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Some new integral inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions

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ABSTRACT. In the present paper, a new class of generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula are given. Moreover, by using new identity via classical integrals some Hermite-Hadamard, Simpson and midpoint type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind are established.

1 Introduction

The following notation are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

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Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The following inequality is well known in the literature as Simpson's inequality:

Theorem 1.2. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be four time differentiable on (a, b) and having the fourth derivative bounded on (a, b), that is $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} |f^{(4)}| < \infty$. Then, we have*

$$\left| \int_a^b f(t)dt - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right| \leq \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^5. \tag{1.2}$$

Inequality (1.2) gives an error bound for the classical Simpson quadrature formula, which is one of the most used quadrature formulae in practical applications.

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [7]-[9],[13]). For other recent results concerning Simpson type inequalities (see [12],[20]).
Now, let us evoke some definitions.

Definition 1.3. (see [2]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be P-function or P-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \ t \in [0, 1].$$

Definition 1.4. (see [6]) A function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be a Godunova-Levin function or $f \in Q(I)$, if f is nonnegative and for all $x, y \in I$, $t \in (0, 1)$, we have that

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

The class $Q(I)$ was firstly described in (see [6]) by Godunova and Levin. Some further properties of it are given in (see [2],[14],[15]). Among others, it is noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

Definition 1.5. (see [17]) A function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be (s, m)-Godunova-Levin functions of first kind or $f \in Q^1_{(s,m)}$, if $\forall s, m \in (0, 1]$, we have

$$f(tx + m(1-t)y) \leq \frac{f(x)}{t^s} + \frac{mf(y)}{1-t^s}, \quad \forall x, y \in I, \ t \in (0, 1).$$

We would like to mention that Definition 1.5 is also introduced and studied by Li et al. (see [10]) independently. For $m = 1$ in Definition 1.5 we have the definition of s-Godunova-Levin functions of first kind, which is introduced and investigated by Noor et al. (see [16]).

Definition 1.6. (see [17]) A function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be (s, m)-Godunova-Levin functions of the second kind or $f \in Q^2_{(s,m)}$, if $s \in [0, 1]$, $m \in (0, 1]$, we have

$$f(tx + m(1-t)y) \leq \frac{f(x)}{t^s} + \frac{mf(y)}{(1-t)^s}, \quad \forall x, y \in I, \ t \in (0, 1).$$

It is obvious that for $s = 0, m = 1$, (s, m)-Godunova-Levin functions of the second kind reduces to Definition 1.3 of P-functions. If $s = 1, m = 1$, it then reduces to Godunova-Levin functions. For $m = 1$, we have the definition of s-Godunova-Levin function of the second kind introduced and studied by Dragomir (see [3],[4]).

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{1.3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$ (see [21]).

Recently, Liu (see [11]) obtained several integral inequalities for the left hand side of (1.3) under the Definition 1.3 of P-function. Also in (see [18]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.3) via some kinds of convexity.

Motivated by these results, the aim of this paper is to establish left-hand side type inequalities of (1.3) and to generalized Hermite-Hadamard, Simpson and midpoint type inequalities using new identity for classical integrals given in Section 3. The paper is organized as follows: In Section 2, a new class of generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind is introduced and some new integral inequalities for the left-hand side of (1.3) involving generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind along with beta function are given. In Section 3, some new Hermite-Hadamard, Simpson and midpoint type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind via classical integrals are given. In Section 4, some conclusions and future research are given. These results obtained provide new estimates on generalizations of Hermite-Hadamard, Simpson and midpoint type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind.

2 New integral inequalities

Definition 2.1. (see [1]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true (see [1],[22]).

Definition 2.2. (see [19]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0, 1]$, we have that

$$f(x + t\eta(y, x)) \leq (1 - t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

Definition 2.3. (see [5]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 2.4. In Definition 2.3, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next give new definition, to be referred as generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind.

Definition 2.5. Let $\varphi : I \longrightarrow K$ be a continuous function and $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. For $f : K \longrightarrow \mathbb{R}$ and some fixed $s \in [0,1]$, $m \in (0,1]$, if

$$f(m\varphi(y) + t\eta(\varphi(x), \varphi(y), m)) \leq \frac{f(\varphi(x))}{t^s} + \frac{mf(\varphi(y))}{(1-t)^s}, \tag{2.1}$$

is valid for all $x, y \in I$, $t \in (0,1)$, then we say that f is a generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind with respect to η. The set of these functions is denoted by $Q^{\star 2}_{(s,m,\varphi)}$.

