}{27}) \cup (\frac{7}{27}, \frac{8}{27}) \cup (\frac{19}{27}, \frac{20}{27}) \cup (\frac{25}{27}, \frac{26}{27})$

$$C_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup [\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right]$$

When n=4

$$C_4 = \frac{C_3}{3} \cup \left(\frac{2}{3} + \frac{C_3}{3} \right)$$

$$C_4 = \frac{[0,\frac{1}{27}] \cup [\frac{2}{27},\frac{1}{9}] \cup [\frac{2}{9},\frac{7}{27}] \cup [\frac{8}{27},\frac{1}{3}] \cup [\frac{2}{3},\frac{19}{27}] \cup [\frac{20}{27},\frac{7}{9}] U [\frac{8}{9},\frac{25}{27}] \cup [\frac{26}{27},1]}{3} \cup$$
$$\left(\frac{2}{3} + \frac{[0,\frac{1}{27}] U [\frac{2}{27},\frac{1}{9}] \cup [\frac{2}{9},\frac{7}{27}] \cup [\frac{8}{27},\frac{1}{3}] \cup [\frac{2}{3},\frac{19}{27}] \cup [\frac{20}{27},\frac{7}{9}] \cup [\frac{8}{9},\frac{25}{27}] \cup [\frac{26}{27},1]}{3} \right)$$

The observation now is that, it is becoming more and more cumbersome already and much concentration is needed.

Open middle third $I_4 = (\frac{1}{81}, \frac{2}{81}) \cup (\frac{7}{81}, \frac{8}{81}) \cup (\frac{19}{81}, \frac{20}{81}) \cup (\frac{25}{81}, \frac{26}{81}) \cup (\frac{61}{81}, \frac{62}{81}) \cup (\frac{73}{81}, \frac{74}{81}) \cup (\frac{79}{81}, \frac{80}{81})$

$$C_4 = \left[0, \frac{1}{81}\right] \cup \left[\frac{2}{81}, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{7}{81}\right] \cup \left[\frac{8}{81}, \frac{1}{9}\right] U [\frac{2}{9}, \frac{19}{81}\right] \cup \left[\frac{20}{81}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{25}{81}\right] \cup \left[\frac{26}{81}, \frac{1}{3}\right] \cup$$
$$\left[\frac{2}{3}, \frac{55}{81}\right] \cup \left[\frac{56}{81}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{61}{81}\right] \cup \left[\frac{62}{81}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{73}{81}\right] \cup \left[\frac{74}{81}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, \frac{79}{81}\right] \cup \left[\frac{80}{81}, 1\right]$$

As n increases per the formula, it gets more and more cumbersome and care must be taken to note the technicalities that arise as seen throughout.The danger may be that one may leave out a particular middle third which would in-turn affect the answer.

Let's see when n=5

$$C_5 = \frac{C_4}{3} \cup (\frac{2}{3} + \frac{C_4}{3})$$

We write out C_4 from the above and substitute into C_5, Open middle third I_5 is

$(\frac{1}{243}, \frac{2}{243}) \cup (\frac{7}{243}, \frac{8}{243}) \cup (\frac{19}{243}, \frac{20}{243}) \cup (\frac{25}{243}, \frac{26}{243}) \cup (\frac{61}{243}, \frac{62}{243}) \cup (\frac{73}{243}, \frac{74}{243}) \cup$

$(\frac{79}{243}, \frac{80}{243}) \cup (\frac{163}{243}, \frac{164}{243}) \cup (\frac{169}{243}, \frac{170}{243}) \cup (\frac{181}{243}, \frac{182}{243}) \cup (\frac{187}{243}, \frac{188}{243}) \cup (\frac{217}{243}, \frac{218}{243}) \cup$

$(\frac{223}{243}, \frac{224}{243}) \cup (\frac{235}{243}, \frac{236}{243}) \cup (\frac{241}{243}, \frac{242}{243})$

$C_5 = [0, \frac{1}{243}] \cup [\frac{2}{243}, \frac{1}{81}] \cup [\frac{2}{81}, \frac{7}{243}] \cup [\frac{8}{243}, \frac{1}{27}] \cup [\frac{2}{27}, \frac{19}{243}] \cup [\frac{20}{243}, \frac{7}{81}] \cup$

