International Journal of Advances in Mathematics
Volume 2017, Number 6, Pages 82-100, 2017
eISSN 2456-6098

A Study on Rank of Hexagonal Fuzzy Number Matrices

D.Stephen Dinagar[1] and U.Hari Narayanan*
*,1 PG and Research Department of Mathematics,T.B.M.L. College, Porayar-609307, India.
E-mail:hari.narayanan23@gmail.com

ABSTRACT. A hexagonal fuzzy matrix has various type of ranks that is, row rank, column rank, fuzzy rank and fuzzy full rank. In this article, some methods are described to find these types of ranks for Hexagonal fuzzy matrices (HFMs) and investigated the relationship between them under the algorithm for Row Reduced Echelon Form (RREF). We have been studied the cross vector and schein rank under the relationship is illustrated with suitable example. A relevant results are presented in fuzzy rank using the definition of scalar multiplication of an Hexagonal fuzzy matrix.

1 Introduction

Fuzzy matrices play an important role to model several uncertain systems. Throughout this paper we deal with fuzzy number matrices that is matrices over fuzzy algebra F. For $A \in \mathbb{F}_{mn}$, $\mathbb{R}(A)$, $\mathbb{C}(A)$,$\rho_r(A)$, $\rho_c(A)$ and $\rho_f(A)$ denotes the row space, column space, row rank, column rank, and fuzzy rank under Hexagonal fuzzy Matrix respectively. Ismail and Morsi [5] established fuzzy rank of fuzzy matrix in the product of two fuzzy matrices cannot exceed the fuzzy rank either fuzzy matrix. Kaguei and Ohsato[6] investigated the new type of rank matrix using fuzzy number matrices. Wang and Zhongxiong [20] proposed the concept of determine the solution of fuzzy matrix rank. Hongxu and Chongxin [21] established the concept of the minimum row (column) space of fuzzy matrix. The recent development of Rank of Interval-valued fuzzy matrices have been studied by [10] . Also, Latha.et.al [9] studied the rank of Type-2 Triangular fuzzy matrix.

* Corresponding Author.
Received August 01, 2017; revised September 22, 2017; accepted September 24, 2017.
2010 Mathematics Subject Classification: 15B15,94D.
Key words and phrases: Hexagonal fuzzy number (HFN), Hexagonal fuzzy matrix(HFM),Row rank, Column rank, Fuzzy rank, Fuzzy Full rank, cross Vector, Schein rank.

The concept of fuzzy matrix (FM) [16] is one of the recent topic to develop for dealing with uncertainties present in engineering, agriculture, science, social science and also in most of our real life situations. Fuzzy matrices defined first time by Thomson [19] and discussed about the convergence of the powers of a fuzzy matrix. The decomposition theorem of fuzzy sets played an important role in the fuzzy set theory. It help us to develop many branches such as fuzzy algebra, fuzzy measure and integral, fuzzy analysis and so on. In FMs, rows and columns are taken as uncertain. He also investigated different properties of these type of matrices along with application. Shyamal and Pal [17] introduced the concept of triangular fuzzy matrix, Pal [**?**] defined as new types of IVFM whose rows and columns are uncertain along with uncertain elements.
Kim and Roush [7] introduced generalized fuzzy matrices where they investigated the standard basis, dimension of fuzzy vector space, row and column basis, row and column rank can be defined for fuzzy matrices. Many authors were involved in extending the basic concepts and results from the crisp vector space to the fuzzy vector space [1, 2, 4] and its dimension [8, 11, 12, 15], crisp matrix to fuzzy matrix.
It is well known that the rank of a crisp matrix is very important topics of linear algebra and it has many use. Motivated, from the theory and use of rank crisp matrix, we define rank of Hexagonal fuzzy matrices. The concept of Hexagonal fuzzy number [18] have been developed under the convexity condition. Due to the flexibility of Hexagonal fuzzy matrices, three different type of ranks are defined.
In this article, in section 2, we give some basic definitions and recall Hexagonal fuzzy number then its operations. In section 3, we have reviewed the definition of Hexagonal fuzzy matrix and its operations. In section 4, the main objective of the article, viz, row space, column space, row rank, column rank, fuzzy rank and fuzzy full rank of Hexagonal fuzzy matrices are investigated, Also some properties are developed here. In section 5, cross vectors and schein rank have been studied and it has been shown in our proposed matrix. In section 6, scalar multiplication of Hexagonal fuzzy matrix is defined and some of the properties are investigated. In section 7, we represent the conclusion for this work.