Remark 2.6. In Definition 2.5, it is worthwhile to note that generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind is an (s, m)-Godunova-Levin functions of the second kind on $K = I$ with respect to $\eta(\varphi(x), \varphi(y), m) = \varphi(x) - m\varphi(y)$, $\varphi(x) = x$, $\forall x \in I$.

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard, Simpson, midpoint type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind along with beta function, we need the following new interesting lemma:

Lemma 2.7. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that a function $f : K = [m\varphi(a), m\varphi(a) + \xi_1] \longrightarrow \mathbb{R}$ is continuous with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. Then for some fixed $m \in (0,1]$ and any fixed $p, q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a) + \xi_1 - x)^q f(x)dx = \xi_1^{p+q+1} \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\xi_1)dt, \tag{2.2}$$

where

$$\xi_1 = \eta(\varphi(b), \varphi(a), m).$$

Proof. It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a) + \xi_1 - x)^q f(x)dx$$

$$= \xi_1 \int_0^1 (m\varphi(a) + t\xi_1 - m\varphi(a))^p (m\varphi(a) + \xi_1 - m\varphi(a) - t\xi_1)^q f(m\varphi(a) + t\xi_1)dt$$

$$= \xi_1^{p+q+1} \int_0^1 t^p (1-t)^q f(m\varphi(a) + t\xi_1)dt.$$

□

The following definition will be used in the sequel.

Definition 2.8. The Euler beta function is defined for $x, y > 0$ as

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.9. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that a function $f : K = [m\varphi(a), m\varphi(a)+\xi_1] \longrightarrow \mathbb{R}$ is continuous on K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $\xi_1 > 0$. If $k > 1$ and $|f|^{\frac{k}{k-1}}$ is generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on K for some fixed $m \in (0,1]$, $s \in [0,1)$, then for any fixed $p,q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a)+\xi_1 - x)^q f(x)dx$$

$$\leq \frac{\xi_1^{p+q+1}}{(1-s)^{\frac{k-1}{k}}} \Big[\beta(kp+1, kq+1)\Big]^{\frac{1}{k}} \left(m|f(\varphi(a))|^{\frac{k}{k-1}} + |f(\varphi(b))|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}. \tag{2.3}$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on K, combining with Lemma 2.7, Hölder inequality and property of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a)+\xi_1 - x)^q f(x)dx$$

$$\leq |\xi_1|^{p+q+1} \left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}} \left[\int_0^1 \left|f(m\varphi(a)+t\xi_1)\right|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}$$

$$\leq \xi_1^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}} \left[\int_0^1 \left(\frac{m|f(\varphi(a))|^{\frac{k}{k-1}}}{(1-t)^s} + \frac{|f(\varphi(b))|^{\frac{k}{k-1}}}{t^s}\right)dt\right]^{\frac{k-1}{k}}$$

$$= \frac{|\xi_1|^{p+q+1}}{(1-s)^{\frac{k-1}{k}}} \Big[\beta(kp+1, kq+1)\Big]^{\frac{1}{k}} \left(m|f(\varphi(a))|^{\frac{k}{k-1}} + |f(\varphi(b))|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

The proof of Theorem 2.9 is completed. □

Theorem 2.10. *Let $\varphi : I \longrightarrow K$ be a continuous function. Assume that a function $f : K = [m\varphi(a), m\varphi(a)+\xi_1] \longrightarrow \mathbb{R}$ is continuous on the interval of real numbers K° with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ and $\xi_1 > 0$. If $l \geq 1$ and $|f|^l$ is generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on K for some fixed $m \in (0,1]$, $s \in [0,1]$, then for any fixed $p,q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a)+\xi_1 - x)^q f(x)dx$$

$$\leq \xi_1^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}} \tag{2.4}$$

$$\times\Big[m|f(\varphi(a))|^l \beta\left(p+1, q-s+1\right) + |f(\varphi(b))|^l \beta\left(p-s+1, q+1\right)\Big]^{\frac{1}{l}}.$$