$[\frac{8}{81}, \frac{25}{243}] \cup [\frac{26}{243}, \frac{1}{9}] \cup [\frac{2}{9}, \frac{55}{243}] \cup [\frac{56}{243}, \frac{19}{81}] \cup [\frac{20}{81}, \frac{61}{243}] \cup [\frac{62}{243}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{73}{243}] \cup [\frac{74}{243}, \frac{25}{81}]$

$\cup [\frac{26}{81}, \frac{79}{243}] \cup [\frac{80}{243}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{163}{243}] \cup [\frac{164}{243}, \frac{55}{81}] \cup [\frac{56}{81}, \frac{169}{243}] \cup [\frac{170}{243}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{181}{243}] \cup [\frac{182}{243}, \frac{61}{81}]$

$\cup [\frac{62}{81}, \frac{187}{243} \cup [\frac{188}{243}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{217}{243}] \cup [\frac{218}{243}, \frac{73}{81}] \cup [\frac{74}{81}, \frac{223}{243}] \cup [\frac{224}{243}, \frac{25}{27}] \cup [\frac{26}{27}, \frac{235}{243}] \cup [\frac{236}{243}, \frac{79}{81}]$

$\cup [\frac{80}{81}, \frac{241}{243}] \cup [\frac{242}{243}, 1]$

3.2 The Lower Bound Numerator Generator as an Alternative Approach

The Cantor set is always made up of disconnected closed interval. Every interval is bounded which implies it has both an upper bound and lower bound. The name of this approach really suggest where the focal point of this approach is. The focus is mainly on the patterns established in the numerator which are in turn used to obtain other Cantor sets. The following observation were made in line with the approach used.

1. The first term of every Cantor set is derived from $C_1 = \left[0, \frac{1}{3^n}\right]$
2. The second term of every Cantor set is derived from $C_2 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right]$
3. The last term of every Cantor set is derived from $C_{2^n} = \left[\frac{3^n-1}{3^n}, 1\right]$
4. The Cantor set has an even number of segment for all values of n
5. The segments of Cantor set for a particular n can be group into pairs of two being odd(C_{2n-1}) and even(C_{2n}) which implies there is always the same number of even segments and odd segments for every n
6. For an odd segment , the difference between the numerator of its upper bound and the numerator of the lower bound of the even segment that follows is always 1.

7. For an even segment , the difference between the numerator of its upper bound and the numerator of the lower bound of the odd segment that follows is always a multiple of 3.

The The Lower Bound Numerator Generator(LBNG) is now considered in detailed below with examples

3.2.1 The Lower Bound Numerator Generator(LBNG)

This approach is derived in relation to the numerators of the Cantor set without reducing them to their least form. A general pattern was observed among the numerators of the Cantor set but this trend and relation is mostly lost when the values are reduced to their least forms. The fact that , all Cantor sets have a denominator of 3^n is preserved in arriving at this approach. Even though , some numerators are reducible, these numbers were ignored in that regard. The steps and examples for the Lower Bound Numerator Generator Approach is outline below.

Steps

First we deal with a few key notation that will be dominant in this work.

Let C_n represent a Cantor set of segment n

Let $C_n = \left[\frac{L_n}{3^n}, \frac{U_n}{3^n}\right]$

Also $\frac{L_n}{3^n}$ is the lower bound of the Cantor set.

We call L_n the Lower bound numerator .

The steps for the Lower Bound Numerator Generator approach is outline as follows

Given the interval $[0, 1]$ and with $n \geq 2$

1. Find the interval of the Cantor set using $\frac{1}{3^n}$ and the number of segments for the Cantor set using 2^n
2. The first segment $C_1 = \left[0, \frac{1}{3^n}\right]$
3. The second segment of every Cantor set is derived by $C_2 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right]$
4. The last term of every Cantor set is derived by $C_{2^n} = \left[\frac{3^n - 1}{3^n}, 1\right]$

Now due to the fact that the construction will also introduce another variable n, we dummy the n variable associated with the interval 3^n to become 3^m. Here the m is the n used to generate the interval and the number of segments, whiles the new n will be used for the recursive construction of the cantor set

5. There are now $2^n - 3$ segments to be computed. This is done using the formula below to generate the remaining set in odd and even pairs as

   (a) $C_{2n-1} = \left[\frac{3L_n}{3^m}, \frac{3L_n+1}{3^m}\right]$

   (b) $C_{2n} = \left[\frac{3L_n+2}{3^m}, \frac{3L_n+3}{3^m}\right]$

   for $n \geq 2, ..., 2^n - 1$

Examples

In the examples that follow, we do the construction of the Cantor set exhausting every step outlined in the previous session.