2 Perliminaries

In this section, some basic definitions are recalled for Hexagonal fuzzy number.

Definition 2.1 (Fuzzy Set)

A Fuzzy set is charaterized by a membership function mapping the elements of a domin, space or universe of discourse to the unit interval $[0, 1]$.
A fuzzy set A in a universe of discourse X is defined as the following set of pairs

$$A = \{(x, \mu_A(x); x \in X)\}$$

Here $\mu_A : X \to [0, 1]$ *is mapping called the degree of membership function of the fuzzy set A and* $\mu_A(x)$ *is called the membership value of* $x \in X$ *in the fuzzy set A. These membership grades are often represented by real ranging from [0,1].*

Definition 2.2 (Convex Fuzzy Set)

A fuzzy set $A = \{(x, \mu_A(x)\} \subseteq X$ *is called convex set in all* A_α *are convex set (i,e) for every element* $x_1 \in A_\alpha$ *and*

$x_2 \in A_\alpha$ for every $\alpha \in [0,1]$. $\lambda x_1 + (1-\lambda)\, x_2 \in A_\alpha$ for all $\lambda \in [0,1]$. Otherwise the fuzzy set is called non-convex fuzzy set.

Definition 2.3 (Fuzzy Number)

A fuzzy set $\tilde{A}$, defined on the set of real numbersR is said to be fuzzy number if its membership function has the following characteristics

1. *$\tilde{A}$ is normal.*
2. *$\tilde{A}$ is convex set.*
3. *The support of $\tilde{A}$ is closed and bounded then $\tilde{A}$ is called fuzzy number.*

Definition 2.4 (Hexagonal Fuzzy Number)

A fuzzy number on $\tilde{A}_h$ is hexagonal fuzzy number denoted by $\tilde{A}_h = (a_1, a_2, a_3, a_4, a_5, a_6)$, where $(a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6)$ are real number satisfying $a_2 - a_1 \leq a_3 - a_2$ and $a_5 - a_4 \geq a_6 - a_5$ and its membership function $\mu_{\tilde{A}_h}(x)$ is given by

$$\mu_{\tilde{A}_h}(x) = \begin{cases} 0 & ;x < a_1 \\ \frac{1}{2}\left(\frac{x-a_1}{a_2-a_1}\right) & ;a_1 \leq x \leq a_2 \\ \frac{1}{2} + \frac{1}{2}\left(\frac{x-a_2}{a_3-a_2}\right) & ;a_2 \leq x \leq a_3 \\ 1 & ;a_3 \leq x \leq a_4 \\ 1 - \frac{1}{2}\left(\frac{x-a_4}{a_5-a_4}\right) & ;a_4 \leq x \leq a_5 \\ \frac{1}{2}\left(\frac{a_6-x}{a_6-a_5}\right) & ;a_5 \leq x \leq a_6 \\ 0 & ;x \geq a_6. \end{cases}$$

The hexagonal fuzzy number $\tilde{A}_h$ becomes trapezoidal fuzzy number if $a_2 - a_1 = a_3 - a_2$ and $a_5 - a_4 = a_6 - a_5$.
The hexagonal fuzzy number $\tilde{A}_h$ becomes non-convex fuzzy number if $a_2 - a_1 > a_3 - a_2$ and $a_5 - a_4 < a_6 - a_5$.

