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(α, β)-Doubt Fuzzy Ideals of BG-Algebras

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ABSTRACT. In this paper, we introduced the concept of (α, β)-doubt fuzzy ideals of BG-algebras and investigated some of their related properties. We also defined doubt cartesian product of (α, β)-doubt fuzzy ideals and studied their properties.

1 Introduction

The concept of fuzzy sets was first proposed by Zadeh([18]) in 1965. Rosenfeld ([12]) was the first who consider the case of a groupoid in terms of fuzzy sets. Since then these ideas have been applied to other algebraic structures such as group, semigroup, ring, vector spaces etc. Imai and Iseki ([5]) introduced BCK-algebra as a generalization of notion of the concept of set theoretic difference and propositional calculus and in the same year Iseki ([7]) introduced the notion of BCI-algebra which is a generalization of BCK-algebra. Xi([17]) applied the concept of fuzzy set to BCK-algebra. and discussed some properties. Since then B-algebras was introduced in [11] by Neggers and Kim and which is related to several classes of algebras such as BCI/BCK-algebras. The generalization of B-algebra is BG-algebra, introduced by Kim and Kim in [9]. Fuzzy subalgebras of BG-algebras introduced in [1] by Ahn and Lee and the fuzzification of ideals of BG-algebras were studied in [10] by R. Muthuraj et al. Huang [4] fuzzified BCI-algebras in little different ways. Jun et al. [3, 8] renamed Huang's definition as doubt(anti) fuzzy ideals in BCK/BCI-algebras. The concept of doubt fuzzy BF- algebras was introduced by Saeid in [13]. Biswas [2] introduced the concept of anti fuzzy subgroup. In [14, 15, 16] Sharma introduced the notion of (α, β)-anti fuzzy set

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and (α,β)-anti fuzzy subgroups of a group G. Modifying their idea, in this paper we apply the combined idea of (α,β)-anti fuzzy set and doubt(anti) fuzzy ideals to BG-algebras and introduced the notion of (α,β)-doubt fuzzy ideals of BG-algebras and establish some of their basic properties. We show that a fuzzy subset is (α,β)-DFI iff its complement $\mu^c_{\alpha,\beta}$ is fuzzy ideal. We also study union and intersection of two (α,β)-doubt fuzzy ideals and image and pre image of (α,β)-doubt fuzzy ideals under homomorphic mapping.

2 Preliminaries

Definition 2.1. *[1, 9]* A BG-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(i) $x * x = 0,$

(ii) $x * 0 = x,$

(iii) $(x * y) * (0 * y) = x \; \forall \; x, y \in X.$

For simplicity, we also call X a BG-algebra. We can define a partial ordering $'\leq'$ by $x \leq y$ if and only if $x * y = 0$.

Example 2.2. *Let X = {0, 1, 2, 3, 4} with the following cayley table Then (X, ∗, 0) is a BG algebra .*

Table 1: Example of BG-algebra.

*	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

A non-empty subset S of a BG-algebra X is called a subalgebra of X if x ∗ y ∈ S, for all x, y ∈ S. A fuzzy subset μ of a BG-algebra X is called a fuzzy subalgebra of X if $\mu(x * y) \geq min\{\mu(x), \mu(y)\}$ for all x, y ∈ S. A nonempty subset I of a BG-algebra X is called a BG-ideal of X if (I1) $0 \in I$, (I2) $x * y \in I,\; y \in I \Rightarrow x \in I \; \forall \; x, y \in X$.
A fuzzy set μ in X is called a fuzzy BG-ideal of X if it satisfies the following conditions:
(FI1) $\mu(0) \geq \mu(x)$, (FI2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \; \forall \; x, y \in X$.

Definition 2.3. Let μ be a fuzzy subset of a BG-algebra X and $\alpha \in [0, 1]$ then the fuzzy set μ^α and μ_α of X are respectively called the α-fuzzy subset and α -doubt fuzzy subset of X with respect to fuzzy set μ and is defined as $\mu^\alpha(x) = min\{\mu(x), \alpha\}$ and $\mu_\alpha(x) = max\{\mu(x), 1-\alpha\}$ for all $x \in X$.
Clearly $\mu^1 = \mu, \quad \mu^0 = \tilde{0}, \quad \mu_1 = \mu, \quad \mu_0 = \tilde{1}$

Definition 2.4. A fuzzy subset μ of BG-algebra X is called α-fuzzy ideal (α-FI) of X if

$$\begin{aligned}(i)\mu^{\alpha}(0) &\geq \mu^{\alpha}(x)\\ (ii)\mu^{\alpha}(x) &\geq min\{\mu^{\alpha}(x*y), \mu^{\alpha}(y)\} \quad \forall x, y \in X, \alpha \in [0,1]\end{aligned}$$

Definition 2.5. A fuzzy subset μ of BG-algebra X is called α-doubt fuzzy ideal (α-DFI) of X if

$$\begin{aligned}(i)\mu_{\alpha}(0) &\leq \mu_{\alpha}(x)\\ (ii)\mu_{\alpha}(x) &\leq max\{\mu_{\alpha}(x*y), \mu_{\alpha}(y)\} \quad \forall x, y \in X, \alpha \in [0,1]\end{aligned}$$

Proposition 2.6. *If μ is fuzzy ideal of BG-algebra X, then μ is also α-FI as wel as α-DFI of X .*

Definition 2.7. A fuzzy subset μ of BG-algebra X is called α- fuzzy subalgebra of X if $\mu^{\alpha}(x*y) \geq min\{\mu^{\alpha}(x), \mu^{\alpha}(y)\} \quad \forall x, y \in X, \alpha \in [0,1]$