1. Construct the the cantor the Cantor set with $n = 3$

   Step 1

   Find the interval of the Cantor set using $\frac{1}{3^n}$ and the number of segments for the Cantor set using 2^n
   For $n = 3$, the interval is $\frac{1}{3^n} = \frac{1}{3^3} = \frac{1}{27}$

   For $n = 3$, the number of segments is $2^n = 2^3 = 8$

   Step 2

   The first segment $C_1 = \left[0, \frac{1}{3^n}\right]$

   $C_1 = \left[0, \frac{1}{3^n}\right] = \left[0, \frac{1}{3^3}\right] = \left[0, \frac{1}{27}\right]$

   Step 3

   The second segment of every Cantor set is derived by $C_2 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right]$

   $C_2 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right] = \left[\frac{2}{3^3}, \frac{3}{3^3}\right] = \left[\frac{2}{27}, \frac{3}{27}\right]$

   Step 4

   The last term of every Cantor set is derived by $C_{2^n} = \left[\frac{3^n - 1}{3^n}, 1\right]$

   $C_{2^n} = C_8 = \left[\frac{3^n - 1}{3^n}, 1\right] = \left[\frac{3^3 - 1}{3^3}, 1\right] = \left[\frac{26}{27}, 1\right]$

   After Step 4, the Cantor set for $n = 3$ looks like this

   $$\left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{3}{27}\right] \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup \left[\frac{26}{27}, 1\right].$$

We complete the construction process using step 5

Step 5

There are now $2^n - 3$ segments to be computed. This is done using the formula below to generate the remaining Cantor set in odd and even pairs as

(a) $C_{2n-1} = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right]$

(b) $C_{2n} = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right]$

for $n \geq 2, ..., 2^n - 1$

For $n = 2$ and $m = 3$

Also $L_n = L_2 = 2$

$$C_{2n-1} = C_3 = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_2}{3^3}, \frac{3L_2 + 1}{3^3}\right] = \left[\frac{3 \times 2}{27}, \frac{(3 \times 2) + 1}{27}\right]$$

$$C_3 = \left[\frac{6}{27}, \frac{7}{27}\right]$$

$$C_{2n} = C_4 = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_2 + 2}{3^3}, \frac{3L_2 + 3}{3^3}\right] = \left[\frac{(3 \times 2) + 2}{27}, \frac{(3 \times 2) + 3}{27}\right]$$

$$C_4 = \left[\frac{8}{27}, \frac{9}{27}\right]$$

For $n = 3$ and $m = 3$

Also $L_n = L_3 = 6$

$$C_{2n-1} = C_5 = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_3}{3^3}, \frac{3L_3 + 1}{3^3}\right] = \left[\frac{3 \times 6}{27}, \frac{(3 \times 6) + 1}{27}\right]$$

$$C_5 = \left[\frac{18}{27}, \frac{19}{27}\right]$$

$$C_{2n} = C_6 = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_3 + 2}{3^3}, \frac{3L_3 + 3}{3^3}\right] = \left[\frac{(3 \times 6) + 2}{27}, \frac{(3 \times 6) + 3}{27}\right]$$

$$C_6 = \left[\frac{20}{27}, \frac{21}{27}\right]$$

For $n = 4$ and $m = 3$

Also $L_n = L_4 = 8$

$$C_{2n-1} = C_7 = \left[\frac{3L_n}{3^m}, \frac{3L_n+1}{3^m}\right] = \left[\frac{3L_4}{3^3}, \frac{3L_4+1}{3^3}\right] = \left[\frac{3\times 8}{27}, \frac{(3\times 8)+1}{27}\right]$$

$$C_7 = \left[\frac{24}{27}, \frac{25}{27}\right]$$

$$C_{2n} = C_8 = \left[\frac{3L_n+2}{3^m}, \frac{3L_n+3}{3^m}\right] = \left[\frac{3L_4+2}{3^3}, \frac{3L_3+4}{3^3}\right] = \left[\frac{(3\times 8)+2}{27}, \frac{(3\times 8)+3}{27}\right]$$

$$C_8 = \left[\frac{26}{27}, \frac{27}{27}\right] = \left[\frac{26}{27}, 1\right]$$

