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The Averaging Method for Uncertain Differential Equation

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ABSTRACT. In this paper, we consider the averaging principle for the general uncertain differential equations under a Lipschitz condition. The solutions of convergence in mean square and convergence in uncertain measure between standard uncertain differential equation and the corresponding averaged uncertain delay differential equation are considered.

1 Introduction

The method of averaging is a powerful tool for the investigation of many perturbation problems in nonlinear oscillations, and some of celestial mechanics. There is a rich literature for ordinary differential equations case ([5, 6] and the references cited therein).

Uncertain process is a sequence of uncertain variables indexed by time or space. Following uncertain process, uncertain calculus [10] was established to deal with differentiation and integration of function of uncertain processes. One of the most important uncertain process is canonical process, which occupies a considerable space in uncertain calculus. The uncertain differential equation is just a type of differential equations driven by canonical process. Liu [2] first investigated uncertain differential equations, and they gave the solution of some equations. Later, Liu et.al. [11] gave another existence and uniqueness theorem for the homogeneous uncertain differential equations when the coefficients satisfy Osgood conditions. Liu and Fei [12] proved an existence and uniqueness

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theorem of solutions for uncertain functional differential equations under the uniform Lipschitz condition and the linear growth condition. Fei [13] studied uniqueness of solution to fuzzy differential equations driven by Liu's process with non-Lipschitz coefficients. Furthermore, Lupulescu et.al. [14] proved a local existence and uniqueness result for fuzzy delay differential equations driven by Liu process.

Uncertain differential equation (UDE) is an important tool to deal with dynamic systems in uncertain environments. In many applications, one assumes that the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. However, under closer scrutiny, it becomes apparent that the principle of causality is often only a first approximation to the true situation and that a more realistic model would include some of the past states of the system. Uncertain delay differential equations (UDDE) give a mathematical formulation for such system. Thus, UDE and UDDE have attracted great attention recently [1], but averaging principles for UDE have not been considered so far. Motivated by the previous paper, we consider the averaging principles for general UDE and UDDE.

The following sections are organized as follows. In Section 2 we give a detailed description of the averaging process of general uncertain differential equations. Section 3 shows the averaging principles for uncertain delay differential equation.

2 Case 1 : Uncertain differential equation

Consider the uncertain differential equation as follows

$$dX(t) = f(t, X(t))dt + g(t, X(t))dC(t) \tag{2.1}$$

with initial data $X(0) = X_0$, where $f : [0, \infty) \times \mathscr{R} \to \mathscr{R}$, $g : [0, \infty) \times \mathscr{R} \to \mathscr{R}$ are both continuous. By definition of uncertain differential, this equation (2.1) is equivalent to the following uncertain integral equation

$$X(t) = X_0 + \int_0^t f(s, X(s))dt + \int_0^t g(s, X(s))dC(t) \tag{2.2}$$

In order to guarantee the existence and uniqueness of the solution to (2.1), we impose a condition on the coefficient functions.

(A1) Lipschitz continuous condition

$$|f(t, X) - f(t, Y)| + |g(t, X) - g(t, Y)| \leq L|X - Y|, \ \ \forall X, Y \in \mathscr{R}, \ t \in [0, T], L > 0.$$

(A2) Linear growth condition

$$|f(t, X)| + |g(t, X)| \leq L_0(1 + |X|), \ \ \forall X \in \mathscr{R}, \ t \in [0, T], L_0 > 0.$$

(A3) X_0 is independent of $C(t)$ and satisfies $E(|X_0|^2) < \infty$.

It is known from Theorem 4.1 in [2] that under the condition (A1) - (A3), there exists a unique solution $X(t)$ to (2.1) with the initial data X_0.

The standard form of (2.1) is

$$X_\epsilon(t) = X_0 + \epsilon \int_0^t f(s, X_\epsilon(s))dt + \sqrt{\epsilon} \int_0^t g(s, X_\epsilon(s))dC(s) \tag{2.3}$$

where X_0 and the coefficients have the same conditions as in (2.1), and $\epsilon \in (0, \epsilon_0]$ is a positive small parameter with ϵ_0 a fixed number.

According to the existence and uniqueness theorem of differential equations, the equation (2.3) also has a unique solution $X_\epsilon(t), t \in [0, T]$ for every fixed $\epsilon \in (0, \epsilon_0]$. In order to find out whether the solution $X_\epsilon(t)$ will be approximated with small ϵ to some other simpler process, we impose some conditions on the coefficients.

Let $\overline{f}(X) : \mathscr{R} \to \mathscr{R}, \overline{g}(X) : \mathscr{R} \to \mathscr{R}$ be continuous, satisfying (A1)-(A2) with respect to X as $f(t, X)$ and $g(t, X)$. Moreover, we assume that the following inequalities are satisfied : For $X \in \mathscr{R}$ and $T_1 \in [0, T]$,

(A4)

$$\frac{1}{T_1} \int_0^{T_1} |f(s, X) - \overline{f}(X)| ds \leq \psi_1(T_1)(1 + |X|),$$

(A5)

$$\frac{1}{T_1} \int_0^{T_1} |g(s, X) - \overline{g}(X)|^2 ds \leq \psi_2(T_1)(1 + |X|^2),$$

where $\psi_i(T_1), i = 1, 2$ are positive bounded functions with $\lim_{T_1 \to \infty} \psi_i(T_1) = 0$.

