International Journal of Advances in Mathematics
Volume 2018, Number 1, Pages 109-119, 2018
eISSN 2456-6098

Generalizations of Eneström-Kakeya Theorem

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ABSTRACT. If $P(z) := \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j, j=0,1,...,n are real numbers. In this paper, we prove some extensions and generalizations of the classical results of the well known Eneström-Kakeya theorem by imposing restrictions on the complex coefficients of a polynomial in order to give bounds concerning the number of zeros in a specific region of the complex plane. Also, a variety of interesting results can be deduced from them by a uniform procedure.

1 Introduction and Statement of Results

Determining zero bounds for real and complex polynomials is a classical problem that has been proven essential in various disciplines such as engineering, mathematical chemistry. Numerical methods have been developed that can be applied to polynomials with real or complex roots [1, 2]. Moreover, upper and lower bounds have been investigated to locate the zeros of polynomials [3, 4]. Over last five decades, a large number of research papers, e.g,[5-8, 9, 10, 11, 12] and monographs [13-15] have been published. Gauss and Cauchy were the earliest contributors in the theory of the location of zeros of a polynomial, and since then an extensive research has been done in this subject by many people. There is always a need for refinement of results in this subject because of its application in many areas, including signal processing, communication theory, and control theory.
The following result is well known in the theory of the distribution of zeros of polynomials.

Theorem A(Eneström-Kakeya). *If* $P(z) := \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n such that*

* Corresponding Author.
Received August 11, 2017; revised November 11, 2017; accepted November 28, 2017.
2010 Mathematics Subject Classification: 30C10, 30C15.
Key words and phrases: Polynomial, Zeros, Eneström-Kakeya theorem.

$$a_n \geq a_{n-1} \geq a_{n-2} \geq ... \geq a_1 \geq a_0 \geq 0,$$

then $P(z)$ does not vanish in $|z| > 1$.

There already exist in literature[1-19], certain generalizations and refinements of Eneström-Kakeya theorem. The Eneström-Kakeya theorem is a very strong tool to find the region in the complex plane containing all the zeros of a class of polynomials. It has been used to analyze overflow oscillation of discrete-time dynamical system, to investigate the properties of orthogonal wavelets, and to determine the asymptotic behaviour of zeros of the Daubechies filter, in addition for application to a model of high energy collisions. Joyal *et al.* [10] obtained the following generalization, by considering the coefficients to be real, instead of being only positive.

Theorem B. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq a_{n-2} \geq ... \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$|z| \leq \frac{1}{|a_n|}\{a_n - a_0 + |a_0|\}.$$

Dewan and Bidkham [8] generalized Theorem B and proved the following:

Theorem C. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $t > 0$ and $0 < \lambda \leq n$,*

$$a_n t^n \leq a_{n-1} t^{n-1} \leq ... \leq a_\lambda t^\lambda \geq a_{\lambda-1} t^{\lambda-1} \geq ... \geq t a_1 \geq a_0,$$

then $P(z)$ has all the zeros in the circle

$$|z| \leq \frac{t}{|a_n|}\{(\frac{2a_\lambda}{t^{n-\lambda}} - a_n) + \frac{1}{t^n}(|a_0| - a_0)\}.$$

Aziz and Zargar [5] also relaxed the hypothesis of Eneström-Kakeya theorem in a different way and proved the following results.

Theorem D. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$,*

$$k a_n \geq a_{n-1} \geq a_{n-2} \geq ... \geq a_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$|z + k - 1| \leq k.$$

Theorem E. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$,*

$$k a_n \geq a_{n-1} \geq a_{n-2} \geq ... \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$|z + k - 1| \leq \frac{k a_n - a_0 + |a_0|}{|a_n|}.$$

The following result concerning the number of zeros in a closed disk is due to Mohammad [16].