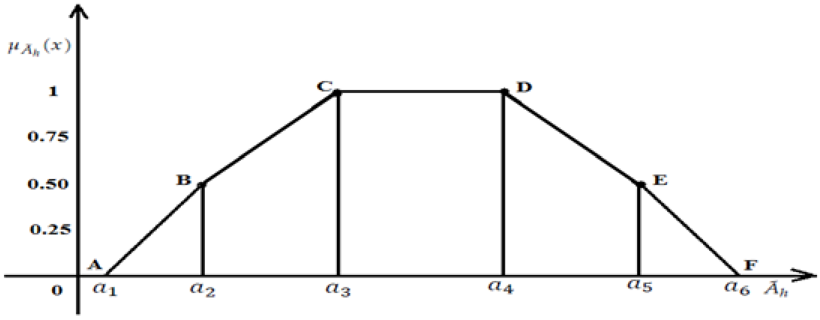

Figure 1: Hexagonal Fuzzy Number $\widetilde{A}_h$

2.1 Arithmetic Operations On Hexagonal Fuzzy Numbers(HFNs)

The arithmetic operations between hexagonal fuzzy numbers (HFNs) are proposed given below.

Let us consider $\tilde{A}_h = (a_1, a_2, a_3, a_4, a_5, a_6)$ and $\tilde{B}_h = (b_1, b_2, b_3, b_4, b_5, b_6)$ be two hexagonal fuzzy numbers then,

$(\mathbf{i})$**Addition** :

$\tilde{A}_h\,(+)\,\tilde{B}_h = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, a_6 + b_6)\,.$

$(\mathbf{ii})$**Subtraction** :

$\tilde{A}_h\,(-)\,\tilde{B}_h = (a_1 - b_6, a_2 - b_5, a_3 - b_4, a_4 - b_3, a_5 - b_2, a_6 - b_1)\,.$

$(\mathbf{iii})$**Multiplication** :

$\tilde{A}_h\,(\times)\,\tilde{B}_h = \left(\frac{a_1}{6}\sigma_b, \frac{a_2}{6}\sigma_b, \frac{a_3}{6}\sigma_b, \frac{a_4}{6}\sigma_b, \frac{a_5}{6}\sigma_b, \frac{a_6}{6}\sigma_b\right).$

Where $\sigma_b = (b_1 + b_2 + b_3 + b_4 + b_5 + b_6)\,.$

$(\mathbf{iv})$**Division** :

$\tilde{A}_h\,(\div)\,\tilde{B}_h = \left(\frac{6a_1}{\sigma_b}, \frac{6a_2}{\sigma_b}, \frac{6a_3}{\sigma_b}, \frac{6a_4}{\sigma_b}, \frac{6a_5}{\sigma_b}, \frac{6a_6}{\sigma_b}\right).$

Where $\sigma_b = (b_1 + b_2 + b_3 + b_4 + b_5 + b_6)\,.$

$(\mathbf{v})$**ScalarMultiplication** :

If $k \neq 0$ is scalar $k\tilde{A}_h$ is defined as

$$k\tilde{A}_h = \begin{cases} (ka_1, ka_2, ka_3, ka_4, ka_5, ka_6) & \text{if} k \geq 0 \\ (ka_6, ka_5, ka_4, ka_3, ka_2, ka_1) & \text{if} k < 0 \end{cases}$$

Definition 2.5 (Ranking Function)

We define a ranking function $\check{R} : F\,(R) \to R$ which maps each fuzzy numbers to real line $F(R)$ represented the set of all hexagonal fuzzy numbers. If R be any linear ranking functions.

$$\check{R}\,(\tilde{A}_h) = \left(\frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6}\right).$$

Also we define the order on $F(R)$ by

$\check{R}\,(\tilde{A}_h) \geq \check{R}\,(\tilde{B}_h)$ *if and only if* $\tilde{A}_h \underset{\breve{R}}{\geq} \tilde{B}_h$

$\check{R}\,(\tilde{A}_h) \leq \check{R}\,(\tilde{B}_h)$ *if and only if* $\tilde{A}_h \underset{\breve{R}}{\leq} \tilde{B}_h$

$\check{R}\,(\tilde{A}_h) = \check{R}\,(\tilde{B}_h)$ *if and only if* $\tilde{A}_h \underset{\breve{R}}{=} \tilde{B}_h$

Definition 2.6 (Zero Hexagonal Fuzzy Number)

If $\tilde{A}_h = (0,0,0,0,0,0)$ *then* $\tilde{A}_h$ *is said to be zero hexagonal fuzzy number. It is denoted by* 0.

