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New Type of Generalized Difference Sequence Space

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Abstract. The space $r^q(u, p, s)$ of non-absolute type have recently been introduced and studied (see, M. F. Rahman and A. B. M. R. Karim, Generalized Riesz sequence space of non-absolute type and some matrix mappings, Pure and Applied Math. Journal, 4(3)(2015), 90-95). In the present paper, we introduce the space $r^q(\triangle_u^p(s))$, we show its completeness property , prove that the space $r^q(\triangle_u^p(s))$ and $l(p)$ are linearly isomorphic and compute their Köthe-duals. Furthermore, we construct its basis and in our last section, we have characterized some matrix classes of infinite matrices.

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1 Preliminaries, Background and Notation:

We denote the set of all sequences(real or complex) by ω. Any subspace of ω is called the sequence space. Throughout the paper N, R and C denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let l_∞, c and c_0, respectively, denotes the space of all bounded sequences , the space of convergent sequences and the sequences converging to zero. Also, by cs, l_1 and $l(p)$ we denote the spaces of all convergent, absolutely and p-absolutely convergent series, respectively.

Let X, Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the A-transformation from X *into* Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk}x_k$. For simplicity in

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notation, here and in what follows, the summation without limits runs from 0 to ∞. By $A \in (X : Y)$ we mean the characterizations of matrices from X to Y *i.e.*, $A : X \to Y$. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) : x = (x_k) \in \omega\}. \tag{1.1}$$

Kizmaz (see, [14]) defined the difference sequence spaces $Z(\triangle)$ as follows

$$Z(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in Z\}$$

where, $Z \in \{l_\infty, c, c_0\}$ and $\triangle x_k = x_k - x_{k+1}$.

Başar and Altay (see, [2]) has studied the sequence space as

$$bv_p = \left\{ x = (x_k) \in \omega : \sum_k |x_k - x_{k-1}|^p < \infty \right\},$$

where $1 \leq p < \infty$. With the notation of (1), the space bv_p can be redefined as

$$bv_p = (l_p)_\triangle, 1 \leq p < \infty$$

where, $\triangle$ denotes the matrix $\triangle = (\triangle_{nk})$ defined as

$$\triangle_{nk} = \begin{cases} (-1)^{n-k}, & \text{if } n-1 \leq k \leq n, \\ 0 & \text{if } k < n-1 \text{ or } k > n. \end{cases}$$

Let (q_k) be a sequence of positive numbers and let us write,$Q_n = \sum_{k=0}^{n} q_k$ for $n \in \mathbb{N}$. Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean (R, q_n) is given by

$$r_{nk} = \begin{cases} \frac{q_k}{Q_n}, & \text{if } k \leq n, \\ 0 & \text{if } k > n \end{cases}$$

The Riesz mean (R, q_n) is regular if and only if $Q_n \to \infty$ as $n \to \infty$ (see, [26]).

Recently, Ganie et al (see, [30]) introduced and studied the space $r^q(\triangle^p_u)$ as follows:

$$r^q(\triangle^p_u) = \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j \triangle x_j \right|^{p_k} < \infty \right\}.$$

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors. They introduced the sequence spaces $(l_\infty)_{N_q}$ and c_{N_q} (see, [32]),

$\left(l_p\right)_{C_1} = X_p$ and $(l_\infty)_{C_1} = X_\infty$ (see, [25]), $(l_\infty)_{R^t} = r^t_\infty$, $(c)_{R^t} = r^t_c$ and $(c_o)_{R^t} = r^t_0$ (see, [19]), $\left(l_p\right)_{R^t} = r^t_p$ (see, [1]), $(c_0)_{E^r} = e^r_0$ and $(c)_{E^r} = e^r_c$ (see, [3]), $\left(l_p\right)_{E^r} = e^r_p$ and $(l_\infty)_{E^r} = e^r_\infty$ (see, [4]), $(c_0)_{A^r} = a^r_0$ and $c_{A^r} = a^r_c$ (see, [4]), $[c_0(u,p)]_{A^r} = a^r_0(u,p)$ and $[c(u,p)]_{A^r} = a^r_c(u,p)$ (see, [5]), $\left(l_p\right)_{A^r} = a^r_p$ and $(l_\infty)_{A^r} = a^r_\infty$ (see, [6]), $(c_0)_{C_1} = \widehat{c}_0$, $c_{C_1} = \widehat{c}$ (see, [28]), $c^\lambda_0(\triangle) = \left(c^\lambda_0\right)_\triangle$ and $c^\lambda(\triangle) = \left(c^\lambda\right)_\triangle$ (see, [24]), $\mu_G = Z(u,v,\mu)$ (see, [20]), $r^q(u,p) = \{l(p)\}_{R^q}$ (see, [29]); etc.