Theorem F. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0,$$

then the number of zeros of P(z) in $|z| \leq \frac{1}{2}$ does not exceed

$$1 + \frac{1}{\log 2} \log \frac{a_n}{a_0}. \tag{1}$$

Bidkham and Dewan [6] generalized the above theorem for different classes of polynomials and among other things proved the following result:

Theorem G. *Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \leq a_{n-1} \leq ... \leq a_{\lambda+1} \leq a_\lambda \geq a_{\lambda-1} \geq ... \geq a_1 \geq a_0$$

for some λ, $0 \leq \lambda \leq n$, then the number of zeros of P(z) in $|z| \leq \frac{1}{2}$ does not exceed

$$\frac{1}{\log 2}\{\log \frac{|a_n| + |a_0| - a_n - a_0 + 2a_\lambda}{|a_0|}\}. \tag{2}$$

The aim of this paper is to prove some extensions of Eneström-Kakeya theorem (Theorem A) by relaxing the hypothesis in different ways. In this paper, we also generalizes Theorem F.

Here, we shall first prove the following generalization of Theorem D which is an interesting extensions of Theorem A.

Theorem 1. *Let $P(z) := \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j, j=0,1,...,n are real numbers.Then for some real numbers t_1, t_2, k_1, k_2, ρ_1, ρ_2, $0 \leq r \leq n-1$ and $a_{n-r} \neq 0$, $b_{n-r} \neq 0$,*

$$t_1 a_n \geq a_{n-1} \geq ... \geq a_{n-r+1} \geq k_1 a_{n-r} \geq a_{n-r-1} \geq ... \geq a_1 \geq \rho_1 a_0,$$

$$t_2 b_n \geq b_{n-1} \geq ... \geq b_{n-r+1} \geq k_2 b_{n-r} \geq b_{n-r-1} \geq ... \geq b_1 \geq \rho_2 b_0,$$

if $a_{n-r-1} > a_{n-r}$, $b_{n-r-1} > b_{n-r}$, then all the zeros of $P(z)$ lie in the disk $|z| \leq R_1$, where R_1 is the greatest positive root of the equation

$$D^{r+1} - \delta_1 D^r - |\tau_1| = 0,$$

where

$\tau_1 = \frac{(k_1-1)a_{n-r}+i(k_2-1)b_{n-r}}{\alpha_n}$

and

$$\delta_1 = \frac{1}{|\alpha_n|}\Big\{t_1a_n + |t_1-1||a_n| + (k_1-1)a_{n-r} - \rho_1 a_0 + |\rho_1-1||a_0| + |a_0| + t_2b_n + |t_2-1||b_n| + (k_2-1)b_{n-r} - \rho_2 b_0 + |\rho_2-1||b_0| + |b_0|\Big\}.$$

If $a_{n-r} > a_{n-r+1}$, $b_{n-r} > b_{n-r+1}$, then all the zeros of $P(z)$ lie in the disk $|z| \le R_2$, where R_2 is the greatest positive root of the equation

$$D^r - \delta_2 D^{r-1} - |\tau_2| = 0,$$

where

$\tau_2 = \frac{(1-k_1)a_{n-r}+i(1-k_2)b_{n-r}}{\alpha_n}$

and

$$\delta_2 = \frac{1}{|\alpha_n|}\Big\{t_1a_n + |t_1-1||a_n| + (1-k_1)a_{n-r} - \rho_1 a_0 + |\rho_1-1||a_0| + |a_0| + t_2b_n + |t_2-1||b_n| + (1-k_2)b_{n-r} - \rho_2 b_0 + |\rho_2-1||b_0| + |b_0|\Big\}.$$