Definition 2.8. A fuzzy subset μ of BG-algebra X is called α-doubt fuzzy subalgebra of X if $\mu_{\alpha}(x*y) \leq max\{\mu_{\alpha}(x), \mu_{\alpha}(y)\} \quad \forall x, y \in X, \alpha \in [0,1]$

Definition 2.9. Let X and Y be two non empty sets and $f : X \longrightarrow Y$ be a mapping. Let μ and ν be two fuzzy subsets of X and Y respectively . Then the image of μ under the map f is denoted by $f(\mu)$ and is defined by $f(\mu)(y)$, where $f(\mu)(y) = \begin{cases} \sup_{x\in f^{-1}(y)} \mu(x), & \text{if} \quad f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases}$

also pre image of ν under f is denoted by $f^{-1}(\nu)$ and is defined as $f^{-1}(\nu)(x) = \nu(f(x)); \forall x \in X$

3 (α, β)-Doubt fuzzy ideals

Definition 3.1. Let μ^{α} and μ_{β} denote respectively the α-fuzzy set and β-doubt fuzzy set of the BG-algebra X (with respect to fuzzy set μ). Then the fuzzy set $\mu_{(\alpha,\beta)}$ defined by $\mu_{(\alpha,\beta)}(x) = max\{(\mu^{\alpha})^{c}(x), \mu_{\beta}(x)\} \quad \forall x \in X$ is called (α, β)-doubt fuzzy set of X (with respect to fuzzy set μ),where $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$

Remark 3.2. (i) $\mu_{(1,0)}(x) = max\{(\mu^{1})^{c}(x), \mu_{0}(x)\} = max\{\mu^{c}(x), 1\} = 1$
(ii) $\mu_{(1,0)}(x) = max\{(\mu^{0})^{c}(x), \mu_{1}(x)\} = max\{1, \mu(x)\} = 1$

Definition 3.3. Given a (α, β)-doubt fuzzy set $\mu_{(\alpha,\beta)}$ and $t \in (0\,1]$, let $\mu_{(\alpha,\beta)_t} = \{x \in X | \mu_{(\alpha,\beta)}(x) \geq t\}$ may be empty set. The set $\mu_{(\alpha,\beta)_t} \neq \phi$ is called the (α, β)-t-confidence set of μ.
Again for (α, β)-doubt fuzzy set $\mu_{(\alpha,\beta)}$ and $t \in (0\,1]$, let $\mu^{t}_{(\alpha,\beta)} = \{x \in X | \mu_{(\alpha,\beta)}(x) \leq t\}$. may be empty set. The set $\mu^{t}_{(\alpha,\beta)} \neq \phi$ is called the (α, β)-t-doubt set of μ.

Remark 3.4. If $t \leq \mu(x) \leq max(\alpha, \beta) \leq 1 - t$ then the set $\mu_{(\alpha,\beta)_t} = \{x \in X | \mu_{(\alpha,\beta)}(x) \geq t\}$ is non empty and if $1 - t \leq \mu(x) \leq 1 - min(\alpha, \beta)$ then the set $\mu^{t}_{(\alpha,\beta)} = \{x \in X | \mu_{(\alpha,\beta)}(x) \leq t\}$ is non empty.

Definition 3.5. Let μ be a (α, β)-doubt fuzzy set of BG-algebra X (with respect to fuzzy set μ),then μ is called (α, β)-doubt fuzzy ideal ((α, β)-DFI) of X if the following condition hold:

$$\begin{aligned} (i)\, \mu_{(\alpha,\beta)}(0) &\leq \mu_{(\alpha,\beta)}(x) \\ (ii)\, \mu_{(\alpha,\beta)}(x) &\leq max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(y)\} \quad \forall\, x, y \in X, \alpha \in [0,1] \end{aligned}$$

Theorem 3.6. *If μ is α-FI and β-DFI of BG-algebra X, then μ is also (α, β)-DFI of X.*

Proof. Since μ is α-FI of X,

$$\begin{aligned} (i)\, \mu^{\alpha}(0) &\geq \mu^{\alpha}(x) \\ (ii)\, \mu^{\alpha}(x) &\geq min\{\mu^{\alpha}(x*y), \mu^{\alpha}(y)\} \quad \forall x, y \in X, \alpha \in [0,1] \end{aligned}$$

Now

$$(i) \qquad \Rightarrow 1 - \mu^{\alpha}(0) \leq 1 - \mu^{\alpha}(x)$$

$$\Rightarrow (\mu^{\alpha})^{c}(0) \leq (\mu^{\alpha})^{c}(x) \tag{3.1}$$

and (3.2)

$$\begin{aligned} (ii) \qquad &\Rightarrow \mu^{\alpha}(x) \geq min\{\mu^{\alpha}(x*y), \mu^{\alpha}(y)\} \\ &\Rightarrow 1 - \mu^{\alpha}(x) \leq 1 - min\{\mu^{\alpha}(x*y), \mu^{\alpha}(y)\} \\ &\Rightarrow 1 - \mu^{\alpha}(x) \leq max\{1 - \mu^{\alpha}(x*y), 1 - \mu^{\alpha}(y)\} \end{aligned}$$

$$\Rightarrow (\mu^{\alpha})^{c}(x) \leq max\{(\mu^{\alpha})^{c}(x*y), (\mu^{\alpha})^{c}(y)\} \tag{3.3}$$