2 The Riesz Sequence space $r^q(\triangle^p_u(s))$ of non-absolute type:

In this section, we define the Riesz sequence space $r^q(\triangle^p_u(s))$, and prove that the space $r^q(\triangle^p_u(s))$ is a complete paranormed linear space and show it is linearly isomorphic to the space $l(p)$.

A linear Topological space X over the field of real numbers $\mathbb{R}$ is said to be a paranormed space if there is a sub-additive function $h : X \to R$ such that $h(\theta) = 0$, $h(-x) = h(x)$ and scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \to 0$ and $h(x_n - x) \to 0$ imply $h(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha's$ in R and $x's$ in X , where θ is a zero vector in the linear space X. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup_k p_k = H$and $M = max\{1, H\}$. Then, the linear spaces $l(p)$ and $l_\infty(p)$ were defined by Maddox (see, [16]) (see also, [29, 31]) as follows :

$$l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$$
$$l_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\}$$

which are complete spaces paranormed by

$$h_1(x) = \left[\sum_k |x_k|^{p_k}\right]^{1/M} \quad and \quad h_2(x) = \sup_k |x_k|^{p_k/M}$$

if and only if $\inf p_k > 0$.

We shall assume throughout that $p_k^{-1} + \{p'_k\}^{-1}$ provided $1 < inf p_k \leq H < \infty$ and we denote the collection of all finite subsets of $\mathbb{N}$ by F.

Quite recently, the space $r^q(u,p,s)$ has been introduced by Rahman et al (see, [27]) and is defined as follows:

$$r^q(u,p,s) = \left\{x = (x_k) \in \omega : \sum_k \left|\frac{1}{Q^{s+1}_k}\sum_{j=0}^k u_j q_j x_j\right|^{p_k}\right\} < \infty.$$

Following Başar and Altay (see, [2]), Başar, Altay and Mursaleen (see, [3]), Choudhary and Mishra (see, [8]), Ganie et al (see, [9, 10, 11, 12]), Gross Erdmann (see, [13]), Lascarides (see, [15]), Mursaleen et al (see, [21, 22, 23, 24]), we introduce the space $r^q(\triangle^p_u(s))$, which is defined as the set of all sequences such that $R^q_{\triangle^s_u}$ transform of it is in the space $l(p)$, that is,

$$r^q(\triangle^p_u(s)) = \left\{x = (x_k) \in \omega : \sum_k \left|\frac{1}{Q^{s+1}_k}\sum_{j=0}^k u_k q_j \triangle x_j\right|^{p_k} < \infty\right\}$$

where, $0 < p_k \leq H < \infty$.

Remark 2.0 : Incase $(u_k) = e = (1, 1, ...)$ and $s = 0$,the sequence spaces $r^q(\triangle_u^p(s))$ reduces to the sequence spaces $r^q(\triangle, p)$ Başarir and Öztürk (see, [7]) and for $s = 0$, it reduces to $r^q(\triangle, u, p)$, introduced by Sheikh and Ganie (see, [30]).

With the notation of (1), we can write

$$r^q(\triangle_u^p(s)) = \{l(p)\}_{R^q_{\triangle_u^p(s)}}.$$

Define the sequence $y = (y_k)$, which will be used, by the $R^q_{\triangle_u^p(s)}$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \frac{1}{Q_k^{s+1}} \sum_{j=0}^{k} u_k q_j \triangle x_j. \tag{2.1}$$

Now, we begin with the following theorem which is essential in the text.