We now consider the following averaged uncertain equation which corresponds to the original standard form (2.3)

$$Y_\epsilon(t) = Y_0 + \epsilon \int_0^t f(s, Y_\epsilon(s))dt + \sqrt{\epsilon} \int_0^t g(s, Y_\epsilon(s))dC(s) \tag{2.4}$$

Obviously, the equation (2.4) also has a unique solution $Y_\epsilon(t)$ under similar conditions as (2.3) for the solution $X_\epsilon(t)$. Now, we consider the connections between the processes $X_\epsilon(t)$ and $Y_\epsilon(t)$. The convergence in mean square and convergence in probability between the standard form and the averaged form of (2.2) are especially considered.

The following two theorems give the connections between the processes $X_\epsilon(t)$ and $Y_\epsilon(t)$.

Theorem 2.1. Assume that the conditions (A1) - (A5) are satisfied. For a given arbitrarily small number $\delta_1 > 0$ and a constant $Q > 0, \alpha \in (0, 1)$, there exists a number $\epsilon_1 \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \epsilon_1]$, we have

$$E\Big(\sup_{t \in [0, Q\epsilon^{-\alpha}]} |X_\epsilon(t) - Y_\epsilon(t)| \Big) \leq \delta_1.$$

Proof. By (2.3) and (2.4), we have

$$X_\epsilon(t) - Y_\epsilon(t) = \epsilon \int_0^t \Big(f(s, X_\epsilon(s)) - f(s, Y_\epsilon(s)) \Big) dt + \sqrt{\epsilon} \int_0^t \Big(g(s, X_\epsilon(s)) - g(s, Y_\epsilon(s)) \Big) dC(s)$$

For $u \in [0, T]$, we obtain

$$\begin{aligned} \sup_{t \in [0,u]} |X_\epsilon(t) - Y_\epsilon(t)| \leq 2\epsilon^2 \sup_{t \in [0,u]} \Big| \int_0^t \Big(f(s, X_\epsilon(s)) - f(Y_\epsilon(s)) \Big) ds \Big|^2 \\ + 2\epsilon \sup_{t \in [0,u]} \Big| \int_0^t \Big(g(s, X_\epsilon(s)) - g(Y_\epsilon(s)) \Big) ds \Big|^2 \end{aligned}$$

Consider the first term, using elementary inequality $|x_1 + x_2|^2 \leq 2|x_1|^2 + 2|x_2|^2$, we get

$$\begin{aligned}
J_1^2 &:= 2\epsilon^2 \sup_{t\in[0,u]} \Big| \int_0^t \Big(f(s, X_\epsilon(s)) - f(Y_\epsilon(s)) \Big) ds \Big|^2 \\
&= 2\epsilon^2 \sup_{t\in[0,u]} \Big| \int_0^t \Big(f(s, X_\epsilon(s)) - f(s, Y_\epsilon(s)) + f(s, Y_\epsilon(s)) - f(Y_\epsilon(s)) \Big) ds \Big|^2 \\
&\leq 4\epsilon^2 \sup_{t\in[0,u]} \Big| \int_0^t \Big(f(s, X_\epsilon(s)) - f(s, Y_\epsilon(s)) \Big) ds \Big|^2 + 4\epsilon^2 \sup_{t\in[0,u]} \Big| \int_0^t \Big(f(s, Y_\epsilon(s)) - f(Y_\epsilon(s)) \Big) ds \Big|^2 \\
&:= J_{11}^2 + J_{12}^2.
\end{aligned}$$

Applying the Cauchy - Schwarz inequality and the condition (A1), taking expectation on J_{11}^2 yields

$$\begin{aligned}
E(|J_{11}^2|) &\leq 4\epsilon^2 E\Big(\sup_{t\in[0,u]} t \Big| \int_0^t \Big(f(s, X_\epsilon(s)) - f(s, Y_\epsilon(s)) \Big) ds \Big|^2 \Big) \\
&\leq 4\epsilon^2 u L \int_0^u E(|X_\epsilon(s) - Y_\epsilon(s)|^2) ds.
\end{aligned}$$

For J_{12}^2 , taking the expectation and using the condition (A4), we get

$$\begin{aligned}
E(|J_{11}^2|) &\leq 4\epsilon^2 E\Big(\sup_{t\in[0,u]} t^2 \Big| \frac{1}{t} \int_0^t \Big(f(s, X_\epsilon(s)) - \overline{f}(Y_\epsilon(s)) \Big) ds \Big|^2 \Big) \\
&\leq 4\epsilon^2 E\Big(t^2 \psi_1^2(t) \Big[1 + E(\sup_{s\in[0,t]} Y_\epsilon(s)) \Big]^2 \Big) \\
&\leq 8\epsilon^2 u^2 \psi_1^2(u) \Big[1 + E(\sup_{t\in[0,u]} Y_\epsilon(t)) \Big]^2.
\end{aligned}$$