Theorem 2. *Let $P(z) := \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n with complex coefficients. If $\alpha_j = a_j + ib_j$ where a_j and b_j, j=0,1,...,n are real numbers and for some real numbers $t_1,\ t_2,\ k_1,\ k_2 \ge 1$, $0 < \rho_1,\ \rho_2 \le 1$ and $0 \le r \le n$,*

$$t_1a_n \ge a_{n-1} \ge ... \ge a_{r+1} \ge k_1a_r \ge a_{r-1} \ge ... \ge a_1 \ge \rho_1 a_0,$$

$$t_2b_n \ge b_{n-1} \ge ... \ge b_{r+1} \ge k_2b_r \ge b_{r-1} \ge ... \ge b_1 \ge \rho_2 b_0,$$

then the number of zeros of $P(z)$ in $\frac{|\alpha_0|}{M_1} \le |z| \le \tau$, $0 < \tau < 1$ does not exceed

$$\frac{1}{\log\frac{1}{\tau}} \log\frac{1}{|\alpha_0|}\Big\{t_1(|a_n|+a_n) + (k_1-1)(|a_r|+a_r) - \rho_1(|a_0|+a_0) + 2|a_0| + t_2(|b_n|+b_n) + (k_2-1)(|b_r|+b_r) - \rho_2(|b_0|+b_0) + 2|b_0|\Big\},$$

where

$$M_1 = \Big\{t_1(|a_n|+a_n) + (k_1-1)(|a_r|+a_r) - \rho_1(|a_0|+a_0) + |a_0| + t_2(|b_n|+b_n) + (k_2-1)(|b_r|+b_r) - \rho_2(|b_0|+b_0) + |b_0|\Big\}.$$

2 Lemmas

For the proof of Theorem 2, we need the following lemma

Lemma 1: *If f(z) is regular, $f(0) \neq 0$ and $|f(z)| \leq M$ in $|z| \leq 1$, then the number of zeros of f(z) in $|z| \leq \tau$, $0 < \tau < 1$, does not exceed*

$$\frac{1}{\log \frac{1}{\tau}} \log \frac{M}{|f(0)|} \quad (see[17]).$$

3 Proofs of Theorems

Proof of Theorem 1. If $a_{n-r-1} > a_{n-r}$, $b_{n-r-1} > b_{n-r}$, then $a_{n-r+1} > a_{n-r}$, $b_{n-r+1} > b_{n-r}$.

Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= -a_n z^{n+1} - (k_1-1)a_{n-r}z^{n-r} + [(t_1 a_n - a_{n-1}) - (t_1 a_n - a_n)]z^n + ... \\
&\quad +(a_{n-r+1} - a_{n-r})z^{n-r+1} + (k_1 a_{n-r} - a_{n-r-1})z^{n-r} + (a_{n-r-1} - a_{n-r-2})z^{n-r-1} + ... \\
&\quad +[(a_1 - \rho_1 a_0) + (\rho_1 a_0 - a_0)]z + a_0 \\
&\quad +i\Big\{ - b_n z^{n+1} - (k_2-1)b_{n-r}z^{n-r} + [(t_2 b_n - b_{n-1}) - (t_2 b_n - b_n)]z^n + ... \\
&\quad +(b_{n-r+1} - b_{n-r})z^{n-r+1} + (k_2 b_{n-r} - b_{n-r-1})z^{n-r} + (b_{n-r-1} - b_{n-r-2})z^{n-r-1} + ... \\
&\quad +[(b_1 - \rho_2 b_0) + (\rho_2 b_0 - b_0)]z + b_0 \Big\}.
\end{aligned}$$