Also since μ is β-DFI of X, therefore

$$(iii)\, \mu_{\beta}(0) \leq \mu_{\beta}(x) \tag{3.4}$$

$$(iv)\, \mu_{\beta}(x) \leq max\{\mu_{\beta}(x*y), \mu_{\beta}(y)\} \quad \forall x, y \in X, \alpha \in [0,1] \tag{3.5}$$

Now

$$\begin{aligned} (v)\, \mu_{(\alpha,\beta)}(0) &= max\{(\mu^{\alpha})^{c}(0), \mu_{\beta}(0)\} \\ &\leq max\{(\mu^{\alpha})^{c}(x), \mu_{\beta}(x)\} \quad \text{By Eqn(3.1)and(3.4)} \\ &= \mu_{(\alpha,\beta)}(x) \\ (vi)\, \mu_{(\alpha,\beta)}(x) &= max\{(\mu^{\alpha})^{c}(x), \mu_{\beta}(x)\} \\ &\leq max\{max\{(\mu^{\alpha})^{c}(x*y), (\mu^{\alpha})^{c}(y)\}, max\{\mu_{\beta}(x*y), \mu_{\beta}(y)\}\} \quad \text{By Eqn(3.3)and(3.5)} \\ &= max\{max\{(\mu^{\alpha})^{c}(x*y), \mu_{\beta}(x*y)\}, max\{(\mu^{\alpha})^{c}(y), \mu_{\beta}(y)\}\} \\ &= max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(y)\} \end{aligned}$$

Hence from (v) and (vi) μ is (α, β)-DFI of X. □

Theorem 3.7. *If μ be a FI of BG-algebra X, then μ is also (α, β)-DFI of X.*

Proof. Since μ is FI of BG-algebra X, therefore by Proposition 2.6 μ is α-FI and β-DFI of X. Hence by above Theorem μ is (α,β)-DFI of X. □

Remark 3.8. The converse of above Theorem need not be true i.e. A fuzzy set μ of a BG-algebra X can be (α,β)-DFI without being FI of X.

Example 3.9. *Consider a BG-algebra $X = \{0,1,2\}$ with the following cayley table: Define a fuzzy set μ by $\mu(0) =$*

Table 2: Example of (α,β)-DFI of BG-algebra X.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

$0.4, \mu(1) = 0.5, \mu(2) = 0.3$ *take* $\alpha = 0.5, \beta = 0.4$. *Now we have* $\mu_{\alpha,\beta}(x) = max\{(\mu^{\alpha})^c(x), \mu_{\beta}(x)\} = max\{1 - (\mu^{\alpha})(x), \mu_{\beta}(x)\} = max\{1 - min(\mu(x),\alpha), max(\mu(x), 1-\beta)\}$, *Therefore*
$\mu_{(0.5,0.4)}(0) = max\{1 - min(\mu(0), 0.5), max(\mu(0), 0.6)\} = max\{0.6, 0.6\} = 0.6$
$\mu_{(0.5,0.4)}(1) = max\{1 - min(\mu(1), 0.5), max(\mu(1), 0.6)\} = max\{0.5, 0.6\} = 0.6$
$\mu_{(0.5,0.4)}(2) = max\{1 - min(\mu(2), 0.5), max(\mu(2), 0.6)\} = max\{0.7, 0.6\} = 0.7$
one can be easily verify that μ is $(0.4, 0.7)$-DFI of X.

Proposition 3.10. *Let μ be a fuzzy subset of BG-algebra X, and let $\alpha \le p$ and $\beta \le 1-q$ where $p = inf\{\mu(x) : \forall x \in X\}$ and $q = sup\{\mu(x) : \forall x \in X\}$ then μ is (α,β)-DFI of X.*

Proof. Here $\alpha \le p$ and $\beta \le 1-q$, therefore $\alpha + \beta \le p + 1 - q \le q + 1 - q = 1$. Also $p = inf\{\mu(x) : \forall x \in X\} \ge \alpha \Rightarrow \mu(x) \ge \alpha \quad \forall x \in X$.
Therefore $\mu^{\alpha}(x) = min\{\mu(x), \alpha\} = \alpha \quad \forall x \in X$.
Similarly we can show $\mu_{\beta}(x) = 1 - \beta, \forall x \in X$.
Now $\mu_{(\alpha,\beta)}(x) = max\{(\mu^{\alpha})^c(x), \mu_{\beta}(x)\} = max\{1-\alpha, 1-\beta\} \quad \forall x \in X$.
Hence both $\mu_{(\alpha,\beta)}(0) \le \mu_{(\alpha,\beta)}(x)$ and $\mu_{(\alpha,\beta)}(x) \le max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(y)\} \quad \forall x,y \in X, \alpha \in [0,1]$ satisfied.
Hence μ is (α,β) -DFI of X. □

Proposition 3.11. *Let μ be an (α,β)-DFI of BG-algebra X, then the following hold:*
(i) If $x \le y$ then $\mu_{(\alpha,\beta)}(x) \le \mu_{(\alpha,\beta)}(y)$
(ii) If $x * y \le z$ then $\mu_{(\alpha,\beta)}(x) \le max\{\mu_{(\alpha,\beta)}(y), \mu_{(\alpha,\beta)}(z)\}$

Proof. (i) We have $x \leq y$, therefore $(x * y) = 0$

$$\begin{aligned} \mu_{(\alpha,\beta)}(x) &\leq max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(y)\} \\ &= max\{\mu_{(\alpha,\beta)}(0), \mu_{(\alpha,\beta)}(y)\} \\ &= max\{\mu_{(\alpha,\beta)}(y) \quad [\text{ Since } \mu_{(\alpha,\beta)}(0) \leq \mu_{(\alpha,\beta)}(y), \quad \forall y \in X.] \end{aligned}$$