Proof. Since $|f|^l$ is generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind on K, combining with Lemma 2.7 and the well-known power mean inequality and property of the modulus, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\xi_1} (x - m\varphi(a))^p (m\varphi(a)+\xi_1 - x)^q f(x)dx$$

$$= \xi_1^{p+q+1}\int_0^1 \Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}} \Big[t^p(1-t)^q\Big]^{\frac{1}{l}} f(m\varphi(a)+t\xi_1)dt$$

$$\leq |\xi_1|^{p+q+1}\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}} \left[\int_0^1 t^p(1-t)^q \left|f(m\varphi(a)+t\xi_1)\right|^l dt\right]^{\frac{1}{l}}$$

$$\leq \xi_1^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}} \left[\int_0^1 t^p(1-t)^q \left(\frac{m|f(\varphi(a))|^l}{(1-t)^s} + \frac{|f(\varphi(b))|^l}{t^s}\right)dt\right]^{\frac{1}{l}}$$

$$= \xi_1^{p+q+1}\left[\beta(p+1,q+1)\right]^{\frac{l-1}{l}}$$
$$\times\left[m|f(\varphi(a))|^l\beta\left(p+1,q-s+1\right)+|f(\varphi(b))|^l\beta\left(p-s+1,q+1\right)\right]^{\frac{1}{l}}.$$

The proof of Theorem 2.10 is completed. □

3 Some new integral inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions

In this section, in order to prove our main results regarding some new generalization of Hermite-Hadamard, Simpson and midpoint type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind via classical integrals, we need the following new interesting lemma:

Lemma 3.1. *Let $\varphi : I \longrightarrow K$ be a continuous function. Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and $\eta_1 > 0$. Assume that $f : K = [m\varphi(b), m\varphi(b) + \eta_1] \longrightarrow \mathbb{R}$ is a differentiable function on K° and f' is integrable on K. Then for all $\lambda, \mu \in [0,1]$, we have*

$$\lambda\left(\mu f(m\varphi(b)+\eta_1)+(1-\mu)f(m\varphi(b))\right)+(1-\lambda)f(m\varphi(b)+\mu\eta_1)-\frac{1}{\eta_1}\int_{m\varphi(b)}^{m\varphi(b)+\eta_1}f(x)dx$$
$$= -\eta_1\left[\int_0^{\mu}(-t+\lambda(1-\mu))f'(m\varphi(b)+t\eta_1)dt+\int_{\mu}^{1}(-t+(1-\mu\lambda))f'(m\varphi(b)+t\eta_1)dt\right], \tag{3.1}$$

where

$$\eta_1 = \eta(\varphi(a), \varphi(b), m).$$

Proof. A simple proof of the equality can be done by performing an integration by parts in the integrals and changing the variable. The details are left to the interested reader. □

Let denote

$$S_{f,\eta_1}(\lambda,\mu,m) = -\eta_1\left[\int_0^{\mu}(-t+\lambda(1-\mu))f'(m\varphi(b)+t\eta_1)dt+\int_{\mu}^{1}(-t+(1-\mu\lambda))f'(m\varphi(b)+t\eta_1)dt\right]. \tag{3.2}$$

Using Lemma 3.1 and the relation (3.2), the following results can be obtained for the corresponding version for power of the absolute value of the first derivative.

Theorem 3.2. *Let $\varphi : I \longrightarrow A$ be a continuous function. Let $A \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $s \in [0,1), m \in (0,1]$ and $\eta_1 > 0$. Assume that $f : A = [m\varphi(b), m\varphi(b) + \eta_1] \longrightarrow \mathbb{R}$ is a differentiable function on A°. If $|f'|^q$ is generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind on A, $q > 1$, $p^{-1} + q^{-1} = 1$ and $|f'| \leq M$, then for all $\lambda, \mu \in [0,1]$, we have*