By the properties of solutions to uncertain differential equations, we know that if $E(|X_0|^2) < \infty$, then for each $t \geq 0$, $E(|X(t)|^2) < \infty$. Following the discussion of [3], this property combines with the fact that $\lim_{T_1\to\infty} \psi_1(T_1) = 0$. We can further estimate that there exists a constant M_1 such that

$$E(|J_{11}^2|) \leq 8\epsilon^2 u^2 M_1.$$

Consequently,

$$E(|J_1|^2) \leq 4\epsilon^2 u L \int_0^u E(|X_\epsilon(s) - Y_\epsilon(s)|^2) ds + 8\epsilon^2 u^2 M_1. \tag{2.5}$$

Next, we consider the second term, taking expectation on it, using the inequality [4] and elementary inequality again, we get

$$\begin{aligned}
E(|J_2^2|) &\leq 8\epsilon E\Big(\int_0^u \Big| \Big(g(s, X_\epsilon(s)) - \overline{g}(Y_\epsilon(s)) \Big) \Big|^2 ds \Big) \\
&\leq 16\epsilon E\Big(\int_0^u \Big| \Big(g(s, X_\epsilon(s)) - \overline{g}(s, Y_\epsilon(s)) \Big) \Big|^2 ds \Big) \\
&+ 16\epsilon E\Big(\int_0^u \Big| \Big(g(s, Y_\epsilon(s)) - \overline{g}(Y_\epsilon(s)) \Big) \Big|^2 ds \Big) \\
&:= E(|J_{21}^2|) + E(|J_{22}^2|).
\end{aligned}$$

Using the condition (A1), we have

$$\begin{aligned}
E(|J_{21}^2|) &= 16\epsilon E\Big(\int_0^u \Big| \Big(g(s, X_\epsilon(s)) - \overline{g}(s, Y_\epsilon(s)) \Big) \Big|^2 ds \Big) \\
&\leq 16\epsilon L \int_0^u E\Big(|X_\epsilon(s) - Y_\epsilon(s)|^2 \Big) ds.
\end{aligned}$$

Using the condition (A4), we have

$$E(|J_{22}^2|) = 16\epsilon E\Big(\int_0^u \Big|\Big(g(s, Y_\epsilon(s)) - \overline{g}(Y_\epsilon(s))\Big)\Big|^2 ds\Big)$$
$$\leq 16\epsilon u\psi_2(u)\Big[1 + E(\sup_{t\in[0,u]} Y_\epsilon(t))\Big]^2.$$

As a similar way of dealing with $E(|J_{12}^2|)$, there exists a constant M_2 such that

$$E(|J_{22}^2|) \leq 16\epsilon u M_2.$$

Then

$$E(|J_2^2|) \leq 16\epsilon L \int_0^u E\Big(|X_\epsilon(s) - Y_\epsilon(s)|^2\Big)ds + 16\epsilon u M_2. \tag{2.6}$$

From (2.5) and (2.6), we have

$$E\Big(\sup_{t\in[0,u]} |X_\epsilon(t) - Y_\epsilon(t)|^2\Big)$$
$$\leq 4\epsilon L(\epsilon u + 1)\int_0^u E(|X_\epsilon(s) - Y_\epsilon(s)|^2)ds + 8\epsilon u(\epsilon u M_1 + 2M_2)$$
$$\leq 4\epsilon L(\epsilon u + 1)\int_0^u E(\sup_{s_1\in[0,s]} |X_\epsilon(s_1) - Y_\epsilon(s_1)|^2)ds + 8\epsilon u(\epsilon u M_1 + 2M_2).$$

Applying Gronwall inequality then yields

$$E\Big(\sup_{t\in[0,u]} |X_\epsilon(t) - Y_\epsilon(t)|^2\Big) \leq 8\epsilon u(\epsilon u M_1 + 2M_2)\exp\{4\epsilon L(\epsilon u + 1)\}.$$

Choose $\alpha \in (0,1)$ and $Q > 0$ such that for every $t \in [0, Q\epsilon^{-\alpha}] \subset [0, T]$.

$$E\Big(\sup_{t\in[0,Q\epsilon^{-\alpha}]} |X_\epsilon(t) - Y_\epsilon(t)|^2\Big) \leq MQ\epsilon^{1-\alpha},$$

where $C = 8Q\epsilon^{1-\alpha}(Q\epsilon^{1-\alpha}M_1 + 2M_2)\exp\{4L\epsilon(Q\epsilon^{1-\alpha} + 1)\}$.