Let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1$, $0 \leq j < n$ and we have

$$\begin{aligned}
|F(z)| \geq |\alpha_n z^{n+1} + [(k_1-1)a_{n-r} + i(k_2-1)b_{n-r}]z^{n-r}| - |z|^n \Big\{ |t_1 a_n - a_{n-1}| + |t_1 - 1||a_n| + ... \\
+ \frac{|a_{n-r+1} - a_{n-r}|}{|z|^{r-1}} + \frac{|k_1 a_{n-r} - a_{n-r-1}|}{|z|^r} + \frac{|a_{n-r-1} - a_{n-r-2}|}{|z|^{r+1}} + ... \\
+ \frac{|a_1 - \rho_1 a_0|}{|z|^{n-1}} + \frac{|\rho_1 - 1||a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}
\end{aligned}$$

$$+|t_2 b_n - b_{n-1}| + |t_2 - 1||b_n| + ...$$

$$\begin{aligned}
+ \frac{|b_{n-r+1} - b_{n-r}|}{|z|^{r-1}} + \frac{|k_2 b_{n-r} - b_{n-r-1}|}{|z|^r} + \frac{|b_{n-r-1} - b_{n-r-2}|}{|z|^{r+1}} + ... \\
+ \frac{|b_1 - \rho_2 b_0|}{|z|^{n-1}} + \frac{|\rho_2 - 1||b_0|}{|z|^{n-1}} + \frac{|b_0|}{|z|^n} \Big\}
\end{aligned}$$

$$\geq |\alpha_n z^{n+1} + [(k_1-1)a_{n-r} + i(k_2-1)b_{n-r}]z^{n-r}| - |z|^n \Big\{ t_1 a_n - a_{n-1} + |t_1-1||a_n| + ...$$

$$+(a_{n-r+1} - a_{n-r}) + (k_1 a_{n-r} - a_{n-r-1}) + (a_{n-r-1} - a_{n-r-2}) + ...$$

$$+(a_1 - \rho_1 a_0) + |\rho_1 - 1||a_0| + |a_0|$$

$$+t_2 b_n - b_{n-1} + |t_2-1||b_n| + ...$$

$$+(b_{n-r+1} - b_{n-r}) + (k_2 b_{n-r} - b_{n-r-1}) + (b_{n-r-1} - b_{n-r-2}) + ...$$

$$+(b_1 - \rho_2 b_0) + |\rho_2 - 1||b_0| + |b_0| \Big\}$$

$$= |z|^{n-r}|\alpha_n z^{r+1} + [(k_1-1)a_{n-r} + i(k_2-1)b_{n-r}]| - |z|^n \Big\{ t_1 a_n + |t_1-1||a_n| + (k_1-1)a_{n-r} - \rho_1 a_0 + |\rho_1-1||a_0| + |a_0|$$
$$+ t_2 b_n + |t_2-1||b_n| + (k_2-1)b_{n-r} - \rho_2 b_0 + |\rho_2-1||b_0| + |b_0| \Big\}$$
$$> 0,$$

if $|z^{r+1} + \tau_1| - |z|^r \delta_1 > 0$,

where

$\tau_1 = \frac{[(k_1-1)a_{n-r} + i(k_2-1)b_{n-r}]}{\alpha_n}$

and

$$\delta_1 = \frac{1}{|\alpha_n|} \Big\{ t_1 a_n + |t_1-1||a_n| + (k_1-1)a_{n-r} - \rho_1 a_0 + |\rho_1-1||a_0| + |a_0|$$
$$+ t_2 b_n + |t_2-1||b_n| + (k_2-1)b_{n-r} - \rho_2 b_0 + |\rho_2-1||b_0| + |b_0| \Big\}.$$

This inequality holds if

$$|z|^{r+1} - \delta_1|z|^r - |\tau_1| > 0.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disk $|z| \leq R_1$, where R_1 is the greatest positive root of the equation

$$D^{r+1} - \delta_1 D^r - |\tau_1| = 0.$$

But the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disk $|z| \leq R_1$, where $R_1 > 1$ (see [7]. Remark 1), this proves the first part of the theorem.