Definition 2.7 (Zero-Equivalent Hexagonal Fuzzy Number)

A hexagonal fuzzy number $\tilde{A}_h$ *is said to be zero-equivalent hexagonal fuzzy number if* $\check{R}(\tilde{A}_h)$*=0. It is denoted by* $\tilde{0}$.

Definition 2.8 (Unit Hexagonal Fuzzy Number)

If $\tilde{A}_h = (1,1,1,1,1,1)$ *then* $\tilde{A}_h$ *is said to be unit hexagonal fuzzy number. It is denoted by* 1.

Definition 2.9 (Unit-Equivalent Hexagonal Fuzzy Number)

A hexagonal fuzzy number $\tilde{A}_h$ *is said to be unit-equivalent hexagonal fuzzy number if* $\check{R}(\tilde{A}_h)$*=1. It is denoted by* $\tilde{1}$.

Definition 2.10 (Inverse Hexagonal Fuzzy Number)

If $\tilde{a}_h$ *is hexagonal fuzzy number and* $\tilde{a}_h \neq 0$ *then we define* $\tilde{a}_h^{-1} = \frac{\tilde{1}}{\tilde{a}}$.

3 Hexagonal Fuzzy Matrices(HFMs)

In this section, we proposed new definition of Hexagonal Fuzzy Matrix and corresponding its matrix operations.

Definition 3.1 (Hexagonal Fuzzy Matrix)

A fuzzy matrix $\hat{A} = (a_{hij})_{m\times n}$ *of order* $m \times n$ *is called a Hexagonal fuzzy matrix if the elements of the matrix are Hexagonal fuzzy numbers, i.e., of the form* $(a_{ij1}, a_{ij2}, a_{ij3}, a_{ij4}, a_{ij5}, a_{ij6})$.

3.1 Operations on Hexagonal Fuzzy Matrices(HFMs)

Through classical matrix algebra, we achieve some algebraic operations of HFM. Let $\hat{A} = (\tilde{a}_{hij})_{m\times n}$ and $\hat{B} = (\tilde{b}_{hij})_{m\times n}$ be two HFMs of same order. Then we have the following

1. $\hat{A} + \hat{B} = \left(\tilde{a}_{hij} + \tilde{b}_{hij}\right)$
2. $\hat{A} - \hat{B} = \left(\tilde{a}_{hij} - \tilde{b}_{hij}\right)$
3. For $\hat{A} = (\tilde{a}_{hij})_{m\times n}$ and $\hat{B} = (\tilde{b}_{hij})_{n\times k}$ then $\hat{A}\hat{B} = (\tilde{c}_{hij})_{m\times k}$ where $(\tilde{c}_{hij})_{m\times k}$= $\sum_{p=1}^{n} \tilde{a}_{hip}\tilde{b}_{hpj}$, i=1,2,...,m and j=1,2...,k
4. $\hat{A}^T or \hat{A}' = (\tilde{a}_{hji})$
5. $k\hat{A} = (k\tilde{a}_{hij})$,where k is scalar.

Definition 3.2 (Zero Hexagonal Fuzzy Matrix)

A Hexagonal Fuzzy Matrix(HFM) is said to be a zero HFM if all its entries are 0 *and it is denoted by* $\hat{O}$.

Definition 3.3 (Zero Hexagonal Fuzzy Matrix)

The square HFM is said to be a unit HFM if the diagonal elements are 1 *and the rest of elements are* 0*. It is denoted by* $\hat{I}$

4 Rank of Hexagonal Fuzzy Matrices(HFMs)

In theory of matrix, rank is one of the fundamental concepts. There are three types of ranks, like row rank, column rank, and matrix rank.

In general, for crisp matrices row rank, column rank, matrix rank are equal. But the fuzzy matrices in row rank and column rank need not be equal also be in the hexagonal fuzzy matrices. There is a third rank called fuzzy rank which has great importance in fuzzy matrix theory.

Here we define various type of ranks associated with hexagonal fuzzy matrix using the elementary operations and Row Reduced Echelon form (RREF) of matrix. Three types of rank decompositions are defined and investigated the relation between them. Another fuzzy full rank matrix have been defined with suitable examples.