$$|S_{f,\eta_1}(\lambda,\mu,m)| \leq M\left(\frac{1}{p+1}\right)^{\frac{1}{p}}|\eta_1| \tag{3.3}$$
$$\times\begin{cases}\left[\vartheta_1^{\frac{1}{p}}\sigma_1^{\frac{1}{q}}+\vartheta_3^{\frac{1}{p}}\sigma_2^{\frac{1}{q}}\right], & \lambda(1-\mu)\leq\mu\leq1-\lambda\mu;\\ \left[\vartheta_2^{\frac{1}{p}}\sigma_1^{\frac{1}{q}}+\vartheta_3^{\frac{1}{p}}\sigma_2^{\frac{1}{q}}\right], & \mu\leq\lambda(1-\mu)\leq1-\lambda\mu;\\ \left[\vartheta_1^{\frac{1}{p}}\sigma_1^{\frac{1}{q}}+\vartheta_4^{\frac{1}{p}}\sigma_2^{\frac{1}{q}}\right], & \lambda(1-\mu)\leq1-\lambda\mu\leq\mu,\end{cases}$$

where

$$\sigma_1 = \frac{\mu^{1-s} + m\left(1 + (-1)^{-s}(\mu-1)^{1-s}\right)}{1-s},$$

$$\sigma_2 = \frac{1-\mu^{1-s}}{1-s} + m\left(\beta(1,1-s) + \frac{1+(-1)^{-s}(\mu-1)^{1-s}}{s-1}\right),$$

$$\vartheta_1 = [\lambda(1-\mu)]^{p+1} + [\mu - \lambda(1-\mu)]^{p+1}, \quad \vartheta_2 = [\lambda(1-\mu)]^{p+1} - [\lambda(1-\mu) - \mu]^{p+1},$$

$$\vartheta_3 = [1 - \lambda\mu - \mu]^{p+1} + [\lambda\mu]^{p+1}, \quad \vartheta_4 = [\lambda\mu]^{p+1} - [\mu - 1 + \lambda\mu]^{p+1}.$$

Proof. Suppose that $q > 1$. Using Lemma 3.1, the fact that $|f'|^q$ is a generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind, Hölder inequality and property of the modulus, we have

$$|S_{f,\eta_1}(\lambda,\mu,m)| \le |\eta_1| \left[\int_0^{\mu} |-t+\lambda(1-\mu)||f'(m\varphi(b)+t\eta_1)|dt + \int_{\mu}^1 |-t+(1-\mu\lambda)||f'(m\varphi(b)+t\eta_1)|dt\right]$$

$$\le |\eta_1| \Bigg\{ \left(\int_0^{\mu} |-t+\lambda(1-\mu)|^p dt\right)^{\frac{1}{p}} \left(\int_0^{\mu} |f'(m\varphi(b)+t\eta_1)|^q dt\right)^{\frac{1}{q}}$$

$$+ \left(\int_{\mu}^1 |-t+(1-\mu\lambda)|^p dt\right)^{\frac{1}{p}} \left(\int_{\mu}^1 |f'(m\varphi(b)+t\eta_1)|^q dt\right)^{\frac{1}{q}} \Bigg\}. \tag{3.4}$$

Since $|f'|^q$ is generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind and $|f'| \le M$, we have

$$\int_0^{\mu} |f'(m\varphi(b)+t\eta_1)|^q dt \le \int_0^{\mu} \left(\frac{|f'(\varphi(a))|^q}{t^s} + \frac{m|f'(\varphi(b))|^q}{(1-t)^s}\right) dt \le M^q \sigma_1, \tag{3.5}$$

$$\int_{\mu}^1 |f'(m\varphi(b)+t\eta_1)|^q dt \le \int_{\mu}^1 \left(\frac{|f'(\varphi(a))|^q}{t^s} + \frac{m|f'(\varphi(b))|^q}{(1-t)^s}\right) dt \le M^q \sigma_2. \tag{3.6}$$

By simple computations

$$\int_0^{\mu} |-t+\lambda(1-\mu)|^p dt = \begin{cases} \dfrac{\vartheta_1}{p+1}, & \mu \ge \lambda(1-\mu); \\ \dfrac{\vartheta_2}{p+1}, & \mu \le \lambda(1-\mu), \end{cases} \tag{3.7}$$

and

$$\int_{\mu}^1 |-t+(1-\mu\lambda)|^p dt = \begin{cases} \dfrac{\vartheta_3}{p+1}, & \mu \le 1-\lambda\mu; \\ \dfrac{\vartheta_4}{p+1}, & \mu \ge 1-\lambda\mu. \end{cases} \tag{3.8}$$

