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On the Candid Appraisal of the Proof of $\prod_1(S, x_o)$ as a Fundamental Group with respect to "$\circ$"

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ABSTRACT. It is known that a homeomorphism between two topological spaces preserves all the topological properties such as connectedness and compactness.To a map of spaces we can associate a homomorphism of groups such that compositions of maps yield compositions of homomorphisms of groups. Then any thing we can say about a topological situation gives information about the algebraic one. In this paper we studied and proved with self explained diagrams that $\prod_1(S, x_0)$ is a fundamental group with respect to "$\circ$"from the homotopical point of view. We made it more comprehensible, self explanatory and user friendly with clear diagrams. This is of interest because it can be used as a great tool in mathematics to prove many facts.

1 Introduction

The central idea behind algebraic topology[1,2,4,5,8,12,14,15,18] is to associate a topological situation to an algebraic situation, and study the simpler algebraic setup. To each topological space a group can be associated, such that homeomorphic spaces give rise to isomorphic groups. To a map of spaces we can associate a homomorphism of groups such that compositions of maps yield compositions of homomorphisms of groups. Then any thing we can say about a topological situation gives information about the algebraic one. The fundamental group is a tool used for describing what a topological space looks like. It creates an algebraic image of the space using loops in the space. However, the group does not tell us everything about a space.

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In this paper, we defined homotopy and presented some group properties. We then described the fundamental group and its related properties such as group homomorphisms. We also looked at useful definitions and theorems that plays an important role in computing fundamental groups, and our main result was actually built upon these theorems. In the end, we proved the fundamental group theorem using self explained diagrams.

In conclusion, we gave a brief summary of the work and also introduced the notion of homotopy equivalence, another tool used for computing fundamental groups. Throughout this paper we assumed the knowledge of basic group theory and general topology[3,6,7,9,10,11,13,16,17].

2 Preliminaries

2.1 Homotopy:

Definition 2.1. Let X be a topological space. A **path** in X from x_0 to x_1 is a continuous map $f : I \longrightarrow X$ such that $f(0) = x_0$ and $f(1) = x_1$. We say that x_0 is the *initial* point and x_1 the **final point**.

2.2 Homotopic path:

Definition 2.2.1 Let $f : I \longrightarrow X$ be a path in X from x_0 to x_1 and $g : I \longrightarrow X$ be a path in X from x_1 to x_2. Then the **product** $f * g$ is defined to be the path $f * g : I \longrightarrow X$ given by

$$f * g(s) = \begin{cases} f(2s) & for\, s \in [0, \frac{1}{2}], \\ g(2s-1) & for\, s \in [\frac{1}{2}, 1], \end{cases}$$

Example 2.2.2 (Linear homotopies.) Let f and f' be any two paths in $\mathbb{R}^n$ having the same endpoints x_0 and x_1. Then $F(s,t) = (1-t)f(s) + tf'(s)$ is a homotopy between f and f'. We verify this:

(i) $F(s,0) = f(s)$ *and* $F(s,1) = f'(s)$,

(ii) $F(0,t) = (1-t)x_0 + tx_0 = x_0$ *and* $F(1,t) = (1-t)x_1 + tx_1 = x_1$.

During the homotopy each point $f(s)$ travels along a line segment to $f'(s)$ at constant speed. It is called a **straight line homotopy**.

In particular, if $U \subset \mathbb{R}^n$ is convex, then any two paths $f, g : I \longrightarrow U$ with same endpoints are homotopic.

Definition 2.2.3. A **loop** in X is a continuous map $f : I \longrightarrow X$ such that $f(0) = f(1)$.

Then two loops can be combined together in an obvious way; first travel along the first loop, then along the second.

Definition 2.2.4 Let X be a topological space, and x_0 a point in X. The **fundamental group** of X is the set of path homotopy classes $[f]$ of loops $f : I \longrightarrow X$ based at x_0, together with the operation *. We denote it by $\pi(X, x_0)$.

Definition 2.2.5 Given two loop classes $[f]$ and $[g]$ we define:

(i) $[f] * [g] = [f * g]$.

(ii) The inverse of $[f]$ is given by $[f^{-1}]$, that is $[f]^{-1}$, where $f^{-1}(t) = \bar{f}(t) = f(1-t)$.

Theorem 2.2.6. If X is a convex subset of $\mathbb{R}^n$, and if $a, b \in X$, $\alpha_t(s) = (1-t)\alpha_0(s) + t\alpha_1(s)$. This defines a homotopy between α_0 and α_1.