Now, for the second part, If $a_{n-r} > a_{n-r+1}$, $b_{n-r} > b_{n-r+1}$, then $a_{n-r} > a_{n-r-1}$, $b_{n-r} > b_{n-r-1}$ and $F(z)$ can be written as

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= -a_n z^{n+1} - (1-k_1)a_{n-r}z^{n-r+1} + [(t_1a_n - a_{n-1}) - (t_1a_n - a_n)]z^n + ... \\
&\quad +(a_{n-r+1} - k_1a_{n-r})z^{n-r+1} + (a_{n-r} - a_{n-r-1})z^{n-r} + (a_{n-r-1} - a_{n-r-2})z^{n-r-1} + ... \\
&\quad +[(a_1 - \rho_1a_0) + (\rho_1a_0 - a_0)]z + a_0 \\
&\quad +i\Big\{ - b_nz^{n+1} - (1-k_2)b_{n-r}z^{n-r+1} + [(t_2b_n - b_{n-1}) - (t_2b_n - b_n)]z^n + ... \\
&\quad +(b_{n-r+1} - k_2b_{n-r})z^{n-r+1} + (b_{n-r} - b_{n-r-1})z^{n-r} + (b_{n-r-1} - b_{n-r-2})z^{n-r-1} + ... \\
&\quad +[(b_1 - \rho_2b_0) + (\rho_2b_0 - b_0)]z + b_0\Big\}
\end{aligned}$$

Let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1, 0 \leq j < n$ and we have

$$\begin{aligned}
|F(z)| \geq\ & |\alpha_nz^{n+1} + [(1-k_1)a_{n-r} + i(1-k_2)b_{n-r}]z^{n-r+1}| - |z|^n\Big\{|t_1a_n - a_{n-1}| + |t_1 - 1||a_n| + ... \\
& + \frac{|a_{n-r+1} - k_1a_{n-r}|}{|z|^{r-1}} + \frac{|a_{n-r} - a_{n-r-1}|}{|z|^r} + \frac{|a_{n-r-1} - a_{n-r-2}|}{|z|^{r+1}} + ... \\
& + \frac{|a_1 - \rho_1a_0|}{|z|^{n-1}} + \frac{|\rho_1 - 1||a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}
\end{aligned}$$

$$+|t_2b_n - b_{n-1}| + |t_2 - 1||b_n| + ...$$

$$\begin{aligned}
& + \frac{|b_{n-r+1} - k_2b_{n-r}|}{|z|^{r-1}} + \frac{|b_{n-r} - b_{n-r-1}|}{|z|^r} + \frac{|b_{n-r-1} - b_{n-r-2}|}{|z|^{r+1}} + ... \\
& + \frac{|b_1 - \rho_2b_0|}{|z|^{n-1}} + \frac{|\rho_2 - 1||b_0|}{|z|^{n-1}} + \frac{|b_0|}{|z|^n}\Big\}
\end{aligned}$$

$$\geq |\alpha_nz^{n+1} + [(1-k_1)a_{n-r} + i(1-k_2)b_{n-r}]z^{n-r+1}| - |z|^n\Big\{t_1a_n - a_{n-1} + |t_1 - 1||a_n| + ...$$

$$+(a_{n-r+1}-k_1a_{n-r})+(a_{n-r}-a_{n-r-1})+(a_{n-r-1}-a_{n-r-2})+...$$

$$+(a_1-\rho_1a_0)+|\rho_1-1||a_0|+|a_0|$$

$$+t_2b_n-b_{n-1}+|t_2-1||b_n|+...$$

$$+(b_{n-r+1}-k_2b_{n-r})+(b_{n-r}-b_{n-r-1})+(b_{n-r-1}-b_{n-r-2})+...$$

$$+(b_1-\rho_2b_0)+|\rho_2-1||b_0|+|b_0|\Big\}$$

$$\begin{aligned}&=|z|^{n-r+1}|\alpha_nz^{r+1}+[(1-k_1)a_{n-r}+i(1-k_2)b_{n-r}]|-|z|^n\Big\{t_1a_n+|t_1-1||a_n|+(1-k_1)a_{n-r}-\rho_1a_0+|\rho_1-1||a_0|\\&\qquad+|a_0|+t_2b_n+|t_2-1||b_n|+(1-k_2)b_{n-r}-\rho_2b_0+|\rho_2-1||b_0|+|b_0|\Big\}\\&>0,\end{aligned}$$