Theorem 2.1 : $r^q(\triangle_u^p(s))$ is a complete linear metric space paranormed by $h_{\triangle_u^s}$, defined as

$$h_{\triangle_u^s}(x) = \left[\sum_k \left|\frac{1}{Q_k^{s+1}} \sum_{j=0}^{k-1}(u_j q_j - u_{j+1}q_{j+1})x_j + \frac{q_k u_k}{Q_k^{s+1}} x_k\right|^{p_k}\right]^{\frac{1}{M}}$$

with $0 < p_k \leq H < \infty$.

Proof: The linearity of $r^q(\triangle_u^p(s))$ with respect to the co-ordinatewise addition and scalar multiplication follows from from the inequalities which are satisfied for $z, x \in r^q(\triangle_u^p(s))$(see, [18])

$$\left[\sum_k \left|\frac{1}{Q_k^{s+1}} \sum_{j=0}^{k-1}(u_j q_j - u_{j+1}q_{j+1})(x_j + z_j) + \frac{q_k u_k}{Q_k}(x_k + z_k)\right|^{p_k}\right]^{\frac{1}{M}}$$
$$\leq \left[\sum_k \left|\frac{1}{Q_k^{s+1}} \sum_{j=0}^{k-1}(u_j q_j - u_{j+1}q_{j+1})x_j + \frac{q_k u_k}{Q_k^{s+1}} x_k\right|^{p_k}\right]^{\frac{1}{M}}$$

$$+ \left[\sum_k \left|\frac{1}{Q_k^{s+1}} \sum_{j=0}^{k-1}(u_j q_j - u_{j+1}q_{j+1})z_j + \frac{q_k u_k}{Q_k} z_k\right|^{p_k}\right]^{\frac{1}{M}} \tag{2.2}$$

and for any $\alpha \in \mathbf{R}$ (see, [17])

$$|\alpha|^{p_k} \leq max(1, |\alpha|^M). \tag{2.3}$$

For $\theta = (0, 0, 0, ...)$, we have $h_{\triangle_u^s}(\theta)$=0 and $h_{\triangle_u^s}(x) = h_{\triangle_u^s}(-x)$ for all $x \in r^q(\triangle_u^p(s))$. Also, the inequality (3) and (4) gives the subadditivity of $h_{\triangle_u^s}$ and

$$h_{\triangle_u^s}(\alpha x) \leq max(1, |\alpha|)h_{\triangle_u^s}(x).$$

Let $\{x^n\}$ be any sequence of points of the space $r^q(\triangle_u^p(s))$ such that $h_{\triangle_u^s}(x^n - x) \rightarrow 0$ and (α_n) be any sequence of scalars such that $\alpha_n \rightarrow \alpha$. Then, $\{h_{\triangle_u^s}(x^n)\}$ is bounded, since by subadditivity, the inequality

$$h_{\triangle_u^s}(x^n) \leq h_{\triangle_u^s}(x) + h_{\triangle_u^s}(x^n - x)$$

holds. Thus, we have

$$h_{\triangle_u^s}(\alpha_n x^n - \alpha x) = \left[\sum_k \left| \frac{1}{Q_k^{s+1}} \sum_{j=0}^{k} (u_j q_j - u_{j+1} q_{j+1})(\alpha_n x_j^n - \alpha x_j) \right|^{p_k} \right]^{\frac{1}{M}}$$

$$\leq |\alpha_n - \alpha|^{\frac{1}{M}} h_{\triangle_u^s}(x^n) + |\alpha|^{\frac{1}{M}} h_{\triangle_u^s}(x^n - x)$$

which tends to zero as $n \to \infty$. This shows that the scalar multiplication is continuous. Hence, $h_{\triangle_u^s}$ is a paranorm on the space $r^q(\triangle_u^p(s))$.