(ii) Here $x * y \leq z$ therefore $(x * y) * z = 0$, Now

$$\begin{aligned} \mu_{(\alpha,\beta)}(x*y) &\leq max\{\mu_{(\alpha,\beta)}((x*y)*z), \mu_{(\alpha,\beta)}(z)\} \\ &= max\{\mu_{(\alpha,\beta)}(0), \mu_{(\alpha,\beta)}(z)\} \\ &= max\{\mu_{(\alpha,\beta)}(z) \quad [\text{ Since } \mu_{(\alpha,\beta)}(0) \leq \mu_{(\alpha,\beta)}(x), \quad \forall x \in X.] \\ \text{Now} \quad \mu_{(\alpha,\beta)}(x) &\leq max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(y)\} \\ &= max\{\mu_{(\alpha,\beta)}(z), \mu_{(\alpha,\beta)}(y)\} \end{aligned}$$

□

Theorem 3.12. *If μ is (α,β)-DFI of BG-algebra X, then the set $X_\mu = \{x \in X \mid \mu_{(\alpha,\beta)}(x) = \mu_{(\alpha,\beta)}(0)\}$ is an ideal.*

Proof. Clearly $0 \in X_\mu$. Let $x * y, y \in X_\mu$
$\Rightarrow \mu_{(\alpha,\beta)}(x*y) = \mu_{(\alpha,\beta)}(y) = \mu_{(\alpha,\beta)}(0)$
Now

$$\begin{aligned} \mu_{(\alpha,\beta)}(x) &\leq max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(y)\} \\ &= max\{\mu_{(\alpha,\beta)}(0), \mu_{(\alpha,\beta)}(0)\} \\ &= \mu_{(\alpha,\beta)}(0) \\ \Rightarrow \mu_{(\alpha,\beta)}(x) &\leq \mu_{(\alpha,\beta)}(0) \\ \text{Also } \mu_{(\alpha,\beta)}(0) &\leq \mu_{(\alpha,\beta)}(x) \\ \text{Therefore, } \mu_{(\alpha,\beta)}(x) &= \mu_{(\alpha,\beta)}(0) \Rightarrow x \in X_\mu \end{aligned}$$

Therefore $x * y, y \in X_\mu \Rightarrow x \in X_\mu$. Therefore X_μ is an ideal. □

Definition 3.13. *Let μ be a (α,β)-doubt fuzzy set of BG-algebra X (with respect to fuzzy set μ),then μ is called (α,β)-fuzzy ideal ((α,β)-FI) of X if the following condition holds:*

$$\begin{aligned} (i)\mu_{(\alpha,\beta)}(0) &\geq \mu_{(\alpha,\beta)}(x) \\ (ii)\mu_{(\alpha,\beta)}(x) &\geq min\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(y)\} \quad \forall x,y \in X, \alpha \in [0,1] \end{aligned}$$

Theorem 3.14. *If μ is (α,β)-DFI of BG-algebra X, iff its complement $\mu^c_{\alpha,\beta}$ is fuzzy ideal.*

Proof. Since μ is (α,β)-DFI of X, therefore

$$\begin{aligned} (i)\, \mu_{(\alpha,\beta)}(0) &\leq \mu_{(\alpha,\beta)}(x) \\ (ii)\, \mu_{(\alpha,\beta)}(x) &\leq max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(y)\} \quad \forall x,y \in X, \alpha \in [0,1] \end{aligned}$$

Now (i) Implies

$$\mu_{(\alpha,\beta)}(0) \le \mu_{(\alpha,\beta)}(x)$$
$$\Rightarrow 1-\mu_{(\alpha,\beta)}(0) \ge 1-\mu_{(\alpha,\beta)}(x)$$
$$\Rightarrow \mu^c_{(\alpha,\beta)}(0) \ge \mu^c_{(\alpha,\beta)}(x)$$

Now (ii) Implies

$$\mu_{(\alpha,\beta)}(x) \le max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(y)\}$$
$$\Rightarrow 1-\mu_{(\alpha,\beta)}(x) \ge 1-max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(y)\}$$
$$\Rightarrow 1-\mu_{(\alpha,\beta)}(x) \ge min\{1-\mu_{(\alpha,\beta)}(x*y), 1-\mu_{(\alpha,\beta)}(y)\}$$
$$\Rightarrow \mu^c_{(\alpha,\beta)}(x) \ge min\{\mu^c_{(\alpha,\beta)}(x*y), \mu^c_{(\alpha,\beta)}(y)\}$$

□

Remark 3.15. Note that $\mu^c_{(\alpha,\beta)}(x) = min\{\mu_\alpha(x), \mu^c_\beta(x)\}$

Theorem 3.16. *If μ and ν are two (α,β)-DFIs of BG-algebra X, then $\mu\cup\nu$ is also (α,β)-DFI of X.*