Definition 4.1 (Hexagonal Fuzzy Vector)
A Hexagonal fuzzy vector is an n-tuple of elements from an fuzzy algebra. That is,an Hexagonal fuzzy vector is of the form $(x_{h1}, x_{h2}, \dots, x_{hn})$ *where each element* $x_{hi} \in \mathbb{F}$, *i=1,2,n.*

Definition 4.2 (Linear Combination)
A linear combination of elements of the set of Hexagonal fuzzy vectors S_h *is a finite sum* $\sum a_{hi}\alpha_i$ *where* $\alpha_i \in S_h$ *and* $a_{hi} \in \mathbb{F}$*(the set of scalar). The set of all linear combinations of elements of* S_h *is called the span of S, denoted by* $\langle s \rangle$.

Definition 4.3 (Spanning Set)
Let S and W be two subsets of the fuzzy vector space V. If $< S >= W$, *then S is called spanning set or a set of generator of W.*

Definition 4.4 (Linearly Independent and Dependent)
A vector set $\{x_{h1}, x_{h2}, \dots, x_{ht}\} \subseteq \mathbb{V}_n$ *is linearly independent if and only if there is not* $x_{hi} \in \{x_{h1}, x_{h2}, \dots, x_{ht}\}$ *where i=1,2,,t such that it is represented as a linear combination of elements of* $x_{h1}, \dots, x_{hi-1}, x_{hi+1}, \dots, x_{ht}$. *If there is same* $x_{hi} \in \{x_{h1}, x_{h2}, \dots, x_{ht}\}$ $(1 \leq i \leq t)$ *such that it is a linear combination of elements of* $x_{h1}, \dots, x_{hi-1}, x_{hi+1}, \dots, x_{ht}$, *then set* $\{x_{h1}, x_{h2}, \dots, x_{ht}\}$ *is said to be linearly dependent.*

Definition 4.5 (Row Rank)
The row space $\mathbb{R}(\hat{A})$ *of an* $m \times n$ *Hexagonal fuzzy matrix* $\hat{A}$ *is the subspace of* $\mathbb{V}_n$ *generated by the rows of* $\hat{A}$. *The row rank* $\rho_r(\hat{A})$ *of* $\hat{A}$ *is minimum possible size of a spanning set of* $\mathbb{R}(\hat{A})$.

Definition 4.6 (Column Rank)
The column space $\mathbb{C}(\hat{A})$ *of an* $m \times n$ *Hexagonal fuzzy matrix* $\hat{A}$ *is the subspace of* $\mathbb{V}_m$ *generated by the rows of* $\hat{A}$. *The column rank* $\rho_c(\hat{A})$ *of* $\hat{A}$ *is the smallest possible size of a spanning set of* $\mathbb{C}(\hat{A})$.

Definition 4.7 (Fuzzy Rank)
A Hexagonal fuzzy matrix $\hat{A}$ *is said to be a fuzzy rank of rank r, if* $\rho_r(\hat{A}) = \rho_c(\hat{A}) = r$ *and it is denoted by* $\rho_f(\hat{A})$.
The row rank, column rank and fuzzy rank of a zero matrix $\hat{O}$ *is 0. For a finite matrix* $\hat{A}$, $\rho_r(\hat{A})$ *is the maximum number of linearly independent rows of* $\hat{A}$. *Similarly,* $\rho_c(\hat{A})$ *is maximum number of linearly independent columns of* $\hat{A}$. *If row rank is equal to column rank, then both are called rank and it is denoted by* ρ.

4.1 Elementary Transformation

The elementary transformation of a HFM $\hat{A} = (\tilde{a}_{hij})$ of the following transformation,

1. Interchange of two rows or columns
2. Multiplication of a row(or column) by an arbitrary Hexagonal fuzzy number not equal to $\tilde{0}$
3. Addition of multiple of one row (or column) by Hexagonal fuzzy number not equal to $\tilde{0}$ to another row (column).