It remains to prove the completeness of the space $r^q(\triangle_u^p(s))$. Let $\{x^j\}$ be any Cauchy sequence in the space $r^q(\triangle_u^p(s))$, where $x^i = \{x_0^i, x_1^i, ...\}$. Hence, for a given $\epsilon > 0$, we can find a positive integer $n_0(\epsilon)$ such that

$$h_{\triangle_u^s}(x^i - x^j) < \epsilon \tag{2.4}$$

for all $i, j \geq n_0(\epsilon)$. Using definition of $h_{\triangle_u^s}$ and for each fixed $k \in \mathbb{N}$ that

$$\left| (R^q_{\triangle_u^s} x^i)_k - (R^q_{\triangle_u^s} x^j)_k \right| \leq \left[\sum_k \left| (R^q_{\triangle_u^s} x^i)_k - (R^q \triangle x^j)_k \right|^{p_k} \right]^{\frac{1}{M}} < \epsilon$$

for $i, j \geq n_0(\epsilon)$, which leads us to the fact that $\{(R^q_{\triangle_u^s} x^0)_k, ((R^q_{\triangle_u^s} x^1)_k, \dots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say, $(R^q_{\triangle_u^s} x^i)_k \to (R^q_{\triangle_u^s} x)_k$ as $i \to \infty$. Using these infinitely many limits $(R^q_{\triangle_u^s} x)_0, (R^q_{\triangle_u^s} x)_1, \dots$, we define the sequence $\{(R^q_{\triangle_u^s} x)_0, (R^q_{\triangle_u^s} x)_1, \dots\}$. From (5) for each $m \in \mathbb{N}$ and $i, j \geq n_0(\epsilon)$,

$$\sum_{k=0}^{m} \left| (R^q_{\triangle_u^s} x^i)_k - (R^q_{\triangle_u^s} x^j)_k \right|^{p_k} \leq h_{\triangle_u^s}(x^i - x^j)^M < \epsilon^M. \tag{2.5}$$

Take any $i, j \geq n_0(\epsilon)$. First, let $j \to \infty$ in (6) and then $m \to \infty$, we obtain

$$h_{\triangle_u^s}(x^i - x) \leq \epsilon.$$

Finally, taking $\epsilon = 1$ in (6) and letting $i \geq n_0(1)$. we have by Minkowski's inequality for each $m \in \mathbb{N}$ that

$$\left[\sum_{k=0}^{m} |(R^q x)_k|^{p_k} \right]^{\frac{1}{M}} \leq h_{\triangle_u^s}(x^i - x) + h_{\triangle_u^s}(x^i) \leq 1 + h_{\triangle_u^s}(x^i)$$

which implies that $x \in r^q(\triangle_u^p(s))$. Since $h_{\triangle_u^s}(x - x^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$, hence we have shown that $r^q(\triangle_u^p(s))$ is complete. This completes the proof .

Note that one can easily see the absolute property does not hold on the spaces $r^q(\triangle_u^p(s))$, that is $h_{\triangle_u^s}(x) \neq h_{\triangle_u^s}(|x|)$ for atleast one sequence in the space $r^q(\triangle_u^p(s))$ and this says that $r^q(\triangle_u^p(s))$ is a sequence space of non-absolute type.

Theorem 2.2 : The Riesz sequence space $r^q(\triangle_u^p(s))$ of non-absolute type is linearly isomorphic to the space $l(p)$, where $0 < p_k \leq H < \infty$.

Proof : For the proof of the theorem, we should show the existence of a linear bijection between the spaces $r^q(\triangle_u^p(s))$ and $l(p)$, where $0 < p_k \leq H < \infty$. With the notation of (3), define the transformation T from $r^q(\triangle_u^p(s))$ to $l(p)$ by $x \to y = Tx$. The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y \in l(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1} \left(\frac{1}{u_n q_n} - \frac{1}{u_{n+1} q_{n+1}} \right) Q_k^{s+1} y_k + \frac{Q_k^{s+1}}{u_k q_k} y_k,$$

for $k \in \mathbb{N}$. Then,

$$\begin{aligned} h_{\triangle_u^s}(x) &= \left[\sum_k \left| \frac{1}{Q_k^{s+1}} \sum_{j=0}^{k-1} (u_j q_j - u_{j+1} q_{j+1}) x_j + \frac{q_k u_k}{Q_k^{s+1}} x_k \right|^{p_k} \right]^{\frac{1}{M}} \\ &= \left[\sum_k \left| \sum_{j=0}^{k} \delta_{kj} y_j \right|^{p_k} \right]^{\frac{1}{M}} \\ &= \left[\sum_k |y_k|^{p_k} \right]^{\frac{1}{M}} = h_1(y) < \infty, \end{aligned}$$

where,

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{if } k \neq j \end{cases}$$

Thus, we have $x \in r^q(\triangle_u^p(s))$. Consequently, T is surjective and is paranorm preserving. Hence, T is a linear bijection and this shows that the spaces $r^q(\triangle_u^p(s))$ and $l(p)$ are linearly isomorphic, hence the proof of the result follows.