Hence, using (3.5)-(3.8) in (3.4), we obtain the inequality (3.3). The proof of Theorem 3.2 is completed. □

Corollary 3.3. *Under the same conditions as in Theorem 3.2, if we choose* $\mu = \frac{1}{2}, \lambda = \frac{1}{3}, m = 1$ *and* $\eta_1 = \varphi(a) - \varphi(b)$, *we get the following generalized Simpson type inequality:*

$$\left|\frac{1}{6}\left[f(\varphi(a)) + 4f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + f(\varphi(b))\right] - \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx\right|$$

$$\le M\left(\frac{\varphi(b)-\varphi(a)}{6}\right)\left(\frac{2^{p+1}+1}{6(p+1)}\right)^{\frac{1}{p}}\left(\sigma_3^{\frac{1}{q}} + \sigma_4^{\frac{1}{q}}\right), \tag{3.9}$$

where

$$\sigma_3 = \frac{\left(\frac{1}{2}\right)^{1-s} + \left(1 + (-1)^{-s}\left(-\frac{1}{2}\right)^{1-s}\right)}{1-s},$$

$$\sigma_4 = \frac{1-\left(\frac{1}{2}\right)^{1-s}}{1-s} + \left(\beta(1,1-s) + \frac{1+(-1)^{-s}\left(-\frac{1}{2}\right)^{1-s}}{s-1}\right).$$

Corollary 3.4. *Under the same conditions as in Theorem 3.2, if we choose* $\mu = \frac{1}{2}, \lambda = 1, m = 1$ *and* $\eta_1 = \varphi(a) - \varphi(b)$, *we get the following generalized Hermite-Hadamard type inequality:*

$$\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx\right|$$

$$\leq M(\varphi(b)-\varphi(a)) \left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}} \left(\sigma_3^{\frac{1}{q}} + \sigma_4^{\frac{1}{q}}\right), \tag{3.10}$$

where

$$\sigma_3 = \frac{\left(\frac{1}{2}\right)^{1-s} + \left(1+(-1)^{-s}\left(-\frac{1}{2}\right)^{1-s}\right)}{1-s},$$

$$\sigma_4 = \frac{1-\left(\frac{1}{2}\right)^{1-s}}{1-s} + \left(\beta(1,1-s) + \frac{1+(-1)^{-s}\left(-\frac{1}{2}\right)^{1-s}}{s-1}\right).$$

Corollary 3.5. *Under the same conditions as in Theorem 3.2, if we choose* $\mu = \frac{1}{2}, \lambda = 0, m = 1$ *and* $\eta_1 = \varphi(a) - \varphi(b)$, *we get the following generalized midpoint type inequality:*

$$\left|f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) - \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx\right|$$

$$\leq M(\varphi(b)-\varphi(a)) \left(\frac{1}{2^{p+1}(p+1)}\right)^{\frac{1}{p}} \left(\sigma_3^{\frac{1}{q}} + \sigma_4^{\frac{1}{q}}\right), \tag{3.11}$$

where

$$\sigma_3 = \frac{\left(\frac{1}{2}\right)^{1-s} + \left(1+(-1)^{-s}\left(-\frac{1}{2}\right)^{1-s}\right)}{1-s},$$

$$\sigma_4 = \frac{1-\left(\frac{1}{2}\right)^{1-s}}{1-s} + \left(\beta(1,1-s) + \frac{1+(-1)^{-s}\left(-\frac{1}{2}\right)^{1-s}}{s-1}\right).$$

Theorem 3.6. *Let* $\varphi : I \longrightarrow A$ *be a continuous function. Let* $A \subseteq \mathbb{R}$ *be an open m-invex subset with respect to* $\eta : A \times A \times (0,1] \longrightarrow \mathbb{R}$ *for some fixed* $s \in [0,1)$, $m \in (0,1]$ *and* $\eta_1 > 0$. *Assume that* $f : A = [m\varphi(b), m\varphi(b) + \eta_1] \longrightarrow \mathbb{R}$ *is a differentiable function on* A°. *If* $|f'|^q$ *is generalized* (s, m, φ)*-preinvex Godunova-Levin function of the second kind on* A, $q \geq 1$ *and* $|f'| \leq M$, *then for all* $\lambda, \mu \in [0,1]$, *we have*