The Cantor set for $n = 3$ is an 8 - segment Cantor set with an interval of $\frac{1}{27}$ is given as

$$\left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27} \cup \frac{3}{27}\right] \cup \left[\frac{6}{27}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{9}{27}\right] \cup \left[\frac{18}{27}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{21}{27}\right] \cup \left[\frac{24}{27}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, \frac{27}{27}\right]$$

which can be simplified as

$$\left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27} \cup \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right]$$

2. Construct the the cantor the Cantor set with $n = 5$

Step 1

Find the interval of the Cantor set using $\frac{1}{3^n}$ and the number of segments for the Cantor set using 2^n

For $n = 5$, the interval is $\frac{1}{3^n} = \frac{1}{3^5} = \frac{1}{243}$

For $n = 5$, the number of segments is $2^n = 2^5 = 32$

Step 2

The first segment $C_1 = \left[0, \frac{1}{3^n}\right]$

$$C_1 = \left[0, \frac{1}{3^n}\right] = \left[0, \frac{1}{3^5}\right] = \left[0, \frac{1}{243}\right]$$

Step 3

The second segment of every Cantor set is derived by $C_2 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right]$

$C_2 = \left[\frac{2}{3^n}, \frac{3}{3^n}\right] = \left[\frac{2}{3^5}, \frac{3}{3^5}\right] = \left[\frac{2}{243}, \frac{3}{243}\right]$

Step 4

The last term of every Cantor set is derived by $C_{2^n} = \left[\frac{3^n - 1}{3^n}, 1\right]$

$C_{2^n} = C_{32} = \left[\frac{3^n - 1}{3^n}, 1\right] = \left[\frac{3^5 - 1}{3^5}, 1\right] = \left[\frac{242}{243}, 1\right]$

After Step 4, the Cantor set for $n = 5$ looks like this

$\left[0, \frac{1}{243}\right] \cup \left[\frac{2}{243}, \frac{3}{243}\right] \cup C_3 \cup C_4 \cup C_5 \cup C_6 \cup C_7 \cup C_8 \cup C_9 \cup C_{10}$

$\cup C_{11} \cup C_{12} \cup C_{13} \cup C_{14} \cup C_{15} \cup C_{16} \cup C_{17} \cup C_{18} \cup C_{19} \cup C_{20} \cup C_{21} \cup C_{22} \cup C_{23}$

$\cup C_{24} \cup C_{25} \cup C_{26} \cup C_{27} \cup C_{28} \cup C_{29} \cup C_{30} \cup C_{31} \cup \left[\frac{242}{243}, 1\right].$

We complete the construction process using step 5

Step 5

There are now $2^n - 3$ segments to be computed. This is done using the formula below to generate the remaining Cantor set in odd and even pairs as

(a) $C_{2n-1} = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right]$

(b) $C_{2n} = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right]$

for $n \geq 2, ..., 2^n - 1$

For $n = 2$ and $m = 5$

Also $L_n = L_2 = 2$

$C_{2n-1} = C_3 = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_2}{3^5}, \frac{3L_2 + 1}{3^5}\right] = \left[\frac{3 \times 2}{243}, \frac{(3 \times 2) + 1}{243}\right]$