4.2 Row Reduced Echelon form of HFM

Let $\hat{A} = (\tilde{a}_{hij})_{m\times n}$ be the hexagonal fuzzy matrix then the following steps are,

1. If $\tilde{a}_{h11} = \tilde{0}$ then an interchange of rows and columns will change element in the position $\tilde{a}_{h11}$, so that $\tilde{a}_{h11} \neq \tilde{0}$.
2. Convert the element $\tilde{a}_{h11}$ to $\tilde{1}$ by multiplying the first row by $\dfrac{1}{\tilde{a}_{h11}}$.
3. Subtract from the i^{th},$i > 1$, the first row multiplied by $\tilde{a}_{hi1}$, whenever $\tilde{a}_{hi1} \neq \tilde{0}$, then the element $\tilde{a}_{hi1}(i > 1)$ will be replaced by $\tilde{0}$.
4. Subtract from the j^{th},$j > 1$, the first row multiplied by $\tilde{a}_{h1j}$, whenever $\tilde{a}_{h1j} \neq \tilde{0}$, then the element $\tilde{a}_{h1j}(j > 1)$ will be replaced by $\tilde{0}$.
5. Performing the same manipulations (step 1 to step 4) with the submatrix that remains in the lower right corner and so on. We finally after a finite number of manipulations arrive at a diagonal-equivalent HFM with the same rank as the original HFM $\hat{A}$.

Note

To find the rank of HFM it is necessary to convert the HFM, by means of elementary transformations, to diagonal-equivalent HFM. Finally, we convert this diagonal-equivalent HFM into classical matrix and obtained maximum number of linearly independent rows or columns. The number of units gives the rank of the given HFM. i. Elementary row operations do not change the row space of a Hexagonal fuzzy matrix.

ii. Elementary column operations do not change the column space of a Hexagonal fuzzy matrix. Suppose that $\hat{A}$ and $\hat{B}$ are $m \times n$ HFMs such that $\hat{B}$ is obtained from $\hat{A}$ by an elementary row operation

Let $a_{h1}, a_{h2}, \ldots, a_{hm}$ be the rows of $\hat{A}$ and $b_{h1}, \ldots, b_{hm}$ be the rows of $\hat{B}$. We have to show that,

$$span(a_{h1}, a_{h2}, \ldots, a_{hm}) = span(b_{h1}, b_{h2}, \ldots, b_{hm})$$

Observe that any row b_{hi} for $\hat{B}$ belongs to span$(a_{h1}, a_{h2}, \ldots, a_{hm})$.

Indeed, either $b_{hi} = a_{hj}$ for some $1 \leq j \leq m$, or $b_{hi} = ra_{hi}$ for some scalar $r \neq 0$ or $b_{hi} = a_{hi} + ra_{hj}$ for some $j \neq i$ and $r \in R$.

It follows that $span(b_{h1}, b_{h2}, \ldots, b_{hm}) \subset span(a_{h1}, a_{h2}, \ldots, a_{hm})$.

Now the HFM $\hat{A}$ can also be obtained from $\hat{B}$ by an elementary operations, we get,

$$span(a_{h1}, a_{h2}, \ldots, a_{hm}) \subset span(b_{h1}, b_{h2}, \ldots, b_{hm})$$

It is similar to follow by proof (i).

Theorem 4.8 *The row rank and the column rank of HFM $\hat{A}$ are equal. Let $\hat{A}$ be an $m \times n$ HFM , then*

Let $\hat{A}$ be an $m \times n$ HFM , then

$$\hat{A} = \begin{bmatrix} \tilde{a}_{h11} & \tilde{a}_{h12} & \dots & \tilde{a}_{h1n} \\ \tilde{a}_{h21} & \tilde{a}_{h22} & \dots & \tilde{a}_{h2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{a}_{hm1} & \tilde{a}_{hm2} & \dots & \tilde{a}_{hmn} \end{bmatrix}$$

We my write $\hat{A}$ interms of its columns or rows

(i,e), $\hat{A} = \begin{bmatrix} A_{h1} & , A_{h2} & , \dots & , A_{hn} \end{bmatrix} \begin{bmatrix} a_{h1}^T \\ a_{h2}^T \\ \vdots \\ a_{hm}^T \end{bmatrix}$

Where j^{th} column vector A_{hj} and i^{th} row vector a_{hi} are given by

$A_{hj} = \begin{bmatrix} \tilde{a}_{h1j} \\ \tilde{a}_{h2j} \\ \vdots \\ \tilde{a}_{hnj} \end{bmatrix}, a_{hi} = \begin{bmatrix} \tilde{a}_{hi1} \\ \tilde{a}_{hi2} \\ \vdots \\ \tilde{a}_{him} \end{bmatrix}$