That is, given any number $\delta_1 > 0$, we can choose $\epsilon_1 \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \epsilon_1]$ and for every $t \in [0, Q\epsilon^{-\alpha}]$

$$E\Big(\sup_{t\in[0,Q\epsilon^{-\alpha}]} |X_\epsilon(t) - Y_\epsilon(t)|^2\Big) \leq \delta_1.$$

This completes the proof. □

Theorem 2.2. Assume that the conditions (A1) - (A5) are satisfied. For a given arbitrarily small number $\delta_2 > 0$ and a constant $Q > 0, \alpha \in (0,1)$, there exists a number $\epsilon_1 \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \epsilon_1]$, we have

$$\lim_{\epsilon\to 0} \mathbb{P}\Big(\sup_{t\in[0,Q\epsilon^{-\alpha}]} |X_\epsilon(t) - Y_\epsilon(t)| > \delta_2\Big) = 0.$$

Proof. By Theorem 2.1 and the Markov inequality [1], for any given number $\delta_2 > 0$, we have

$$\lim_{\epsilon\to 0} \mathbb{P}\Big(\sup_{t\in[0,Q\epsilon^{-\alpha}]} |X_\epsilon(t) - Y_\epsilon(t)| > \delta_2\Big) \leq \frac{1}{\delta_2^2} E\Big(\sup_{t\in[0,Q\epsilon^{-\alpha}]} |X_\epsilon(t) - Y_\epsilon(t)|^2\Big) \leq \frac{1}{\delta_2^2} MQ\epsilon^{1-\alpha},$$

Taking limits on both sides of the inequality, one get the required results. □

Example 2.3. Consider the uncertain differential equation as follows

$$dX(t) = \sin tX(t)dt + X(t)dC(t) \tag{2.7}$$

with initial data $X(0) = X_0$ and $E(|X_0|^2) < \infty$, where $C(t)$ is Liu process.

The standard form of (2.7) is

$$dX_\epsilon(t) = \epsilon \sin tX_\epsilon(t)dt + \sqrt{\epsilon}X_\epsilon dC(t) \tag{2.8}$$

Denote

$$f(t, X_\epsilon) = \sin tX_\epsilon, \;\; g(t, X_\epsilon) = X_\epsilon$$

Then

$$\overline{f}(X_\epsilon) = \frac{1}{\pi}\int_0^\pi f(t, X_\epsilon)dt = \frac{2}{\pi}X_\epsilon, \quad \overline{g}(X_\epsilon) = \frac{1}{\pi}\int_0^\pi g(t, X_\epsilon)dt = X_\epsilon$$

In the other hand, we define the averaged uncertain differential equation

$$dY_\epsilon(t) = \frac{2}{\pi}\epsilon Y_\epsilon dt + \sqrt{\epsilon}Y_\epsilon dC(t) \tag{2.9}$$

By [4], the solution of (2.9) is

$$Y(t) = X_0 \exp\Big(\Big(\frac{2}{\pi} - \frac{1}{2}\Big)t + C(t)\Big),$$

It is easy to see that the conditions (A1) - (A5) are satisfied and Theorem 2.1, Theorem 2.2 hold, i.e.,

$$E\Big(\sup_{t\in[0,Q\epsilon^{-\alpha}]} |X_\epsilon(t) - Y_\epsilon(t)|\Big) \leq \delta_1.$$

and $X_\epsilon \to Y_\epsilon$ as $\epsilon \to 0$.

Example 2.4. Consider the uncertain differential equation as follows

$$dX(t) = X(t)dt + X^2(t)dC(t) \tag{2.10}$$

with initial data $X(0) = X_0$ and $E(|X_0|^2) < \infty$, where $C(t)$ is Liu process.

The standard form of (2.10) is

$$dX_\epsilon(t) = \epsilon X_\epsilon(t)dt + \sqrt{\epsilon}X_\epsilon^2 dC(t) \tag{2.11}$$

Denote

$$f(t, X_\epsilon) = X_\epsilon, \;\; g(t, X_\epsilon) = X_\epsilon^2$$

Then

$$\overline{f}(X_\epsilon) = \int_0^1 f(t, X_\epsilon)dt = X_\epsilon, \quad \overline{g}(X_\epsilon) = \int_0^1 g(t, X_\epsilon)dt = X_\epsilon^2$$

In the other hand, we define the averaged uncertain differential equation

$$dY_\epsilon(t) = \epsilon Y_\epsilon dt + \sqrt{\epsilon}Y_\epsilon^2 dC(t) \tag{2.12}$$

By [4], the solution of (2.11) is

$$Y(t) = \exp(t)\Big[X_0^{-1} - \int_0^t \exp(s)dC(s)\Big]^{-1}$$

It is easy to see that the conditions (A1) - (A5) are satisfied and Theorem 2.1, Theorem 2.2 hold, i.e.,

$$E\Big(\sup_{t\in[0,Q\epsilon^{-\alpha}]} |X_\epsilon(t) - Y_\epsilon(t)|\Big) \leq \delta_1.$$

and $X_\epsilon \to Y_\epsilon$ as $\epsilon \to 0$.