$C_3 = \left[\frac{6}{243}, \frac{7}{243}\right]$

$C_{2n} = C_4 = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_2 + 2}{3^5}, \frac{3L_2 + 3}{3^5}\right] = \left[\frac{(3 \times 2) + 2}{243}, \frac{(3 \times 2) + 3}{243}\right]$

$$C_4 = \left[\frac{8}{243}, \frac{9}{243}\right]$$

For $n = 3$ and $m = 5$

Also $L_n = L_3 = 6$

$$C_{2n-1} = C_5 = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_3}{3^5}, \frac{3L_3 + 1}{3^5}\right] = \left[\frac{3 \times 6}{243}, \frac{(3 \times 6) + 1}{243}\right]$$

$$C_5 = \left[\frac{18}{243}, \frac{19}{243}\right]$$

$$C_{2n} = C_6 = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_3 + 2}{3^5}, \frac{3L_3 + 3}{3^5}\right] = \left[\frac{(3 \times 6) + 2}{243}, \frac{(3 \times 6) + 3}{243}\right]$$

$$C_6 = \left[\frac{20}{243}, \frac{21}{243}\right]$$

For $n = 4$ and $m = 5$

Also $L_n = L_4 = 8$

$$C_{2n-1} = C_7 = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_4}{3^5}, \frac{3L_3 + 1}{3^5}\right] = \left[\frac{3 \times 8}{243}, \frac{(3 \times 8) + 1}{243}\right]$$

$$C_7 = \left[\frac{24}{243}, \frac{25}{243}\right]$$

$$C_{2n} = C_8 = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_4 + 2}{3^5}, \frac{3L_3 + 3}{3^5}\right] = \left[\frac{(3 \times 8) + 2}{243}, \frac{(3 \times 8) + 3}{243}\right]$$

$$C_8 = \left[\frac{26}{243}, \frac{27}{243}\right]$$

For $n = 5$ and $m = 5$

Also $L_n = L_4 = 18$

$$C_{2n-1} = C_9 = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_5}{3^5}, \frac{3L_5 + 1}{3^5}\right] = \left[\frac{3 \times 18}{243}, \frac{(3 \times 18) + 1}{243}\right]$$

$$C_9 = \left[\frac{54}{243}, \frac{55}{243}\right]$$

$$C_{2n} = C_{10} = \left[\frac{3L_n+2}{3^m}, \frac{3L_n+3}{3^m}\right] = \left[\frac{3L_5+2}{3^5}, \frac{3L_5+3}{3^5}\right] = \left[\frac{(3\times 18)+2}{243}, \frac{(3\times 18)+3}{243}\right]$$

$$C_{10} = \left[\frac{56}{243}, \frac{57}{243}\right]$$

For $n = 6$ and $m = 5$

Also $L_n = L_6 = 20$

$$C_{2n-1} = C_{11} = \left[\frac{3L_n}{3^m}, \frac{3L_n+1}{3^m}\right] = \left[\frac{3L_6}{3^5}, \frac{3L_6+1}{3^5}\right] = \left[\frac{3\times 20}{243}, \frac{(3\times 20)+1}{243}\right]$$

$$C_{11} = \left[\frac{60}{243}, \frac{61}{243}\right]$$

$$C_{2n} = C_{12} = \left[\frac{3L_n+2}{3^m}, \frac{3L_n+3}{3^m}\right] = \left[\frac{3L_6+2}{3^5}, \frac{3L_6+3}{3^5}\right] = \left[\frac{(3\times 20)+2}{243}, \frac{(3\times 20)+3}{243}\right]$$

$$C_{12} = \left[\frac{62}{243}, \frac{63}{243}\right]$$

For $n = 7$ and $m = 5$

Also $L_n = L_7 = 26$

$$C_{2n-1} = C_{13} = \left[\frac{3L_n}{3^m}, \frac{3L_n+1}{3^m}\right] = \left[\frac{3L_7}{3^5}, \frac{3L_7+1}{3^5}\right] = \left[\frac{3\times 26}{243}, \frac{(3\times 26)+1}{243}\right]$$

$$C_{13} = \left[\frac{72}{243}, \frac{73}{243}\right]$$

$$C_{2n} = C_{14} = \left[\frac{3L_n+2}{3^m}, \frac{3L_n+3}{3^m}\right] = \left[\frac{3L_7+2}{3^5}, \frac{3L_7+3}{3^5}\right] = \left[\frac{(3\times 26)+2}{243}, \frac{(3\times 26)+3}{243}\right]$$

$$C_{14} = \left[\frac{74}{243}, \frac{75}{243}\right]$$

For $n = 8$ and $m = 5$

Also $L_n = L_8 = 26$

$$C_{2n-1} = C_{15} = \left[\frac{3L_n}{3^m}, \frac{3L_n+1}{3^m}\right] = \left[\frac{3L_8}{3^5}, \frac{3L_8+1}{3^5}\right] = \left[\frac{3\times 26}{243}, \frac{(3\times 26)+1}{243}\right]$$