Theorem 2.2.7. Let$X$ and Y be topological spaces, then the homotopy relation between maps from X to Y is an equivalence relation.

Proof. Let $f_1, f_2, f_3 : X \longrightarrow Y$ be maps, then $f_1 \overset{\approx}{F} f_1$, where $F(x,t) = f_1(x)$. If $f_1 \overset{\approx}{f_t} f_2$ then $f_2 \overset{\approx}{f_{-t}} f_1$. Finally, if $f_1 \overset{\approx}{F} f_2$ and $f_2 \overset{\approx}{G} f_3$ then $f_1 \overset{\approx}{H} f_3$ where

$$H(x,t) = \begin{cases} F(x,2t) & : 0 \leq t \leq \frac{1}{2}, \\ G(x,2t-1) & : \frac{1}{2} \leq t \leq 1. \end{cases}$$

Hence H is continuous because $H(x, \frac{1}{2} = F(x,1) = g(x) = G(x,0)$. We denote this equivalence relation by $\sim$.

Definition 2.2.8. The equivalence classes of maps from X to Y as in the Theorem 1.1.3 are called the homotopy classes ad denoted by $[f]$ as the homotopy class of the map f. While we use the notation $\langle\gamma\rangle$ when γ is a loop in X based at $x_0 \in X$.

From now on we assume all spaces are topological spaces.

2.3 Homotopy Classes:

Definition 2.3.1 The equivalent classes $[f]$ determined by homotopy modulo x_0 on the collection $C(S, x_\circ)$ of all closed paths f on S based at $x_\circ \in S$ are called homotopy classes of $C(S, x_\circ)$. The collection of these homotopy classes is denoted by $\prod_1(s, x_0)$.

3 Main Result

Definition 3.1

1. If $f, g \in c(s, x_0)$ we define the juxtaposition $f * g$ of f and g as follows:

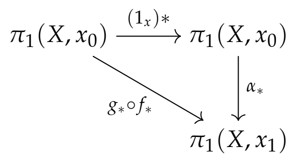
$$(f*g)(s) = \begin{cases} f(2s) & if\ 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & if\ \frac{1}{2} \leq s \leq 1 \end{cases}$$

   Thus $f * g \in c(s, x_0)$ and * is a binary operation on $c(s, x_0)$.

2. If $[f]$, $[g] \in \prod_1(s, x_0)$, then let $[f] \circ [g] = [f * g]$

Theorem 3.1. If X and Y are homotopy equivalent then their fundamental groups are isomorphic.

Proof. Let$f$ and g be s.t $fg \overset{\approx}{F} 1_X$ and $fg \overset{\approx}{H} 1_Y$. In order to apply Theorem 3.1, we have $gf, 1_X : X \longrightarrow X$ which are homotopic maps. Suppose that $x_0 \in X, x_1 = gf(x_0)$ and $\alpha(t) := F(x_0, t), 0 \leq t \leq 1$, hence we have the following commutative diagram,



since $(1_X)_*$ is an isomorphism then so is $g_* \circ f_*$. Mimicking the way above to see that $f_* \circ g_*$ is also an isomorphism. Therefore f_* is an isomorphism $\pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$.

Corollary 3.2. $\pi_1(\mathbb{R}^n \setminus \{0\}) \cong \pi_1(\mathbb{S}^{n-1})$.

Theorem 3.3. $\prod_1(S, x_0)$ is a group with respect to "$\circ$".

Proof.

1. "$\circ$"is associative. We need to show that $(f * g) * k\, \tilde{\tilde{x}}_0\, f * (g * k)$ for $f, g, k \in C(S, x_0)$.

$$[(f*g)*k](s) = \left\{ \begin{pmatrix} f(2s) \\ g(2s-1) \end{pmatrix} * k = \begin{cases} f[2(2s)] \\ g[2(2s)-1] \\ k(2s-1) \end{cases} = \begin{cases} f(4s) & \text{if } 0 \le s \le \frac{1}{4} \\ g(4s-1) & \text{if } \frac{1}{4} \le s \le \frac{1}{2} \\ k(2s-1) & \text{if } \frac{1}{2} \le s \le 1 \end{cases} \right.$$

and

$$[f*(g*k)](s) = \left\{ f * \begin{pmatrix} g(2s) \\ k(2s-1) \end{pmatrix} = \begin{cases} f(2s) \\ g[2(2s-1)] \\ k[2(2s-1)-1] \end{cases} = \begin{cases} f(4s) & if\ 0 \le s \le \frac{1}{2} \\ g(4s-2) & if\ \frac{1}{2} \le s \le \frac{3}{4} \\ k(4s-3) & if\ \frac{3}{4} \le s \le 1 \end{cases} \right.$$