3 Case 2 : Uncertain delay differential equation

For a positive number q, we denote by $\mathscr{C}_q$ the space $C([-q,0],\mathscr{R})$. Then $\mathscr{C}_q$ is a Banach space with respect to the supremum norm

$$||\varphi|| = \sup_{t\in[-q,0]} |\varphi(t)|$$

Let $X : [-q,\infty)\times\Omega \to \mathscr{R}$ be an uncertain process. For each $t \geq 0$ we can define a fuzzy segment process $X_t : [-q,0]\times\Omega \to \mathscr{R}$ given by

$$X_t(t,\omega) = X(t+s,\omega),\ \forall s\in[-q,0],\ \omega\in\Omega.$$

X_t is called an uncertain process with delay of the uncertain process X at moment $t \geq 0$.

Suppose that C is a standard Liu process and $f, g : [0,T]\times\mathscr{C}_q \to \mathscr{R}$ are some given function. Consider the uncertain delay differential equation as follows

$$dX(t) = f(t,X_t)dt + g(t,X_t)dC(t),\ \text{for } t\geq\tau, \tag{3.1}$$

with initial data $X(t) = \varphi(t-\tau)$ for $t\in[\tau-q,\tau]$, which is an $\mathscr{F}_{t_0}$-measurable $\mathscr{C}_q$-valued random variable such that $E(||\xi||^2) < \infty$.

This equation (3.1) is equivalent to the following uncertain delay integral equation

$$X(t) = \begin{cases} \varphi(0) + \int_\tau^t f(s,X_s)ds + \int_\tau^t g(s,X_s)dC(s) & t\in[\tau,T] \\ \varphi(t-\tau) & t\in[\tau-q,\tau] \end{cases} \tag{3.2}$$

In order to guarantee the existence and uniqueness of the solution to (3.1), we impose a condition on the coefficient functions.

(B1) Lipschitz locally condition

$$|f(t,\varphi) - f(t,\overline{\varphi})| + |g(t,\varphi) - g(t,\overline{\varphi})| \leq \overline{L}|\varphi-\overline{\varphi}|,$$

where $t\in[\tau,\tau+h]$ for some $h>0$ and $\varphi,\overline{\varphi}\in\mathscr{B}_q = \{\varphi\in\mathscr{C}_q : ||\varphi||\leq\rho\}$.

(B2) there exists $\underline{L} > 0$ such that

$$\max\{F(t,\varphi), G(t,\varphi)\} \leq \underline{L}$$

where $(t,\varphi)\in[\tau,\tau+h]\times\mathscr{B}_q$.

Consider the standard form of (3.1) is

$$X^\epsilon(t) = \varphi(0) + \epsilon\int_\tau^t f(s,X_s^\epsilon)ds + \sqrt{\epsilon}\int_\tau^t g(s,X_s^\epsilon)dC(s) \tag{3.3}$$

where initial data and the coefficients have the same conditions as (3.1), and $\epsilon\in(0,\epsilon_0]$ is a positive small parameter with ϵ_0 a fixed number.

Let $\overline{f}(X_t) : \mathscr{C}_q \to \mathscr{R}$, $\overline{g}(X_t) : \mathscr{C}_q \to \mathscr{R}$ be continuous, satisfying (B1)-(B2) with respect to X_t as $f(t,X_t)$ and $g(t,X_t)$ for $X_t\in\mathscr{C}_q$. Moreover, we assume that the following inequalities are satisfied : For $X\in\mathscr{R}$ and $T_2, a\in[0,T], a<T_2$,

(B4)

$$\frac{1}{T_2 - a}\int_a^{T_2} |f(s, X_s) - \overline{f}(X_s)|ds \le \psi_3(T_2)(1 + ||X||),$$

(B5)

$$\frac{1}{T_2 - a}\int_a^{T_2} |g(s, X_s) - \overline{g}(X_s)|^2 ds \le \psi_3(T_2)(1 + ||X||^2),$$

where $\psi_i(T_2), i = 3, 4$ are positive bounded functions with $\lim_{T_2\to\infty} \psi_i(T_2) = 0$.

The averaged form of (3.3)

$$Y^\epsilon(t) = \varphi(0) + \epsilon\int_\tau^t f(s, Y_s^\epsilon)ds + \sqrt{\epsilon}\int_\tau^t \overline{g}(Y_s^\epsilon)dC(s) \tag{3.4}$$

Obviously, the equation (3.4) also has a unique solution $Y^\epsilon(t)$ under similar conditions as (3.3) for the solution $X^\epsilon(t)$. Next, we consider the connections between the processes $X^\epsilon(t)$ and $Y^\epsilon(t)$.