$$C_{15} = \left[\frac{78}{243}, \frac{79}{243}\right]$$

$$C_{2n} = C_{16} = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_8 + 2}{3^5}, \frac{3L_8 + 3}{3^5}\right] = \left[\frac{(3 \times 8) + 2}{243}, \frac{(3 \times 8) + 3}{243}\right]$$

$$C_{16} = \left[\frac{80}{243}, \frac{81}{243}\right]$$

For $n = 9$ and $m = 5$

Also $L_n = L_9 = 54$

$$C_{2n-1} = C_{17} = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_9}{3^5}, \frac{3L_9 + 1}{3^5}\right] = \left[\frac{3 \times 54}{243}, \frac{(3 \times 54) + 1}{243}\right]$$

$$C_{17} = \left[\frac{162}{243}, \frac{163}{243}\right]$$

$$C_{2n} = C_{18} = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_9 + 2}{3^5}, \frac{3L_9 + 3}{3^5}\right] = \left[\frac{(3 \times 54) + 2}{243}, \frac{(3 \times 54) + 3}{243}\right]$$

$$C_{18} = \left[\frac{164}{243}, \frac{165}{243}\right]$$

For $n = 10$ and $m = 5$

Also $L_n = L_{10} = 56$

$$C_{2n-1} = C_{19} = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_{10}}{3^5}, \frac{3L_{10} + 1}{3^5}\right] = \left[\frac{3 \times 56}{243}, \frac{(3 \times 56) + 1}{243}\right]$$

$$C_{19} = \left[\frac{168}{243}, \frac{169}{243}\right]$$

$$C_{2n} = C_{20} = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_{10} + 2}{3^5}, \frac{3L_{10} + 3}{3^5}\right] = \left[\frac{(3 \times 56) + 2}{243}, \frac{(3 \times 56) + 3}{243}\right]$$

$$C_{20} = \left[\frac{170}{243}, \frac{171}{243}\right]$$

For $n = 11$ and $m = 5$

Also $L_n = L_{11} = 60$

$$C_{2n-1} = C_{21} = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_{11}}{3^5}, \frac{3L_{11} + 1}{3^5}\right] = \left[\frac{3 \times 60}{243}, \frac{(3 \times 60) + 1}{243}\right]$$

$$C_{21} = \left[\frac{180}{243}, \frac{181}{243}\right]$$

$$C_{2n} = C_{22} = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_{11} + 2}{3^5}, \frac{3L_{11} + 3}{3^5}\right] = \left[\frac{(3 \times 60) + 2}{243}, \frac{(3 \times 60) + 3}{243}\right]$$

$$C_{22} = \left[\frac{182}{243}, \frac{183}{243}\right]$$

For $n = 12$ and $m = 5$

Also $L_n = L_{12} = 62$

$$C_{2n-1} = C_{23} = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_{12}}{3^5}, \frac{3L_{12} + 1}{3^5}\right] = \left[\frac{3 \times 62}{243}, \frac{(3 \times 62) + 1}{243}\right]$$

$$C_{23} = \left[\frac{186}{243}, \frac{187}{243}\right]$$

$$C_{2n} = C_{24} = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_{12} + 2}{3^5}, \frac{3L_{12} + 3}{3^5}\right] = \left[\frac{(3 \times 62) + 2}{243}, \frac{(3 \times 62) + 3}{243}\right]$$

$$C_{24} = \left[\frac{188}{243}, \frac{189}{243}\right]$$

For $n = 13$ and $m = 5$

Also $L_n = L_{13} = 72$

$$C_{2n-1} = C_{25} = \left[\frac{3L_n}{3^m}, \frac{3L_n + 1}{3^m}\right] = \left[\frac{3L_{13}}{3^5}, \frac{3L_{13} + 1}{3^5}\right] = \left[\frac{3 \times 72}{243}, \frac{(3 \times 72) + 1}{243}\right]$$