3 Basis and α-, β- and γ-duals of the space $r^q(\triangle_u^p(s))$:

In this section, we compute α , β- and γ- duals of the space $r^q(\triangle_u^p(s))$ and finally we give the basis for the space $r^q(\triangle_u^p(s))$.

For the sequence space X and Y, define the set

$$S(X:Y) = \{z = (z_k) : xz = (x_k z_k) \in Y\}. \tag{3.1}$$

With the notation of (7), the α-, β- and γ- duals of a sequence space X, which are respectively denoted by X^α and X^β and are defined by

$$X^\alpha = S(X:l_1)\ ,\ X^\beta = S(X:cs) and X^\gamma = S(X:bs).$$

If a sequence space X paranormed by h contains a sequence (b_n) with the property that for every $x \in X$ there is a unique sequence of scalars (α_n) such that

$$\lim_n h(x - \sum_{k=0}^{n} \alpha_k b_k) = 0$$

then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$.

First we first state some lemmas which are needed in proving our theorems .

Lemma 3.1 [[13], Theorem 5.10] :
(i) Let $1 < p_k \le H < \infty$. Then $A \in (l(p) : l_1)$ if and only if there exists an integer $B > 1$ such that

$$\sup_{K\in F} \sum_k \left| \sum_{n\in K} a_{nk} B^{-1} \right|^{p_k'} < \infty.$$

(ii) Let $0 < p_k \le 1$.Then $A \in (l(p) : l_1)$ if and only if

$$\sup_{K\in F} \sup_k \left| \sum_{n\in K} a_{nk} B^{-1} \right|^{p_k} < \infty.$$

Lemma 3.2 [[16], Theorem 1] :
(i) Let $1 < p_k \le H < \infty$.Then $A \in (l(p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that

$$\sup_n \sum_k |a_{nk} B^{-1}|^{p_k'} < \infty. \tag{3.2}$$

(ii) Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : l_\infty)$ if and only if

$$\sup_{n,k} |a_{nk}|^{p_k} < \infty. \tag{3.3}$$

Lemma 3.3 [[16], Theorem 1] : Let $0 < p_k \le H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (l(p) : c)$ if and only if (8) and (9) hold along with

$$\lim_n a_{nk} = \beta_k \ for\ k \in \mathbb{N} \tag{3.4}$$

also holds.

Theorem 3.4 : Let Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Define the sets $D_1^s(u,p)$ and $D_2^s(u,p)$ as follows

$$D_1^s(u,p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_{K \in F} \sum_k \left| \sum_{n \in K} \left(\frac{1}{u_k q_k} - \frac{1}{u_{k+1} q_{k+1}} \right) a_n Q_k^{s+1} + \frac{a_n}{u_n q_n} Q_n^{s+1} B^{-1} \right|^{p_k'} < \infty \right\}$$

and

$$D_2^s(u,p) = \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sum_k \left| \left[\left(\frac{a_k}{u_k q_k} + \left(\frac{1}{u_k q_k} - \frac{1}{u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) Q_k^{s+1} \right] B^{-1} \right|^{p_k'} < \infty \right\}.$$

Then,

$$\left[r^q(\triangle_u^p(s)) \right]^\alpha = D_1^s(u,p) \text{ and } \left[r^q(\triangle_u^p(s)) \right]^\beta = D_2^s(u,p) \cap cs = \left[r^q(\triangle_u^p(s)) \right]^\gamma.$$