We define a homotopy between $(f*g)*k$ and $f*(g*k)$ as follows:

$$h(s,t) = \begin{cases} f(\frac{4s}{1+t}) & if \langle s,t \rangle \in I^2\ and\ t \ge 4s-1 \\ g(4s-t-1) & if \langle s,t \rangle \in I^2\ and\ 4s-1 \ge t \ge 4s-2 \\ k(\frac{4s-t-1}{2-t}) & if \langle s,t \rangle \in I^2\ and\ 4s-2 \ge t \end{cases}$$

Then the following is true:

$$h(s,0) = \begin{cases} f(4s) & if\ 0 \ge 4s-1 & [i.e. 0 \le s \le \frac{1}{4}] \\ g(4s-1) & if\ 4s-1 \ge 0 \ge 4s-2 & [i.e. \frac{1}{4} \le s \le \frac{1}{2}] \\ k(2s-1) & if\ 4s-2 \ge 0 & [i.e. \frac{1}{2} \le s \le 1] \end{cases}$$

and

$$h(s,1) = \begin{cases} f(2s) & if\ 1 \ge 4s-1 & [i.e. 0 \le s \le \frac{1}{2}] \\ g(4s-2) & if\ 4s-1 \ge 1 \ge 4s-2 & [i.e. \frac{1}{2} \le s \le \frac{3}{4}] \\ k(4s-3) & if\ 4s-2 \ge 1 & [i.e. \frac{3}{4} \le s \le 1] \end{cases}$$

Thus $h(s,0) = [(f*g)*k](s)$ and $h(s,1) = [f*(g*k)](s)$ for all $s \in I'$

Also $h(0,t) = f(0) = x_0$ and $h(1,t) = k(1) = x_0$ for all $t \in I'$

Hence $(f*g)*k\ \tilde{\tilde{x}}_0\ f*(g*k)$

2. We show that the constant mapping $c : I \longrightarrow x_0$ is such that [c] is the identity element of $\prod_1(s, x_0)$ w.r.t "$\circ$"

Thus W.T.S $f * c\ \tilde{\tilde{x}}_0\ f$ for any $f \in C(S, x_0)$

Let $h : I^2 \longrightarrow S$ be defined as follows:

$$h(s,t) = \begin{cases} f(\frac{2s}{1+t}) & if\ \langle s,t \rangle \in I^2 & t \ge 2s-1 \\ x_0 & if\ \langle s,t \rangle \in I^2 & 2s-1 \ge t \end{cases}$$

Then

$$(f*c)(s) = h(s,0) = \begin{cases} f(2s) & if\ 0 \ge 2s-1 & [i.e. 0 \le s \le \frac{1}{2}] \\ x_0 & if\ 2s-1 \ge 0 & [i.e. \frac{1}{2} \le s \le 1] \end{cases}$$

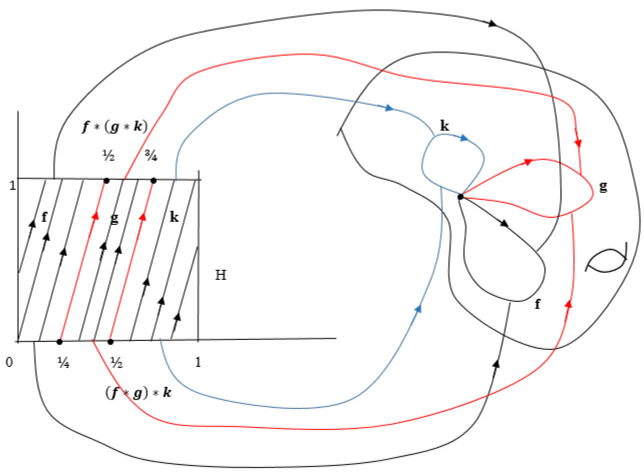


Figure 1: $(f*g)*k \ \tilde{\tilde{x}}_0 \ f*(g*k)$

and $h(s,1) = f(s)$
Thus $h(s,0) = (f*c)(s)$ and $h(s,1) = f(s)$ for all $s \in I'$

More so $h(0,t) = f(0) = x_0$ and $h(1,t) = f(1) = x_0$ for all $t \in I'$
Hence $f*c \ \tilde{\tilde{x}}_0 \ f$