Theorem 3.1. Assume that the conditions (B1) - (B5) are satisfied. For a given arbitrarily small number $\delta_3 > 0$ and a constant $Q_2 > 0, \alpha \in (0, 1)$, there exists a number $\epsilon_1 \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \epsilon_1]$, we have

$$E\Big(\sup_{t\in[0,Q_2\epsilon^{-\alpha}]} |X^\epsilon(t) - Y^\epsilon(t)|^2\Big) \le \delta_3.$$

Proof. Considering the difference $X^\epsilon(t) - Y^\epsilon(t)$, we have

$$X^\epsilon(t) - Y^\epsilon(t) = \epsilon\int_\tau^t \Big(f(s, X_s^\epsilon) - \overline{f}(Y_s^\epsilon)\Big)ds + \sqrt{\epsilon}\int_\tau^t \Big(g(s, X_s^\epsilon) - \overline{g}(Y_s^\epsilon)\Big)dC(s)$$

For $u \in [0, T]$, applying the elementary inequalities $|x_1 + x_2|^2 \le 2|x_1|^2 + 2|x_2|^2$, we have

$$\begin{aligned}\sup_{t\in[0,u]} |X^\epsilon(t) - Y^\epsilon(t)|^2 &\le 2\epsilon^2 \sup_{t\in[0,u]}\Big|\int_\tau^t \Big(f(s, X_s^\epsilon) - \overline{f}(Y_s^\epsilon)\Big)ds\Big|^2\\ &\quad + 2\epsilon \sup_{t\in[0,u]}\Big|\int_\tau^t \Big(g(s, X_s^\epsilon) - \overline{g}(Y_s^\epsilon)\Big)ds\Big|^2\\ &:= K_1^2 + K_2^2.\end{aligned} \tag{3.5}$$

Using elementary computations, we get

$$\begin{aligned}K_1^2 &:= 2\epsilon^2 \sup_{t\in[0,u]}\Big|\int_\tau^t \Big(f(s, X_s^\epsilon) - \overline{f}(Y_s^\epsilon)\Big)ds\Big|^2\\ &= 2\epsilon^2 \sup_{t\in[0,u]}\Big|\int_\tau^t \Big(f(s, X_s^\epsilon(\cdot)) - \overline{f}(Y_s^\epsilon(\cdot))\Big)ds\Big|^2\\ &\le 4\epsilon^2 \sup_{t\in[0,u]}\Big|\int_\tau^t \Big(f(s, X_s^\epsilon(\cdot)) - f(s, Y_s^\epsilon(\cdot))\Big)ds\Big|^2 + 4\epsilon^2\int_\tau^t \sup_{t\in[0,u]}\Big|\Big(f(s, X_s^\epsilon(\cdot)) - \overline{f}(Y_s^\epsilon(\cdot))\Big)ds\Big|^2\\ &:= K_{11}^2 + K_{12}^2.\end{aligned}$$

Applying the Cauchy-Schwarz inequality and condition (B1), taking expectation on K_{11}^2 yields

$$\begin{aligned}
E(|K_{11}|^2) &= 4\epsilon^2 E\Big(\sup_{t\in[0,u]}\Big|\int_\tau^t \Big(f(s,X_s^\epsilon(\cdot)) - f(s,Y_s^\epsilon(\cdot))\Big)ds\Big|^2\Big)\\
&\le 4\epsilon^2 E\Big(\sup_{t\in[0,u]}(t-\tau)\int_\tau^t \Big|f(s,X_s^\epsilon(\cdot)) - f(s,Y_s^\epsilon(\cdot))\Big|^2 ds\Big)\\
&\le 4\epsilon^2(u-\tau)\overline{L}E\Big(\int_\tau^t |X_s^\epsilon(r) - X_s^\epsilon(r)|^2 ds\Big)\\
&= 4\epsilon^2(u-\tau)\overline{L}E\Big(\int_\tau^t \sup_{r\in[-q,0]}|X^\epsilon(s+r) - X^\epsilon(s+r)|^2 ds\Big)\\
&= 4\epsilon^2(u-\tau)\overline{L}E\Big(\int_\tau^t \sup_{\theta\in[s-q,s]}|X^\epsilon(\theta) - X^\epsilon(\theta)|^2 ds\Big).
\end{aligned}$$

Using the condition (B3) and the elementary inequalities $|x_1+x_2|^2 \le 2|x_1|^2 + 2|x_2|^2$, we obtain

$$\begin{aligned}
E(|K_{12}|^2) &= 4\epsilon^2 E\Big(\sup_{t\in[0,u]}(t-\tau)^2\Big|\frac{1}{t-\tau}\int_\tau^t \Big(f(s,X_s^\epsilon(\cdot)) - \overline{f}(X_s^\epsilon(\cdot))\Big)ds\Big|^2\Big)\\
&\le 4\epsilon^2 E\Big(\sup_{t\in[0,u]}\Big[(t-\tau)^2\psi_3^2(t)\Big(1+\sup_{s\in[-q,0]}|X(s)|^2\Big)\Big]\Big)\\
&\le 8\epsilon^2(u-\tau)^2\psi_3^2(u)\Big[1+E\Big(\sup_{s\in[-q,0]}|X(s)|^2\Big)\Big].
\end{aligned}$$