Proof : Let us take any $a = (a_k) \in \omega$ .We can easily derive with (2) that

$$a_n x_n = \sum_{k=0}^{n-1} \left(\frac{1}{u_k q_k} - \frac{1}{u_{k+1} q_{k+1}} \right) a_n Q_k^{s+1} y_k + \frac{a_n}{u_n q_n} Q_n^{s+1} y_n = (Cy)_n \tag{3.5}$$

where, $C = (c_{nk})$ is defined as

$$c_{nk} = \begin{cases} \left(\frac{1}{u_k q_k} - \frac{1}{u_{k+1} q_{k+1}}\right) a_n Q_k^{s+1}, & \text{if } 0 \leq k \leq n-1 \\ \frac{a_n}{u_n q_n} Q_n^{s+1}, & \text{if } k = n \\ 0, & \text{if } k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Thus we observe by combining (11) with (i) of Lemma 3.1 that $ax = (a_n x_n) \in l_1$ whenever $x = (x_n) \in r^q(\triangle_u^p(s))$ if and only if $Cy \in l_1$ whenever $y \in l(p)$. This gives the result that $\left[r^q(\triangle_u^p(s)) \right]^\alpha = D_1^s(u,p)$.

Further, consider the equation,

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[\left(\frac{a_k}{u_k q_k} + \left(\frac{1}{u_k q_k} - \frac{1}{u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) Q_k^{s+1} \right] y_k = (Dy)_n \tag{3.6}$$

where, $D = (d_{nk})$ is defined as

$$d_{nk} = \begin{cases} \frac{a_k}{u_k q_k} + (\frac{1}{u_k q_k} - \frac{1}{u_{k+1} q_{k+1}}) \sum\limits_{i=k+1}^{n} a_i) Q_k^{s+1}, & \text{if } 0 \le k \le n \\ 0, & \text{if } k > n \end{cases}$$

Thus we deduce from Lemma 3.3 with (12) that $ax = (a_n x_n) \in cs$ whenever $x = (x_n) \in r^q(\triangle_u^p(s))$ if and only if $Dy \in c$ whenever $y \in l(p)$. Therefore,

we derive from (8) that

$$\sum_k \left| \left[\left(\frac{a_k}{u_k q_k} + \left(\frac{1}{u_k q_k} - \frac{1}{u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) Q_k^{s+1} \right] B^{-1} \right|^{p_k'} < \infty \tag{3.7}$$

and $\lim_n d_{nk}$ exists and hence shows that that $\left[r^q(\triangle_u^p(s)) \right]^\beta = D_2^s(u,p) \cap cs.$

As this, from Lemma 3.2 together with (12) that $ax = (a_k x_k) \in bs$ whenever $x = (x_n) \in r^q(\triangle_u^p(s))$ if and only if $Dy \in l_\infty$ whenever $y = (y_k) \in l(p)$. Therefore, we again obtain the condition (13) which means that $\left[r^q(\triangle_u^p(s)) \right]^\gamma = D_2^s(u,p) \cap cs$ and the proof of the theorem is complete.

Theorem 3.5 : Let Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Define the sets $D_3^s(u,p)$ and $D_4^s(u,p)$, as follows

$$D_3^s(u,p) = \left\{ a = (a_k) \in \omega : \sup_{K \in F} \sup_k \left| \sum_{n \in K} \left[\left(\frac{1}{u_k q_k} - \frac{1}{u_{k+1} q_{k+1}} \right) a_n Q_k^{s+1} + \frac{a_n Q_n^{s+1}}{q_n u_n} \right] B^{-1} \right|^{p_k} < \infty \right\}$$

and

$$D_4^s(u,p) = \left\{ a = (a_k) \in \omega : \sup_k \left| \left[\left(\frac{a_k}{u_k q_k} + \left(\frac{1}{u_k q_k} - \frac{1}{u_{k+1} q_{k+1}} \right) \sum_{i=k+1}^{n} a_i \right) Q_k^{s+1} \right] B^{-1} \right|^{p_k} < \infty \right\}.$$

Then,

$$\left[r^q(\triangle_u^p(s)) \right]^\alpha = D_3^s(u,p) \text{ and } \left[r^q(\triangle_u^p(s)) \right]^\beta = \left[r^q(\triangle_u^p(s)) \right]^\gamma = D_4(u,p) \cap cs.$$

Proof. This follows by the similar technique as in the proof of Theorem 2.7, above by using second parts of Lemmas 3.1, 3.2 and 3.3 instead of the first parts. So, we omit the details.