$$C_{25} = \left[\frac{216}{243}, \frac{217}{243}\right]$$

$$C_{2n} = C_{26} = \left[\frac{3L_n + 2}{3^m}, \frac{3L_n + 3}{3^m}\right] = \left[\frac{3L_{13} + 2}{3^5}, \frac{3L_{13} + 3}{3^5}\right] = \left[\frac{(3 \times 72) + 2}{243}, \frac{(3 \times 72) + 3}{243}\right]$$

$$C_{26} = \left[\frac{218}{243}, \frac{219}{243}\right]$$

For $n = 14$ and $m = 5$

Also $L_n = L_{14} = 74$

$$C_{2n-1} = C_{27} = \left[\frac{3L_n}{3^m}, \frac{3L_n+1}{3^m}\right] = \left[\frac{3L_{14}}{3^5}, \frac{3L_{14}+1}{3^5}\right] = \left[\frac{3\times 74}{243}, \frac{(3\times 74)+1}{243}\right]$$

$$C_{27} = \left[\frac{222}{243}, \frac{223}{243}\right]$$

$$C_{2n} = C_{28} = \left[\frac{3L_n+2}{3^m}, \frac{3L_n+3}{3^m}\right] = \left[\frac{3L_{14}+2}{3^5}, \frac{3L_{14}+3}{3^5}\right] = \left[\frac{(3\times 74)+2}{243}, \frac{(3\times 74)+3}{243}\right]$$

$$C_{28} = \left[\frac{224}{243}, \frac{225}{243}\right]$$

For $n = 15$ and $m = 5$

Also $L_n = L_{15} = 78$

$$C_{2n-1} = C_{29} = \left[\frac{3L_n}{3^m}, \frac{3L_n+1}{3^m}\right] = \left[\frac{3L_{15}}{3^5}, \frac{3L_{15}+1}{3^5}\right] = \left[\frac{3\times 78}{243}, \frac{(3\times 78)+1}{243}\right]$$

$$C_{29} = \left[\frac{234}{243}, \frac{235}{243}\right]$$

$$C_{2n} = C_{30} = \left[\frac{3L_n+2}{3^m}, \frac{3L_n+3}{3^m}\right] = \left[\frac{3L_{15}+2}{3^5}, \frac{3L_{15}+3}{3^5}\right] = \left[\frac{(3\times 78)+2}{243}, \frac{(3\times 78)+3}{243}\right]$$

$$C_{30} = \left[\frac{236}{243}, \frac{237}{243}\right]$$

For $n = 16$ and $m = 5$

Also $L_n = L_{16} = 80$

$$C_{2n-1} = C_{31} = \left[\frac{3L_n}{3^m}, \frac{3L_n+1}{3^m}\right] = \left[\frac{3L_{16}}{3^5}, \frac{3L_{16}+1}{3^5}\right] = \left[\frac{3\times 80}{243}, \frac{(3\times 80)+1}{243}\right]$$

$$C_{31} = \left[\frac{240}{243}, \frac{241}{243}\right]$$

$$C_{2n} = C_{32} = \left[\frac{3L_n+2}{3^m}, \frac{3L_n+3}{3^m}\right] = \left[\frac{3L_{16}+2}{3^5}, \frac{3L_{16}+4}{3^5}\right] = \left[\frac{(3\times 80)+2}{243}, \frac{(3\times 80)+3}{243}\right]$$

$$C_{32} = \left[\frac{242}{243}, \frac{243}{243}\right] = \left[\frac{242}{243}, 1\right]$$

The Cantor set for $n = 5$ is an 32 - segment Cantor set with an interval of $\frac{1}{243}$ is given as