By the fact that $E(|X_0|^2) < \infty$, for $t \ge 0$, $E(|X(t)|^2) < \infty$. This combines with the fact that $\lim_{T_2\to 0}\psi_3(T_2) = 0$, we can estimate that there exists a constant M_3 such that

$$E(|K_{12}|^2) \le 8\epsilon^2(u-\tau)^2 M_3.$$

Then

$$E(|K_1|^2) \le 8\epsilon^2(u-\tau)^2 M_3 + 4\epsilon^2(u-\tau)\overline{L}E\Big(\int_\tau^t \sup_{\theta\in[s-q,s]}|X^\epsilon(\theta) - X^\epsilon(\theta)|^2 ds\Big). \tag{3.6}$$

Applying the Burkholder - Davis-Gundy inequality and taking expectation on K_2^2, we have

$$\begin{aligned}
E(|K_2|^2) &\le 8\epsilon E\Big(\sup_{t\in[0,u]}\Big|\int_\tau^t \Big(g(s,X_s^\epsilon) - \overline{g}(Y_s^\epsilon)\Big)ds\Big|^2\Big)\\
&\le 16\epsilon E\Big(\sup_{t\in[0,u]}\int_\tau^t \Big|g(s,X_s^\epsilon) - g(s,Y_s^\epsilon)\Big|^2 ds\Big) + 16\epsilon E\Big(\sup_{s\in[0,u]}\int_\tau^t \Big|g(s,Y_s^\epsilon) - \overline{g}(Y_s^\epsilon)\Big|^2 ds\Big)\\
&:= E(|K_{21}|^2) + E(|K_{22}|^2).
\end{aligned}$$

Using the condition (B1), we have

$$\begin{aligned}
E(|K_{21}|^2) &\le 16\epsilon\overline{L}\int_\tau^t E(|X_s^\epsilon - Y_s^\epsilon|^2)ds\\
&= 16\epsilon\overline{L}E\Big(\int_\tau^t \sup_{\theta\in[s-q,s]}|X(\theta) - Y(\theta)|^2 ds\Big).
\end{aligned}$$

Using the condition (B5), we have

$$\begin{aligned}
E(|K_{22}|^2) &\le 16\epsilon\overline{L}E\Big(\int_\tau^t \Big|g(s,Y_s^\epsilon) - \overline{g}(Y_s^\epsilon)\Big|^2\Big)\\
&\le 16\epsilon(u-\tau)\psi_4(u)\Big[1+E\Big(\sup_{s\in[-q,0]}|X(s)|^2\Big)\Big]
\end{aligned}$$

By the fact that $E(|X_0|^2) < \infty$, for $t \geq 0$, $E(|X(t)|^2) < \infty$. This combines with the fact that $\lim_{T_2 \to 0} \psi_4(T_2) = 0$, we can estimate that there exists a constant M_4 such that

$$E(|K_{22}|^2) \leq 16\epsilon(u-\tau)M_4.$$

Then

$$E(|K_2|^2) \leq 16\epsilon(u-\tau)M_4 + 16\epsilon\overline{L}E\Big(\int_\tau^t \sup_{\theta\in[s-q,s]} |X(\theta) - Y(\theta)|^2 ds\Big). \tag{3.7}$$

From (3.5), (3.6) and (3.7), we have

$$\begin{aligned} E(\sup_{t\in[0,u]} |X^\epsilon(t) - Y^\epsilon(t)|^2) &\leq 8\epsilon[\epsilon(u-\tau)M_3 + 2M_4] \\ &\quad + 4\epsilon\overline{L}[4+\epsilon(u-\tau)]E\Big(\int_\tau^t \sup_{\theta\in[s-q,s]} |X(\theta) - Y(\theta)|^2 ds\Big). \end{aligned}$$

Applying the Gronwall inequality, we have

$$E(\sup_{t\in[0,u]} |X^\epsilon(t) - Y^\epsilon(t)|^2) \leq 8\epsilon[\epsilon(u-\tau)M_3 + 2M_4]\exp\{4\epsilon\overline{L}[4+\epsilon(u-\tau)]\}$$

Choose $\alpha \in (0,1)$ and $Q_2 > 0$ such that for every $t \in [0, Q_2\epsilon^{-\alpha}] \subset [0, T]$.

$$E\Big(\sup_{t\in[0,Q_2\epsilon^{-\alpha}]} |X^\epsilon(t) - Y^\epsilon(t)|^2\Big) \leq M_2 Q_2 \epsilon^{1-\alpha},$$

where $M_2 = 8\epsilon^{1-\alpha}[\epsilon^{1-\alpha}(u-\tau)M_3 + 2M_4]\exp\{4\epsilon^{1-\alpha}\overline{L}[4+\epsilon^{1-\alpha}(u-\tau)]\}$.