Theorem 3.6 : Define the sequence $b^{(k)}(q) = \{b_n^{(k)}(q)\}$ of the elements of the space $r^q(\triangle_u^p(s))$ for every fixed $k \in \mathbb{N}$ by

$$b_n^{(k)}(q) = \begin{cases} (\frac{1}{u_n q_n} - \frac{1}{u_{n+1} q_{n+1}})Q_n^{s+1} + \frac{Q_k^{s+1}}{u_k q_k}, & \text{if } 0 \leq n \leq k-1, \\ \\ 0, & \text{if } n > k-1. \end{cases}$$

Then, the sequence $\{b^{(k)}(q)\}$ is a basis for the space $r^q(\triangle_u^p(s))$ and any $x \in r^q(\triangle_u^p(s))$ has a unique representation of

$$x = \sum_k \lambda_k(q) b^{(k)}(q) \tag{3.8}$$

where, $\lambda_k(q) = ((R^q_{\triangle_u^s} x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \leq H < \infty$.

Proof : It is obvious that $b^{(k)}(q) \subset r^q(\triangle_u^p(s))$, since

$$R^q_{\triangle_u^s} b^{(k)}(q) = e^{(k)} \in l(p) \; for \; k \in \mathbb{N} \tag{3.9}$$

and $0 < p_k \leq H < \infty$, where $e^{(k)}$ is the sequence whose only non-zero term is 1 at k^{th} place for each $k \in \mathbb{N}$.

Let $x \in r^q(\triangle_u^p(s))$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) b^{(k)}(q) \tag{3.10}$$

Then, we obtain by applying $R^q_{\triangle_u^s}$ to (16) with (15) that

$$R^q_{\triangle_u^s} x^{[m]} = \sum_{k=0}^{m} \lambda_k(q)(R^q_{\triangle_u^s} b^{(k)}(q)) = \sum_{k=0}^{m} (R^q_{\triangle_u^s} x)_k e^{(k)}$$

and

$$\left(R^q_{\triangle_u^s}\left(x - x^{[m]}\right)\right)_i = \begin{cases} 0, & \text{if } 0 \leq i \leq m \\ (R^q_{\triangle_u^s} x)_i, & \text{if } i > m \end{cases}$$

where $i, m \in \mathbb{N}$. Given $\varepsilon > 0$, there exists an integer m_0 such that

$$\left(\sum_{i=m}^{\infty}|(R^{q}_{\triangle_u^s}x)_i|^{p_k}\right)^{\frac{1}{M}}<\frac{\varepsilon}{2}$$

for all $m \geq m_0$. Hence,

$$h_{\triangle_u^s}\left(x-x^{[m]}\right)=\left(\sum_{i=m}^{\infty}|(R^{q}_{\triangle_u^s}x)_i|^{p_k}\right)^{\frac{1}{M}}$$
$$\leq\left(\sum_{i=m_0}^{\infty}|(R^{q}_{\triangle_u^s}x)_i|^{p_k}\right)^{\frac{1}{M}}<\frac{\varepsilon}{2}<\varepsilon$$

for all $m \geq m_0$, which proves that $x \in r^q(\triangle_u^p(s))$ is represented as (14).

Let us show the uniqueness of the representation for $x \in r^q(\triangle_u^p(s))$ given by (13). Suppose, on the contrary; that there exists a representation $x = \sum_k \mu_k(q)b^k(q)$. Since the linear transformation T from $r^q(\triangle_u^p(s))$ to $l(p)$ used in the Theorem 2.2 is continuous we have
$((R^{q}_{\triangle_u^s}x)_n$

$$=\sum_k \mu_k(q)\left((R^{q}_{\triangle_u^s}b^k(q)\right)_n=\sum_k \mu_k(q)e_n^{(k)}=\mu_n(q)$$

for $n \in \mathbb{N}$, which contradicts the fact that $(R^q x)_n = \lambda_n(q)$ for all $n \in \mathbb{N}$. Hence, the representation (14) is unique. This completes the proof.