if $|z^r+\tau_2|-|z|^{r-1}\delta_2>0$,

where

$\tau_2=\frac{[(1-k_1)a_{n-r}+i(1-k_2)b_{n-r}]}{\alpha_n}$

and

$$\begin{aligned}\delta_2=\frac{1}{|\alpha_n|}&\Big\{t_1a_n+|t_1-1||a_n|+(1-k_1)a_{n-r}-\rho_1a_0+|\rho_1-1||a_0|+|a_0|\\&+t_2b_n+|t_2-1||b_n|+(1-k_2)b_{n-r}-\rho_2b_0+|\rho_2-1||b_0|+|b_0|\Big\}.\end{aligned}$$

This inequality holds if

$$|z|^r-\delta_2|z|^{r-1}-|\tau_2|>0.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disk $|z| \leq R_2$, where R_2 is the greatest positive root of the equation

$$D^r - \delta_2 D^{r-1} - |\tau_2| = 0.$$

But the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disk $|z| \leq R_2$, where $R_2 > 1$ (see [7]. Remark 2), this proves the second part of the theorem.

Proof of Theorem 2. Consider the polynomial

$$\begin{aligned}
H(z) &= (1-z)P(z) \\
&= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_{r+1} - \alpha_r)z^{r+1} + ... + (\alpha_1 - \alpha_0)z + \alpha_0 \\
&= -a_n z^{n+1} + [(t_1 a_n - a_{n-1}) - (t_1 a_n - a_n)]z^n + ... \\
&\quad + (a_{r+1} - a_r)z^{r+1} + [(k_1 a_r - a_{r-1}) - (k_1 a_r - a_r)]z^r + (a_{r-1} - a_{r-2})z^{r-1} ... \\
&\quad + [(a_1 - \rho_1 a_0) + (\rho_1 a_0 - a_0)]z + a_0 \\
&\quad + i\Big\{ - b_n z^{n+1} + [(t_2 b_n - b_{n-1}) - (t_2 b_n - b_n)]z^n + ... \\
&\quad + (b_{r+1} - b_r)z^{r+1} + [(k_2 b_r - b_{r-1}) - (k_2 b_r - b_r)]z^r + (b_{r-1} - b_{r-2})z^{r-1} ... \\
&\quad + [(b_1 - \rho_2 b_0) + (\rho_2 b_0 - b_0)]z + b_0 \Big\}.
\end{aligned}$$

For $|z| \leq 1$ and by using the hypothesis, we get

$$\begin{aligned}
|H(z)| \leq \Big\{ & t_1(|a_n| + a_n) + (k_1 - 1)(|a_r| + a_r) - \rho_1(|a_0| + a_0) + 2|a_0| \\
& + t_2(|b_n| + b_n) + (k_2 - 1)(|b_r| + b_r) - \rho_2(|b_0| + b_0) + 2|b_0| \Big\}.
\end{aligned}$$

Therefore, by using Lemma 1, we conclude that the number of zeros of $H(z)$ and hence $P(z)$ in $|z| \leq \tau$, $0 < \tau < 1$ does not exceed

$$\frac{1}{\log \frac{1}{\tau}} \log \frac{1}{|\alpha_0|} \Big\{ t_1(|a_n| + a_n) + (k_1 - 1)(|a_r| + a_r) - \rho_1(|a_0| + a_0) + 2|a_0| + t_2(|b_n| + b_n) + (k_2 - 1)(|b_r| + b_r) - \rho_2(|b_0| + b_0) + 2|b_0| \Big\}.$$