$$|S_{f,\eta_1}(\lambda,\mu,m)| \leq M|\eta_1| \tag{3.12}$$

$$\times \begin{cases} \left[\epsilon_2^{1-\frac{1}{q}}\delta_2^{\frac{1}{q}} + \epsilon_3^{1-\frac{1}{q}}\delta_3^{\frac{1}{q}}\right], & \lambda(1-\mu) \leq \mu \leq 1-\lambda\mu; \\ \left[\epsilon_1^{1-\frac{1}{q}}\delta_1^{\frac{1}{q}} + \epsilon_3^{1-\frac{1}{q}}\delta_3^{\frac{1}{q}}\right], & \mu \leq \lambda(1-\mu) \leq 1-\lambda\mu; \\ \left[\epsilon_2^{1-\frac{1}{q}}\delta_2^{\frac{1}{q}} + \epsilon_4^{1-\frac{1}{q}}\delta_4^{\frac{1}{q}}\right], & \lambda(1-\mu) \leq 1-\lambda\mu \leq \mu, \end{cases}$$

where

$$\epsilon_1 = -\frac{\mu^2}{2} + \lambda(1-\mu)\mu, \quad \epsilon_2 = [\lambda(1-\mu)]^2 - \epsilon_1,$$

$$\epsilon_3 = (1-\lambda\mu)^2 - (1-\lambda\mu)(1+\mu) + \frac{1+\mu^2}{2}, \quad \epsilon_4 = \frac{1-\mu^2}{2} - (1-\lambda\mu)(1-\mu),$$

$$\delta_1 = \lambda(1-\mu)\frac{\mu^{1-s}}{1-s} + m\lambda(1-\mu)\beta(\mu;1,1-s) - \frac{\mu^{2-s}}{2-s} - m\beta(\mu;2,1-s),$$

$$\delta_2 = \frac{\lambda(1-\mu)^{2-s}}{1-s} + m\lambda(1-\mu)\beta(\lambda(1-\mu);1,1-s) - \frac{(\lambda(1-\mu))^{2-s}}{2-s}$$

$$-m\beta(\lambda(1-\mu);2,1-s) + \frac{\mu^{2-s} - (\lambda(1-\mu))^{2-s}}{2-s} + m(\beta(\mu;2,1-s) - \beta(\lambda(1-\mu);2,1-s))$$

$$-\frac{\lambda(1-\mu)}{1-s}(\mu^{1-s} - (\lambda(1-\mu))^{1-s}) - m\lambda(1-\mu)(\beta(\mu;1,1-s) - \beta(\lambda(1-\mu);1,1-s)),$$

$$\delta_3 = (1-\mu\lambda)\left(\frac{(1-\mu)^{1-s} - \mu^{1-s}}{1-s}\right) + m(1-\mu\lambda)(\beta(1-\mu\lambda;1,1-s) - \beta(\mu;1,1-s))$$

$$-\frac{(1-\mu\lambda)^{2-s} - \mu^{2-s}}{2-s} - m(\beta(1-\mu\lambda;2,1-s) - \beta(\mu;2,1-s))$$

$$+\frac{1-(1-\mu\lambda)^{2-s}}{2-s} + m(\beta(2,1-s) - \beta(1-\mu\lambda;2,1-s))$$

$$-(1-\mu\lambda)\left(\frac{1-(1-\mu\lambda)^{1-s}}{1-s}\right) - m(1-\mu\lambda)(\beta(1,1-s) - \beta(1-\mu\lambda;1,1-s)),$$

$$\delta_4 = \frac{1-\mu^{2-s}}{2-s} + m(\beta(2,1-s) - \beta(\mu;2,1-s)) - (1-\mu\lambda)\left(\frac{1-\mu^{1-s}}{1-s}\right)$$

$$-m(1-\mu\lambda)(\beta(1,1-s) - \beta(\mu;1,1-s)),$$

where $\beta(a;x,y)$ is incomplete Euler beta function for $x,y > 0$ and $0 < a \leq 1$.