4 Matrix Mappings on the Space $r^q(\triangle_u^p(s))$:

In this section, we characterize the matrix mappings from the space $r^q(\triangle_u^p(s))$ to the space l_∞.

Theorem 4.1:(i) Let $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then $A \in \left(r^q(\triangle_u^p(s)) : l_\infty\right)$ if and only if there exists an integer $B > 1$ such that

$$C(B)=\sup_n\sum_k\left|\left[\frac{a_{nk}}{u_kq_k}+\left(\frac{1}{u_kq_k}-\frac{1}{u_{k+1}q_{k+1}}\right)\sum_{i=k+1}^{n}a_{ni}\right]B^{-1}Q_k^{s+1}\right|^{p_k'} \tag{4.1}$$

and $\{a_{nk}\}_{k\in\mathbb{N}} \in cs$ for each $n \in \mathbb{N}$.

(ii) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in \left(r^q(\triangle_u^p(s)) : l_\infty\right)$ if and only if

$$\sup_{n,k}\left|\left[\frac{a_{nk}}{u_kq_k}+\left(\frac{1}{u_kq_k}-\frac{1}{u_{k+1}q_{k+1}}\right)\sum_{i=k+1}^{n}a_{ni}\right]Q_k^{s+1}\right|^{p_k} \tag{4.2}$$

and $\{a_{nk}\}_{k\in\mathbb{N}} \in cs$ for each $n \in \mathbb{N}$.

Proof : We only prove the part (i) and (ii) may be proved in a similar fashion. So, let $A \in \left(r^q\left(\triangle_u^p\right) : l_\infty\right)$ and $1 < p_k \leq H < \infty$ for every $k \in \mathbb{N}$. Then Ax exists for $x \in r^q(\triangle_u^p(s))$ and implies that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{r^q(\triangle_u^p(s))\}^\beta$ for each $n \in \mathbb{N}$. Hence necessity of (17) holds.

Conversely, suppose that the necessities (17) hold and $x \in r^q(\triangle_u^p(s))$,
since $\{a_{nk}\}_{k\in\mathbb{N}} \in \{r^q(\triangle_u^p(s))\}^\beta$ for every fixed $n \in \mathbb{N}$, so the A-transform of x exists. Consider the following equality obtained by using the relation (11) that

$$\sum_{k=0}^{m} a_{nk}x_k$$

$$= \sum_{k=0}^{m}\left[\frac{a_{nk}}{q_k} + \left(\frac{1}{u_kq_k} - \frac{1}{u_{k+1}q_{k+1}}\right)\sum_{i=k+1}^{m} a_{ni}\right] u_k^{-1}Q_k^{s+1}y_k \tag{4.3}$$

Taking into account the assumptions we derive from (19) as $m \to \infty$ that

$$\sum_k a_{nk}x_k$$

$$= \sum_k\left[\frac{a_{nk}}{u_kq_k} + \left(\frac{1}{u_kq_k} - \frac{1}{u_{k+1}q_{k+1}}\right)\sum_{i=k+1}^{\infty} a_{ni}\right] Q_k^{s+1}y_k \tag{4.4}$$

Now, by combining (20) and the inequality which holds for any $B > 0$ and any complex numbers a, b

$$|ab| \leq B\left(\left|aB^{-1}\right|^{p'} + |b|^p\right)$$

with $p^{-1} + p'^{-1} = 1$(see [10]), one can easily see that

$$\sup_{n\in\mathbb{N}}\left|\sum_k a_{nk}x_k\right|$$

$$\leq \sup_{n\in\mathbb{N}}\sum_k\left|\left[\frac{a_{nk}}{u_kq_k} + \left(\frac{1}{u_kq_k} - \frac{1}{u_{k+1}q_{k+1}}\right)\sum_{i=k+1}^{\infty} a_{ni}\right] Q_k^{s+1}\right| |y_k|$$

$$\leq B\left[C(B) + h_1^B(y)\right] < \infty.$$

This shows that $Ax \in l_\infty$ whenever $x \in r^q(\triangle_u^p(s))$. This completes the proof of the theorem.

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