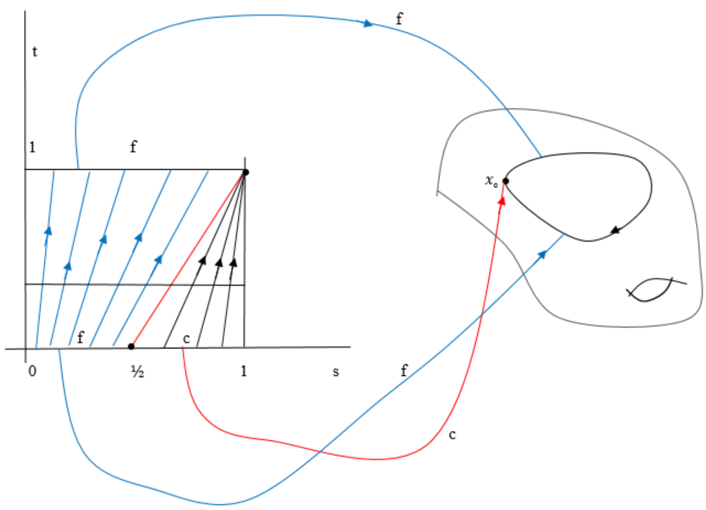


Figure 2: $f*c \ \tilde{\tilde{x}}_0 \ f$

3. Finally w.t.s that each homotopy class $[f] \in \prod_1(S, x_0)$ has an inverse $[g] \in \prod_1(S, x_0)$ s.t $[f] \circ [g] = [c]$

$\longrightarrow f * g \ \tilde{\tilde{x}}_0 \ c$

Let $g(s) = f(1-s) \ \ \forall s \in I'$

Since $g(0) = f(1) = x_0 = f(0) = g(1)$, $g \in C(S, x_0)$

By definition we have

$$(f * g)(s) = \begin{cases} f(2s) & if \ 0 \leq s \leq \frac{1}{2} \\ g(2s-1) = f(2-2s) & if \ \frac{1}{2} \leq s \leq 1 \end{cases}$$

We then define homotopy h between $f * g$ and c as follows:

$$h(s,t) = \begin{cases} x_0 & if \ 0 \leq s \leq \frac{t}{2} \\ g(2s-t) & if \ \frac{t}{2} \leq s \leq \frac{1}{2} \\ g(2s+t-1) & if \ \frac{1}{2} \leq s \leq 1 - \frac{t}{2} \\ x_0 & if \ 1 - \frac{t}{2} \leq s \leq 1 \end{cases}$$

Since f and g are continuous, h is continuous and we have

$$h(s,0) = \begin{cases} f(2s) & if \ 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & if \ \frac{1}{2} \leq s \leq 1 \end{cases}$$

and

$$h(s,0) = \ x_0 \qquad if \ 0 \leq s \leq 1$$

Thus $h(s,0) = (f * g)(s)$ and $h(s,1) = c(s) \ \forall s \in I'$

Also $h(0,t) = h(s,t) = x_0$ for all $t \in I'$

Hence $f * g \ \tilde{\tilde{x}}_0 \ c$.

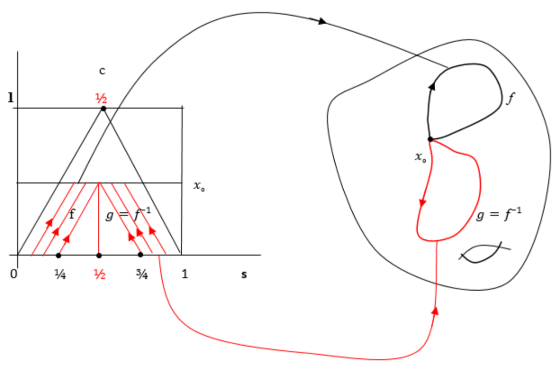


Figure 3: $f * g \ \tilde{\tilde{x}}_0 \ c$

4 Remark

This paper which is an appraisal of the proof $\prod_1(s, x_0)$*is a group with respect to* "o" is an attempt to prove with comprehensible diagrams that it is a fundamental group in full accordance with the principles of homotopy.
In general, $\prod_1(S, x_\circ)$ depends upon $x_\circ$. However, in the case of an arc-wise-connected space S, we can show that $\prod_1(S, x_\circ)$ is independent of $x_\circ$. Also, $\prod_1(S, x_\circ)$ is *not* Abelian in general, even if S is arcwise connected.

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