In order to show that $P(z)$ has no zeros in $\frac{|\alpha_0|}{M_1} \leq |z|$, we consider

$$\begin{aligned}
H(z) &= (1-z)P(z) \\
&= -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_{r+1} - \alpha_r)z^{r+1} + ... + (\alpha_1 - \alpha_0)z + \alpha_0 \\
&= -a_n z^{n+1} + [(t_1 a_n - a_{n-1}) - (t_1 a_n - a_n)]z^n + ... \\
&\quad + (a_{r+1} - a_r)z^{r+1} + [(k_1 a_r - a_{r-1}) - (k_1 a_r - a_r)]z^r + (a_{r-1} - a_{r-2})z^{r-1} ... \\
&\quad + [(a_1 - \rho_1 a_0) + (\rho_1 a_0 - a_0)]z + a_0 \\
&\quad + i\Big\{ - b_n z^{n+1} + [(t_2 b_n - b_{n-1}) - (t_2 b_n - b_n)]z^n + ... \\
&\quad + (b_{r+1} - b_r)z^{r+1} + [(k_2 b_r - b_{r-1}) - (k_2 b_r - b_r)]z^r + (b_{r-1} - b_{r-2})z^{r-1} ... \\
&\quad + [(b_1 - \rho_2 b_0) + (\rho_2 b_0 - b_0)]z + b_0 \Big\}. \\
&\quad = \alpha_0 + E(z),
\end{aligned}$$

where

$$\begin{aligned}E(z) = &-a_n z^{n+1} + [(t_1 a_n - a_{n-1}) - (t_1 a_n - a_n)]z^n + ...\\&+(a_{r+1} - a_r)z^{r+1} + [(k_1 a_r - a_{r-1}) - (k_1 a_r - a_r)]z^r + (a_{r-1} - a_{r-2})z^{r-1}...\\&+[(a_1 - \rho_1 a_0) + (\rho_1 a_0 - a_0)]z\\&+i\Big\{ - b_n z^{n+1} + [(t_2 b_n - b_{n-1}) - (t_2 b_n - b_n)]z^n + ...\\&+(b_{r+1} - b_r)z^{r+1} + [(k_2 b_r - b_{r-1}) - (k_2 b_r - b_r)]z^r + (b_{r-1} - b_{r-2})z^{r-1}...\\&+[(b_1 - \rho_2 b_0) + (\rho_2 b_0 - b_0)]z\Big\}.\end{aligned}$$

For $|z| \leq 1$, we have

$$\begin{aligned}\max_{|z|=1} |E(z)| \leq \Big\{ &t_1(|a_n| + a_n) + (k_1 - 1)(|a_r| + a_r) - \rho_1(|a_0| + a_0) + |a_0|\\&+ t_2(|b_n| + b_n) + (k_2 - 1)(|b_r| + b_r) - \rho_2(|b_0| + b_0) + |b_0|\Big\} = M_1.\end{aligned}$$

Therefore,

$$\begin{aligned}|H(z)| &= |\alpha_0 + E(z)|\\&\geq |\alpha_0| - |E(z)|\\&\geq |\alpha_0| - |z| \max_{|z|=1} |E(z)|\\&\geq |\alpha_0| - |z|M_1\\&> 0 \text{ if } |z| < \tfrac{|\alpha_0|}{M_1}.\end{aligned}$$

This shows that $H(z)$ and hence $P(z)$ has no zeros in $|z| < \frac{|\alpha_0|}{M_1}$. This proves the Theorem 2 completely.

Conclusion: Finding the roots of a polynomial is an interesting area of research for engineers as well as mathematicians because of their applications in linear control system, signal processing, electrical networks, coding theory and several areas of physical sciences, where among others location of zeros and stability problems arise in a natural way. Polynomials in various forms have recently come under extensive revision because of their applications in the above mentioned fields. Many more interesting results can be deduced in a similar procedure.

Acknowledgement: The authors are extremely grateful to the referee(s) for their valuable suggestions.

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