Proof. Using Lemma 3.1, the fact that $|f'|^q$ is a generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind, the well-known power mean inequality and property of the modulus, we have

$$|S_{f,\eta_1}(\lambda,\mu,m)| \leq |\eta_1|\left[\int_0^\mu |-t+\lambda(1-\mu)||f'(m\varphi(b)+t\eta_1)|dt + \int_\mu^1 |-t+(1-\mu\lambda)||f'(m\varphi(b)+t\eta_1)|dt\right]$$

$$\leq |\eta_1|\Bigg\{\left(\int_0^\mu |-t+\lambda(1-\mu)|dt\right)^{1-\frac{1}{q}}\left(\int_0^\mu |-t+\lambda(1-\mu)||f'(m\varphi(b)+t\eta_1)|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_\mu^1 |-t+(1-\mu\lambda)|dt\right)^{1-\frac{1}{q}}\left(\int_\mu^1 |-t+(1-\mu\lambda)||f'(m\varphi(b)+t\eta_1)|^q dt\right)^{\frac{1}{q}}\Bigg\}. \tag{3.13}$$

Since $|f'|^q$ is generalized (s,m,φ)-preinvex Godunova-Levin function of the second kind and $|f'| \leq M$, we have

$$|f'(m\varphi(b)+t\eta_1)|^q \leq \frac{|f'(\varphi(a))|^q}{t^s} + \frac{m|f'(\varphi(b))|^q}{(1-t)^s} \leq M^q\left[\frac{1}{t^s} + \frac{m}{(1-t)^s}\right].$$

By simple computations

$$\int_0^\mu |-t+\lambda(1-\mu)|dt = \begin{cases} \epsilon_1, & \mu \leq \lambda(1-\mu); \\ \epsilon_2, & \mu \geq \lambda(1-\mu), \end{cases} \tag{3.14}$$

and

$$\int_\mu^1 |-t+(1-\mu\lambda)|dt = \begin{cases} \epsilon_3, & \mu \leq 1-\lambda\mu; \\ \epsilon_4, & \mu \geq 1-\lambda\mu. \end{cases} \tag{3.15}$$

Also it's not difficult to show the following inequalities

$$\int_0^{\mu} |-t+\lambda(1-\mu)||f'(m\varphi(b)+t\eta_1)|^q dt \leq \begin{cases} M^q\delta_1, & \mu \leq \lambda(1-\mu); \\ M^q\delta_2, & \mu \geq \lambda(1-\mu), \end{cases} \tag{3.16}$$

and

$$\int_{\mu}^{1} |-t+(1-\mu\lambda)||f'(m\varphi(b)+t\eta_1)|^q dt \leq \begin{cases} M^q\delta_3, & \mu \leq 1-\mu\lambda; \\ M^q\delta_4, & \mu \geq 1-\mu\lambda. \end{cases} \tag{3.17}$$

Hence, using (3.14)-(3.17) in (3.13), we obtain the inequality (3.12). The proof of Theorem 3.6 is completed. □

Corollary 3.7. *Under the same conditions as in Theorem 3.6, if we choose* $\mu = \frac{1}{2}, \lambda = \frac{1}{3}, m = 1$ *and* $\eta_1 = \varphi(a) - \varphi(b)$, *we get the following generalized Simpson type inequality:*

$$\left|\frac{1}{6}\left[f(\varphi(a)) + 4f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) + f(\varphi(b))\right] - \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx\right|$$

$$\leq M(\varphi(b)-\varphi(a))\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left(\delta_2^{\frac{1}{q}} + \delta_3^{\frac{1}{q}}\right), \tag{3.18}$$

where

$$\delta_2 = \frac{\frac{1}{3}\left(\frac{1}{2}\right)^{2-s}}{1-s} - \frac{1}{18}\frac{6-5^{1-s}6^s}{s-1} - \frac{\left(\frac{1}{6}\right)^{2-s}}{2-s} - \frac{1}{18}\frac{5^{1-s}6^s s + 36 - 35\cdot 5^{-s}6^s}{(s-1)(s-2)}$$

$$\frac{\left(\frac{1}{2}\right)^{2-s} - \left(\frac{1}{6}\right)^{2-s}}{2-s} + \frac{1}{4}\frac{2^s s - 3\cdot 2^s + 4}{(s-1)(s-2)} - \frac{1}{6}\frac{\left(\frac{1}{2}\right)^{1-s} - \left(\frac{1}{6}\right)^{1-s}}{1-s} - \frac{1}{12}\frac{2^s-2}{s-1},$$