Proof. Let $x, y \in X$. Now we have

$$\begin{aligned}
(i)\quad (\mu\cup\nu)_{(\alpha,\beta)}(0) &= max\{\mu_{(\alpha,\beta)}(0), \nu_{(\alpha,\beta)}(0)\}\\
&\le max\{\mu_{(\alpha,\beta)}(x), \nu_{(\alpha,\beta)}(x)\}\\
&= (\mu\cup\nu)_{(\alpha,\beta)}(x)\\
\Rightarrow\quad (\mu\cup\nu)_{(\alpha,\beta)}(0) &\le (\mu\cup\nu)_{(\alpha,\beta)}(x)\\
(ii)\quad (\mu\cup\nu)_{(\alpha,\beta)}(x) &= max\{\mu_{(\alpha,\beta)}(x), \nu_{(\alpha,\beta)}(x)\}\\
&\le max\{max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(x)\}, max\{\nu_{(\alpha,\beta)}(x*y), \nu_{(\alpha,\beta)}(x)\}\}\\
&= max\{max\{\mu_{(\alpha,\beta)}(x*y), \nu_{(\alpha,\beta)}(x*y)\}, max\{\mu_{(\alpha,\beta)}(y), \nu_{(\alpha,\beta)}(y)\}\}\\
&= max\{(\mu\cup\nu)_{(\alpha,\beta)}(x*y), (\mu\cup\nu)_{(\alpha,\beta)}(y)\}\\
\Rightarrow\quad (\mu\cup\nu)_{(\alpha,\beta)}(x) &\le max\{(\mu\cup\nu)_{(\alpha,\beta)}(x*y), (\mu\cup\nu)_{(\alpha,\beta)}(y)\}
\end{aligned}$$

Hence $\mu\cup\nu$ is (α,β)-DFI of X. □

Remark 3.17. Intersection of two (α,β)-DFIs of BG-algebra X need not be (α,β)-DFI of X.

Proof. Consider BG algebra X as in Example 3.9, Define two fuzzy sets μ and ν by $\mu(0)=0.3, \mu(1)=0.2, \mu(2)=0.7$ and $\nu(0)=0.6, \nu(1)=0.1, \nu(2)=0.5$ take $\alpha=0.4, \beta=0.7$. Now $\mu_{\alpha,\beta}(x) = max\{(\mu^\alpha)^c(x), \mu_\beta(x)\}$
$= max\{1-(\mu^\alpha)(x), \mu_\beta(x)\} = max\{1-min(\mu(x),\alpha), max(\mu(x), 1-\beta)\}$
$= max\{1-min(\mu(x), 0.4), max(\mu(x), 0.3\}$, therefore
$\mu_{(0.4,0.7)}(0) = max\{1-min(\mu(0), 0.4), max(\mu(0), 0.3)\} = max\{0.7, 0.3\} = 0.7$
$\mu_{(0.4,0.7)}(1) = max\{1-min(\mu(1), 0.4), max(\mu(1), 0.3)\} = max\{0.8, 0.3\} = 0.8$

$\mu_{(0.4,0.7)}(2) = max\{1 - min(\mu(2), 0.4), max(\mu(2), 0.3)\} = max\{0.6, 0.7\} = 0.7$
$\nu_{(0.4,0.7)}(0) = max\{1 - min(\nu(0), 0.4), max(\nu(0), 0.3)\} = max\{0.6, 0.6\} = 0.6$
$\nu_{(0.4,0.7)}(1) = max\{1 - min(\nu(1), 0.4), max(\nu(1), 0.3)\} = max\{0.9, 0.3\} = 0.9$
$\nu_{(0.4,0.7)}(2) = max\{1 - min(\nu(2), 0.4), max(\nu(2), 0.3)\} = max\{0.6, 0.5\} = 0.6$
It can be easily verify that both μ and ν are (0.4, 0.7)-DFI of X. But $(\mu \cap \nu)(x) = min\{\mu(x), \nu(x)\}$ Therefore $(\mu \cap \nu)(0) = 0.3, (\mu \cap \nu)(1) = 0.1, (\mu \cap \nu)(2) = 0.$
$(\mu \cap \nu)_{(0.4,0.7)}(0) = max\{1 - min((\mu \cap \nu)(0), 0.4), max((\mu \cap \nu)(0), 0.3)\} = max\{0.7, 0.3\} = 0.7$
$(\mu \cap \nu)_{(0.4,0.7)}(1) = max\{1 - min((\mu \cap \nu)(1), 0.4), max((\mu \cap \nu)(1), 0.3)\} = max\{0.9, 0.3\} = 0.9$
$(\mu \cap \nu)_{(0.4,0.7)}(2) = max\{1 - min((\mu \cap \nu)(2), 0.4), max((\mu \cap \nu)(2), 0.3)\} = max\{0.6, 0.5\} = 0.6$
Since $(\mu \cap \nu)_{(0.4,0.7)}(0) = 0.7 \nleq (\mu \cap \nu)_{(0.4,0.7)}(2) = 0.6$
Therefore $(\mu \cap \nu)$ is not $(0.4, 0.7)$-DFI of X. □

Theorem 3.18. *A fuzzy subset μ of a BG-algebra X is a (α, β)-DFI iff for every $t \in (0\,1]$,$\mu^t_{(\alpha,\beta)}$ is an ideal of X, when $\mu^t_{(\alpha,\beta)} \neq \phi$*