That is, given any number $\delta_3 > 0$, we can choose $\epsilon_1 \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \epsilon_1]$ and for every $t \in [0, Q_2\epsilon^{-\alpha}]$

$$E\Big(\sup_{t\in[0,Q_2\epsilon^{-\alpha}]} |X^\epsilon(t) - Y^\epsilon(t)|^2\Big) \leq \delta_3.$$

This completes the proof. □

Theorem 3.2. Assume that the conditions (B1) - (B5) are satisfied. For a given arbitrarily small number $\delta_4 > 0$ and a constant $Q_2 > 0, \alpha \in (0,1)$, there exists a number $\epsilon_1 \in (0, \epsilon_0]$ such that for each $\epsilon \in (0, \epsilon_1]$, we have

$$\lim_{\epsilon\to 0} \mathbb{P}\Big(\sup_{t\in[0,Q_2\epsilon^{-\alpha}]} |X^\epsilon(t) - Y^\epsilon(t)| > \delta_4\Big) = 0.$$

Proof. By Theorem 3.1 and the Markov inequality [1], for any given number $\delta_4 > 0$, we have

$$\lim_{\epsilon\to 0} \mathbb{P}\Big(\sup_{t\in[0,Q_2\epsilon^{-\alpha}]} |X^\epsilon(t) - Y^\epsilon(t)| > \delta_4\Big) \leq \frac{1}{\delta_4^2} E\Big(\sup_{t\in[0,Q_2\epsilon^{-\alpha}]} |X^\epsilon(t) - Y^\epsilon(t)|^2\Big) \leq \frac{1}{\delta_4^2} M_2 Q_2 \epsilon^{1-\alpha},$$

Taking limits on both sides of the inequality, we get the required results. □

Example 3.3. Consider the uncertain differential equation as follows

$$\begin{cases} dX^\epsilon(t) = \epsilon\mu X^\epsilon(t-\tau)dt + \sqrt{\epsilon}\sigma dC(t) & \text{for } t \geq 0, \\ X^\epsilon(t) = 1 & \text{for } -\tau \leq t \leq 0 \end{cases} \tag{3.8}$$

where μ and σ are constant, and $C(t)$ is a standard Liu process.

Let

$$\overline{f}(X_t^\epsilon) = \int_0^1 f(s, X_s^\epsilon)ds = \mu X^\epsilon(t-\tau), \quad \overline{g}(X_t^\epsilon) = \int_0^1 g(s, X_s^\epsilon)ds = \sigma.$$

Define the corresponding averaged UDDE (3.8) as follows

$$dY^\epsilon(t) = \epsilon\mu Y^\epsilon(t-\tau)dt + \sqrt{\epsilon}\sigma dC(t) \text{ for } t \geq 0. \tag{3.9}$$

If $t \in [0,\tau]$, then $t-\tau \in [-\tau, 0]$, and $Y^\epsilon(t-\tau) = 1$. Therefore, for $t \in [0,\tau]$, we have the explicit solution of (3.12) is

$$Y(t) = 1 + \mu t + \sigma C(t), \text{ for } t \in [0,\tau]. \tag{3.10}$$

Repeating this procedure over the intervals $[\tau, 2\tau], [2\tau, 3\tau]$ and so on, we can obtain the explicit solution.

It is easy to see that the conditions (B1) - (B5) are satisfied and Theorem 3.1, Theorem 3.2 hold, i.e.,

$$E\Big(\sup_{t\in[0,Q_2\epsilon^{-\alpha}]} |X^\epsilon(t) - Y^\epsilon(t)|^2\Big) \leq \delta_3.$$

and $X_\epsilon \to Y_\epsilon$ as $\epsilon \to 0$.

Example 3.4. Consider the uncertain differential equation as follows

$$\begin{cases} dX^\epsilon(t) = \epsilon\mu\sin^2 tX^\epsilon(t-1)dt + \sqrt{\epsilon}\sigma X^\epsilon(t-1)dC(t) & \text{for } t \geq 0, \\ X^\epsilon(t) = 1 + t & \text{for } -1 \leq t \leq 0 \end{cases} \tag{3.11}$$

where μ and σ are constant, and $C(t)$ is a standard Liu process.

Let

$$\overline{f}(X_t^\epsilon) = \frac{1}{\pi}\int_0^\pi f(s, X_s^\epsilon)ds = \mu X^\epsilon(t-1), \quad \overline{g}(X_t^\epsilon) = \frac{1}{\pi}\int_0^\pi g(s, X_s^\epsilon)ds = \sigma X^\epsilon(t-1).$$

Define the corresponding averaged UDDE (3.12) as follows

$$dY^\epsilon(t) = \epsilon\mu Y^\epsilon(t-1)dt + \sqrt{\epsilon}\sigma Y^\epsilon(t-1)dC(t) \text{ for } t \geq 0. \tag{3.12}$$

If $t \in [0,1]$, we have the explicit solution of (3.12) is

$$Y(t) = Y(0) + \mu\frac{t^2}{2} + \sigma\int_0^t s dC(s). \tag{3.13}$$

Repeating this procedure over the intervals $[1,2], [2,3]$ and so on, we can obtain the explicit solution.

It is easy to see that the conditions (B1) - (B5) are satisfied and Theorem 3.1, Theorem 3.2 hold, i.e.,

$$E\Big(\sup_{t\in[0,Q_2\epsilon^{-\alpha}]} |X^\epsilon(t) - Y^\epsilon(t)|^2\Big) \leq \delta_3.$$

and $X^\epsilon \to Y^\epsilon$ as $\epsilon \to 0$.

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