$$\delta_3 = \frac{5}{18}\frac{6^s-6}{s-1} - \frac{5}{12}\frac{2^s-2}{s-1} - \frac{\left(\frac{5}{6}\right)^{2-s} - \left(\frac{1}{2}\right)^{2-s}}{2-s} - \frac{1}{18}\frac{5\cdot 6^s s - 11\cdot 6^s + 36}{(s-1)(s-2)}$$

$$+\frac{1}{4}\frac{2^s s - 3\cdot 2^s + 4}{(s-1)(s-2)} + \frac{1-\left(\frac{5}{6}\right)^{2-s}}{2-s} + \frac{1}{(s-1)(s-2)} - \frac{5}{6}\frac{1-\left(\frac{5}{6}\right)^{1-s}}{1-s} + \frac{5}{6(s-1)}.$$

Corollary 3.8. *Under the same conditions as in Theorem 3.6, if we choose* $\mu = \frac{1}{2}, \lambda = 1, m = 1$ *and* $\eta_1 = \varphi(a) - \varphi(b)$, *we get the following generalized Hermite-Hadamard type inequality:*

$$\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{1}{\varphi(b)-\varphi(a)}\int_{\varphi(a)}^{\varphi(b)} f(x)dx\right|$$

$$\leq M(\varphi(b)-\varphi(a))\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\sigma_2^{\frac{1}{q}} + \sigma_3^{\frac{1}{q}}\right), \tag{3.19}$$

where

$$\sigma_2 = \frac{\left(\frac{1}{2}\right)^{2-s}}{1-s} + \frac{1}{4}\frac{2^s-2}{s-1} - \frac{\left(\frac{1}{2}\right)^{2-s}}{2-s} - \frac{1}{4}\frac{2^s s - 3\cdot 2^s + 4}{(s-1)(s-2)},$$

$$\sigma_3 = \frac{1}{4}\frac{2^s-2}{s-1} - \frac{1}{4}\frac{2^s s - 3\cdot 2^s + 4}{(s-1)(s-2)} + \frac{1-\left(\frac{1}{2}\right)^{2-s}}{2-s} + \frac{1}{(s-1)(s-2)} - \frac{1}{2}\frac{1-\left(\frac{1}{2}\right)^{1-s}}{1-s} + \frac{1}{2(s-1)}.$$

Corollary 3.9. *Under the same conditions as in Theorem 3.2, if we choose* $\mu = \frac{1}{2}, \lambda = 0, m = 1$ *and* $\eta_1 = \varphi(a) - \varphi(b)$, *we get the following generalized midpoint type inequality:*

$$\left| f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) - \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \right|$$

$$\leq M(\varphi(b)-\varphi(a)) \left(\frac{1}{8}\right)^{1-\frac{1}{q}} \left(\sigma_2^{\frac{1}{q}} + \sigma_3^{\frac{1}{q}}\right), \tag{3.20}$$

where

$$\sigma_2 = \frac{\left(\frac{1}{2}\right)^{2-s}}{2-s} + \frac{1}{4}\frac{2^s s - 3\cdot 2^s + 4}{(s-1)(s-2)},$$

$$\sigma_3 = \frac{1}{1-s} - \frac{1}{2}\frac{2^s-2}{s-1} - \frac{1-\left(\frac{1}{2}\right)^{2-s}}{2-s} - \frac{1}{(s-1)(s-2)} + \frac{1}{4}\frac{2^s s - 3\cdot 2^s + 4}{(s-1)(s-2)}.$$

4 Conclusions

In the present paper, a new class of generalized (s, m, φ)-preinvex Godunova-Levin function of the second kind is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula are given. Moreover, by using new identity via classical integrals some Hermite-Hadamard, Simpson and midpoint type inequalities for generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind are established. Motivated by this new interesting class of generalized (s, m, φ)-preinvex Godunova-Levin functions of the second kind we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard, Simpson, midpoint, trapezoid and Ostrowski type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k-fractional integrals, local fractional integrals, fractional integral operators, q-calculus, (p, q)-calculus, time scale calculus and conformable fractional integrals.

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