Proof. Assume μ is a (α, β)-DFI. Therefore we have $\mu_{(\alpha,\beta)}(0) \leq \mu_{(\alpha,\beta)}(x)$ and $\mu_{(\alpha,\beta)}(x) \leq max\{\mu_{(\alpha,\beta)}(x * y), \mu_{(\alpha,\beta)}(y)\} \quad \forall x, y \in X, \alpha \in [0, 1]$. Let $x \in \mu^t_{(\alpha,\beta)}$ implies $\mu_{(\alpha,\beta)}(x) \leq t$. Now $\mu_{(\alpha,\beta)}(0) \leq \mu_{(\alpha,\beta)}(x) \leq t$ which implis $\mu_{(\alpha,\beta)}(0) \leq t$ and therefore $0 \in \mu^t_{(\alpha,\beta)}$ again let $x * y, y \in \mu^t_{(\alpha,\beta)}$ implies $\mu_{(\alpha,\beta)}(x * y) \leq t$ and $\mu_{(\alpha,\beta)}(y) \leq t$ Now $\mu_{(\alpha,\beta)}(x) \leq max\{\mu_{(\alpha,\beta)}(x * y), \mu_{(\alpha,\beta)}(y)\}$ implies $\mu_{(\alpha,\beta)}(x) \leq t$ which implies $x \in \mu^t_{(\alpha,\beta)}$. Hence $\mu^t_{(\alpha,\beta)}$ is an ideal of X.
Conversely,
Assume $\mu^t_{(\alpha,\beta)}$ is an ideal of X, to prove μ is (α, β) -DFI, if μ is not a (α, β) -DFI, then at least one of $\mu_{(\alpha,\beta)}(0) > \mu_{(\alpha,\beta)}(x)$ and $\mu_{(\alpha,\beta)}(x) > max\{\mu_{(\alpha,\beta)}(x * y), \mu_{(\alpha,\beta)}(y)\}$ must hold for at least some $x, y \in X$. Suppose $\mu_{(\alpha,\beta)}(0) > \mu_{(\alpha,\beta)}(x)$ holds for $x = x'$. then choose $t = \{\mu_{(\alpha,\beta)}(0) + \mu_{(\alpha,\beta)}(x')\}/2 \in (0, 1]$ then

$$\mu_{(\alpha,\beta)}(0) > t > \mu_{(\alpha,\beta)}(x') \tag{3.6}$$

Now (3.6)$\Rightarrow \mu_{(\alpha,\beta)}(x') < t$ which implis $x' \in \mu^t_{(\alpha,\beta)}$ Since $\mu^t_{(\alpha,\beta)}$ is an ideal of X, therefore $0 \in \mu^t_{(\alpha,\beta)} \Rightarrow \mu_{(\alpha,\beta)}(0) < t$ which contradicts (3.6). Again if $\mu_{(\alpha,\beta)}(x) > max\{\mu_{(\alpha,\beta)}(x * y), \mu_{(\alpha,\beta)}(y)\}$ for some x', y' then choose
$t = \{\mu_{(\alpha,\beta)}(x') + max\{\mu_{(\alpha,\beta)}(x' * y'), \mu_{(\alpha,\beta)}(y')\}\}/2 \in (0, 1]$ then

$$\mu_{(\alpha,\beta)}(x') > t > max\{\mu_{(\alpha,\beta)}(x' * y'), \mu_{(\alpha,\beta)}(y')\} \tag{3.7}$$

Now (3.7)$\Rightarrow \mu_{(\alpha,\beta)}(x' * y'), \mu_{(\alpha,\beta)}(y') < t$ which implis $x' * y', y' \in \mu^t_{(\alpha,\beta)}$ Since $\mu^t_{(\alpha,\beta)}$ is an ideal of X. Therefore $x' \in \mu^t_{(\alpha,\beta)} \Rightarrow \mu_{(\alpha,\beta)}(x') < t$ which contradicts (3.7). Hence μ is a (α, β)-DFI of X. □

Definition 3.19. Let f be a mapping defined on a set X. If μ is a fuzzy set in X then the fuzzy set ν in $f(X)$ defined by

$$\nu(y) = \inf_{x \in f^{-1}(y)} \mu(x)$$

for all $y \in f(X)$ is called the doubt image of μ under f. If ν is a fuzzy set in $f(X)$ then the fuzzy set $\mu = \nu o f$ in X (i.e. the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in X$) is called pre image of ν undef f.

4 Homomorphism of (α, β)-Doubt Fuzzy Ideals

In this section, we investigate the image and the preimage of (α, β)-DFI of a BG-algebra under homomorphism.

Lemma 4.1. *[15, 16] Let $f : X \longrightarrow Y$ be a mapping and μ, ν be two fuzzy subsets of X and Y repectively, then*
$(i) f^{-1}(\nu_{(\alpha,\beta)}) = (f^{-1}(\nu))_{(\alpha,\beta)}$
$(ii) f(\mu_{(\alpha,\beta)}) = (f(\mu))_{(\alpha,\beta)}$

Definition 4.2. Let X and Y be two BG-algebras, then a mapping $f : X \longrightarrow Y$ is said to be homomorphism if $f(x * y) = f(x) * f(y),\ \forall x, y \in X$.

Theorem 4.3. *If $f : X \longrightarrow Y$ be a homomorphism of BG-algebras, then $f(0) = 0$.*

Proof. we have $f(0) = f(x * x) = f(x) * f(x) = 0$.

□

Theorem 4.4. *Let $f : X \longrightarrow Y$ be an onto homomorphism of BG-algebras, If ν be an (α, β)-DFI of Y, then the pre image of ν under f is also a DFI of X.*