$$\left[0,\frac{1}{243}\right]\cup\left[\frac{2}{243},\frac{3}{243}\right]\cup\left[\frac{6}{243},\frac{7}{243}\right]\cup\left[\frac{8}{243},\frac{9}{243}\right]\cup\left[\frac{18}{243}\frac{19}{243}\right]\cup\left[\frac{20}{243},\frac{21}{243}\right]$$
$$\cup\left[\frac{24}{243},\frac{25}{243}\right]\cup\left[\frac{26}{243},\frac{27}{243}\right]\cup\left[\frac{54}{243},\frac{55}{243}\right]\cup\left[\frac{56}{243},\frac{57}{243}\right]\cup\left[\frac{60}{243},\frac{61}{243}\right]$$
$$\cup\left[\frac{62}{243},\frac{63}{243}\right]\cup\left[\frac{72}{243},\frac{73}{243}\right]\cup\left[\frac{74}{243},\frac{75}{243}\right]\cup\left[\frac{78}{243},\frac{79}{243}\right]\cup\left[\frac{80}{243},\frac{81}{243}\right]$$
$$\cup\left[\frac{162}{243},\frac{163}{243}\right]\cup\left[\frac{164}{243},\frac{165}{243}\right]\cup\left[\frac{168}{243},\frac{169}{243}\right]\cup\left[\frac{170}{243},\frac{171}{243}\right]\cup\left[\frac{180}{243},\frac{181}{243}\right]$$
$$\cup\left[\frac{182}{243},\frac{183}{243}\right]\cup\left[\frac{186}{243},\frac{187}{243}\right]\cup\left[\frac{188}{243},\frac{189}{243}\right]\cup\left[\frac{216}{243},\frac{217}{243}\right]\cup\left[\frac{218}{243},\frac{219}{243}\right]$$
$$\cup\left[\frac{222}{243},\frac{223}{243}\right]\cup\left[\frac{224}{243},\frac{225}{243}\right]\cup\left[\frac{234}{243},\frac{235}{243}\right]\cup\left[\frac{236}{243},\frac{237}{243}\right]\cup\left[\frac{240}{243},\frac{241}{243}\right]\cup\left[\frac{242}{243},\frac{243}{243}\right]$$

which can be simplified as

$$\left[0,\frac{1}{243}\right]\cup\left[\frac{2}{243},\frac{1}{81}\right]\cup\left[\frac{2}{243},\frac{7}{243}\right]\cup\left[\frac{8}{243},\frac{1}{27}\right]\cup\left[\frac{2}{27}\frac{19}{243}\right]\cup\left[\frac{20}{243},\frac{7}{81}\right]\cup\left[\frac{8}{81},\frac{25}{243}\right]$$
$$\cup\left[\frac{26}{243},\frac{1}{9}\right]\cup\left[\frac{2}{9},\frac{55}{243}\right]\cup\left[\frac{56}{243},\frac{19}{81}\right]\cup\left[\frac{20}{81},\frac{61}{243}\right]\cup\left[\frac{62}{243},\frac{7}{27}\right]\cup\left[\frac{8}{27},\frac{73}{243}\right]\cup\left[\frac{74}{243},\frac{25}{81}\right]$$
$$\cup\left[\frac{26}{81},\frac{79}{243}\right]\cup\left[\frac{80}{243},\frac{1}{3}\right]\cup\left[\frac{2}{3},\frac{163}{243}\right]\cup\left[\frac{164}{243},\frac{55}{81}\right]\cup\left[\frac{55}{81},\frac{169}{243}\right]\cup\left[\frac{170}{243},\frac{19}{27}\right]\cup\left[\frac{20}{27},\frac{181}{243}\right]$$
$$\cup\left[\frac{182}{243},\frac{61}{81}\right]\cup\left[\frac{186}{243},\frac{187}{243}\right]\cup\left[\frac{188}{243},\frac{7}{9}\right]\cup\left[\frac{8}{9},\frac{217}{243}\right]\cup\left[\frac{218}{243},\frac{73}{81}\right]\cup\left[\frac{74}{81},\frac{223}{243}\right]\cup\left[\frac{224}{243},\frac{25}{27}\right]$$
$$\cup\left[\frac{26}{27},\frac{235}{243}\right]\cup\left[\frac{236}{243},\frac{79}{81}\right]\cup\left[\frac{80}{81},\frac{241}{243}\right]\cup\left[\frac{242}{243},1\right]$$

Conclusion

In time past, one of the most familiar problems in dealing with the Cantor set has been the unavailability of a formula that generates the Cantor set with ease. The Lower bound numerator generator approach can generate the Cantor set easily. It is thus far one of the few if not the only formula where higher segment sets are constructed without necessarily having to construct the immediate segment before it. In this approach, an 8-segment Cantor can be constructed without having to first construct a 4-segment cantor. In the Lower bound numerator generator approach, the level of work done is minimal and not as complicated compare to the other approaches. It is by far the fastest approach in construction of the Cantor set.

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