Proof. Let μ be the pre image of ν under f, then $\mu(x) = \nu(f(x))$ for all $x \in X$. Since ν is an (α, β)-DFI of Y, therefore $\nu_{(\alpha,\beta)}(f(0)) \leq \nu_{(\alpha,\beta)}(f(x)) = \mu_{(\alpha,\beta)}(x)$. Also $\nu_{(\alpha,\beta)}(f(0)) = \nu_{(\alpha,\beta)}(f(0)) = \mu_{(\alpha,\beta)}(0)$ and hence $\mu_{(\alpha,\beta)}(0) \leq \mu_{(\alpha,\beta)}(x)$ for all $x \in X$, again

$$\begin{aligned}
\mu_{(\alpha,\beta)}(x) &= f^{-1}(\nu_{(\alpha,\beta)})(x) \\
&= \nu_{(\alpha,\beta)}(f(x)) \\
&\leq max\{\nu_{(\alpha,\beta)}((f(x) * f(y)), \nu_{(\alpha,\beta)}(f(y)\} \\
&= max\{\nu_{(\alpha,\beta)}(f(x * y)), \nu_{(\alpha,\beta)}(f(y)\} \\
&= max\{f^{-1}(\nu_{(\alpha,\beta)})(x * y), f^{-1}(\nu_{(\alpha,\beta)})(y)\} \\
&= max\{(f^{-1}(\nu))_{(\alpha,\beta)}(x * y), (f^{-1}(\nu))_{(\alpha,\beta)}(y)\} \\
&= max\{\mu_{(\alpha,\beta)}(x * y), \mu_{(\alpha,\beta)}(y)\} \\
\text{Therefore} \quad \mu_{(\alpha,\beta)}(x) &\leq max\{\mu_{(\alpha,\beta)}(x * y), \mu_{(\alpha,\beta)}(y)\}
\end{aligned}$$

Since $f(y) \in Y$ is arbitrary and f is onto, therefore $y \in X$ is also, therefore
$\mu_{(\alpha,\beta)}(x) \leq max\{\mu_{(\alpha,\beta)}(x * y), \mu_{(\alpha,\beta)}(y)\}$ is true for all $x, y \in X$.
Hence $\mu = f^{-1}(\nu)$ is (α, β)-DFI of X. □

Definition 4.5. A fuzzy set μ of X has inf property if for any subset T of X, there exists $t_0 \in T$ such that

$$\mu(t_0) = \inf_{t \in T} \mu(t)$$

Theorem 4.6. *Let $f : X \longrightarrow Y$ be a onto homomorphism of BG-algebras, If μ be an (α, β)-DFI of X with inf property, then $f(\mu)$ is an (α, β)-DFI of Y.*

Proof. Let μ be an (α, β)-DFI of X with inf property and $y \in Y$ Since f is onto, therefore there exists $x \in X$ such that $y = f(x)$, now

$$\begin{aligned}
(f(\mu))_{(\alpha,\beta)}(f(0)) &= \inf_{t \in f^{-1}(0')} \mu_{(\alpha,\beta)}(t) \\
&= \mu_{(\alpha,\beta)}(0) \\
&\leq \mu_{(\alpha,\beta)}(x) \\
&= \inf_{t \in f^{-1}(x)} \mu_{(\alpha,\beta)}(t) \\
&= f(\mu_{(\alpha,\beta)})(f(x)) \\
\Rightarrow (f(\mu))_{(\alpha,\beta)}(f(0)) &\leq f(\mu_{(\alpha,\beta)})(f(x))
\end{aligned}$$

Again let $x', y' \in Y$, let $x_0 \in f^{-1}(x'), y_0 \in f^{-1}(y')$ be such that

$$\mu_{(\alpha,\beta)})(x_0) = \inf_{t \in f^{-1}(x')} \mu_{(\alpha,\beta)})(t), \quad \mu_{(\alpha,\beta)})(y_0) = \inf_{t \in f^{-1}(y')} \mu_{(\alpha,\beta)})(t)$$
$$\mu_{(\alpha,\beta)})(x_0 * y_0) = \inf_{t \in f^{-1}(x' * y')} \mu_{(\alpha,\beta)})(t)$$

Now

$$\begin{aligned}
(f(\mu))_{(\alpha,\beta)}(x') &= \inf_{t \in f^{-1}(x')} \mu_{(\alpha,\beta)})(t) \\
&= \mu_{(\alpha,\beta)})(x_0) \\
&\leq max\{\mu_{(\alpha,\beta)})(x_0 * y_0), \mu_{(\alpha,\beta)})(y_0)\} \\
&= max\{\inf_{t \in f^{-1}(x' * y')} \mu_{(\alpha,\beta)})(t), \inf_{t \in f^{-1}(y')} \mu_{(\alpha,\beta)})(t)\} \\
&= max\{(f(\mu_{(\alpha,\beta)}))(x' * y'), \mu_{(\alpha,\beta)}(y')\}
\end{aligned}$$

Hence $f(\mu)$ is an (α, β)-DFI of Y. □

5 Doubt Cartesian Product of (α, β)-Doubt Fuzzy Ideals

Definition 5.1. Let μ and ν be two (α, β)-doubt fuzzy sets in a set X. Then their doubt cartesian product $\mu \times \nu : X \times X \longrightarrow [0, 1]$ is defined by $(\mu \times \nu)(x y) = max\{\mu(x), \nu(x)\}$ for all $x, y \in X$.

Theorem 5.2. *If μ and ν are two (α, β)-doubt fuzzy ideals of a BG-algebra X, then $\mu \times \nu$ is also an (α, β)-doubt fuzzy ideals of $X \times X$.*

Proof. Let $x, y \in X$. Now we have

$$
\begin{aligned}
(i)\quad (\mu\times\nu)_{(\alpha,\beta)}(0) &= max\{\mu_{(\alpha,\beta)}(0), \nu_{(\alpha,\beta)}(0)\}\\
&\le max\{\mu_{(\alpha,\beta)}(x), \nu_{(\alpha,\beta)}(x)\}\\
&= (\mu\times\nu)_{(\alpha,\beta)}(x)\\
\Rightarrow\quad (\mu\times\nu)_{(\alpha,\beta)}(0) &\le (\mu\times\nu)_{(\alpha,\beta)}(x)\\
(ii)\quad (\mu\times\nu)_{(\alpha,\beta)}(x) &= max\{\mu_{(\alpha,\beta)}(x), \nu_{(\alpha,\beta)}(x)\}\\
&\le max\{max\{\mu_{(\alpha,\beta)}(x*y), \mu_{(\alpha,\beta)}(x)\}, max\{\nu_{(\alpha,\beta)}(x*y), \nu_{(\alpha,\beta)}(x)\}\}\\
&= max\{max\{\mu_{(\alpha,\beta)}(x*y), \nu_{(\alpha,\beta)}(x*y)\}, max\{\mu_{(\alpha,\beta)}(y), \nu_{(\alpha,\beta)}(y)\}\}\\
&= max\{(\mu\times\nu)_{(\alpha,\beta)}(x*y), (\mu\times\nu)_{(\alpha,\beta)}(y)\}\\
\Rightarrow\quad (\mu\times\nu)_{(\alpha,\beta)}(x) &\le max\{(\mu\times\nu)_{(\alpha,\beta)}(x*y), (\mu\times\nu)_{(\alpha,\beta)}(y)\}
\end{aligned}
$$

Hence $(\mu\times\nu)$ is (α,β)-DFI of $X\times X$. □

Theorem 5.3. *If μ and ν be two (α,β)-doubt fuzzy subsets in a BG-algebra X such that $\mu\times\nu$ is a (α,β)-doubt fuzzy ideals of $X\times X$, then*
(i) either $\mu_{(\alpha,\beta)}(x) \ge \mu_{(\alpha,\beta)}(0)$ or $\nu_{(\alpha,\beta)}(x) \ge \nu_{(\alpha,\beta)}(0)$ for all $x\in X$.
(ii) if $\mu_{(\alpha,\beta)}(x) \ge \mu_{(\alpha,\beta)}(0)$ for all $x\in X$, then either $\mu_{(\alpha,\beta)}(x) \ge \nu_{(\alpha,\beta)}(0)$ or $\nu_{(\alpha,\beta)}(x) \ge \nu_{(\alpha,\beta)}(0)$.
(iii) if $\nu_{(\alpha,\beta)}(x) \ge \nu_{(\alpha,\beta)}(0)$ for all $x\in X$, then either $\mu_{(\alpha,\beta)}(x) \ge \mu_{(\alpha,\beta)}(0)$ or $\nu_{(\alpha,\beta)}(x) \ge \mu_{(\alpha,\beta)}(0)$

Proof. (i) Suppose that $\mu_{(\alpha,\beta)}(x) < \mu_{(\alpha,\beta)}(0)$ and $\nu_{(\alpha,\beta)}(x) < \nu_{(\alpha,\beta)}(0)$ for some $x, y\in X$ Then

$$
\begin{aligned}
(\mu\times\nu)_{(\alpha,\beta)}(0) &= max\{\mu_{(\alpha,\beta)}(0), \nu_{(\alpha,\beta)}(0)\}\\
&\ge max\{\mu_{(\alpha,\beta)}(x), \nu_{(\alpha,\beta)}(x)\}\\
&= (\mu\times\nu)_{(\alpha,\beta)}(x)
\end{aligned}
$$

Which contradicts the fact that $\mu\times\nu$ is a (α,β) -doubt fuzzy ideals of $X\times X$.
(ii) Assume that there exist $x, y\in X$ such that $\mu_{(\alpha,\beta)}(x) < \nu_{(\alpha,\beta)}(0)$ and $\nu_{(\alpha,\beta)}(x) < \nu_{(\alpha,\beta)}(0)$.
. Then

$$
\begin{aligned}
(\mu\times\nu)_{(\alpha,\beta)}(0) &= max\{\mu_{(\alpha,\beta)}(0), \nu_{(\alpha,\beta)}(0)\} = \nu_{(\alpha,\beta)}(0)\\
(\mu\times\nu)_{(\alpha,\beta)}(x) &= max\{\mu_{(\alpha,\beta)}(x), \nu_{(\alpha,\beta)}(x)\}\\
&< max\{\nu_{(\alpha,\beta)}(0), \nu_{(\alpha,\beta)}(0)\}\\
&= \nu_{(\alpha,\beta)}(0) = (\mu\times\nu)_{(\alpha,\beta)}(0)\\
\Rightarrow (\mu\times\nu)_{(\alpha,\beta)}(x) &< (\mu\times\nu)_{(\alpha,\beta)}(0)
\end{aligned}
$$

Which contradicts the fact that $\mu\times\nu$ is a (α,β) -doubt fuzzy ideals of $X\times X$. Hence (ii) holds.
(iii) Similar to proof of(ii)

□

Theorem 5.4. *If μ and ν be two (α, β)-doubt fuzzy subsets in a BG-algebra X such that $\mu \times \nu$ is a (α, β)-doubt fuzzy ideals of $X \times X$, then either μ or ν is an (α, β)-doubt fuzzy ideal of X.*

Proof. Straightforward. □

6 Conclusion

In this article, we introduced the notion of (α, β)-doubt fuzzy ideals in BG-algebra. The same idea we can apply to other algebric systems (BCK/BCI/BF/BH/BCH-algebras) also. In [4] Huang defined fuzzy ideal of BCK-algebras in another way, now this ideal some author called as doubt fuzzy ideal and some other called as anti fuzzy ideal i.e., doubt fuzzy and anti fuzzy are same meaning. But the (α, β)-doubt fuzzy ideals is little different from doubt fuzzy ideal. The most of the properties of (α, β)-doubt fuzzy ideals and doubt fuzzy ideals are same (see [3, 8]).

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