The column vectors of $\hat{A}$ span a space which is a subset of R^n and its column space $\hat{A}$ is,

$$\mathbb{C}(\hat{A}) = span(A_{h1}, A_{h2}, \dots, A_{hn})$$

Then the row vectors of $\hat{A}$ span a space which is a subset of R^m and its row space $\hat{A}$ is,

$$\mathbb{R}(\hat{A}) = span(a_{h1}, a_{h2}, \dots, a_{hm})$$

The dimension of $\mathbb{C}(\hat{A})$ is called the column rank of $\hat{A}$ and it is denoted by $C = \rho_c(\hat{A})$. Similarly the dimension of $\mathbb{R}(\hat{A})$ is called row rank of$\hat{A}$ and it is denoted by $R = \rho_r(\hat{A})$.

Case(i)

Let $\hat{B} = [b_{h1}, b_{h2}, \dots, b_{hC}]$ where the columns of $\hat{B}$ form a basis for the column space of $\hat{A}$. Then there is $C \times n$ matrix $\hat{D}$ such that,

$$\hat{A} = \hat{B}\hat{D}.$$

It follows that the rows of $\hat{A}$ are linearly combinations of the rows of $\hat{D}$ and hence,

$$\mathbb{R}(\hat{A}) \subseteq \mathbb{R}(\hat{D})$$

Hence,

$$R = dim[\mathbb{R}(\hat{A}) \leq \mathbb{R}(\hat{D})] \leq C$$

(i,e), The row rank of $\hat{A}$ is less than or equal to the column rank of $\hat{A}$.

Case(ii)

$\hat{E} = \begin{bmatrix} e_{h1}^T \\ e_{h2}^T \\ \vdots \\ e_{hR}^T \end{bmatrix}$ *Where the rows of $\hat{E}$ form a basis for the row space of $\hat{A}$. Then there is a $m \times R$ matrix $\hat{K}$ such that,*

$$\hat{A} = \hat{K}\hat{E}.$$

It follows that the rows of $\hat{A}$ are linearly combinations of the rows of $\hat{K}$ and hence,

$$\mathbb{C}(\hat{A}) \subseteq \mathbb{C}(\hat{K})$$

Hence,

$$C = dim[\mathbb{C}(\hat{A}) \leq \mathbb{C}(\hat{K})] \leq R$$

(i,e), The column rank of $\hat{A}$ is less than or equal to the row rank of $\hat{A}$.
Hence we have the row and column rank of HFM are equal, then the common value is called the fuzzy rank of $\hat{A}$ and it is denoted by $\rho = \rho_f(\hat{A})$.

Example 4.9 *Consider the HFM,*

$\hat{A} = \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,2,4,6) & (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) \\ (-1,0,1,2,4,6) & (1,2,3,4,6,8) & (-2,-1,0,1,2,6) & (-6,-4,-2,-1,0,1) \\ (-1,0,1,3,6,9) & (-1,1,6,8,10,12) & (-1,0,1,3,6,9) & (-12,-10,-8,-7,-6,1) \end{bmatrix}$ *into RREF.*

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,2,4,6) & (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) \\ (-1,0,1,2,4,6) & (1,2,3,4,6,8) & (-2,-1,0,1,2,6) & (-6,-4,-2,-1,0,1) \\ (-1,0,1,3,6,9) & (-1,1,6,8,10,12) & (-1,0,1,3,6,9) & (-12,-10,-8,-7,-6,1) \end{bmatrix}$$

$$R_1 \rightarrow \frac{R_1}{\tilde{a}_{h11}} \text{ , where } \tilde{a}_{h11} \neq \tilde{0}$$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,2,4,6) & (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) \\ (-1,0,1,2,4,6) & (1,2,3,4,6,8) & (-2,-1,0,1,2,6) & (-6,-4,-2,-1,0,1) \\ (-1,0,1,3,6,9) & (-1,1,6,8,10,12) & (-1,0,1,3,6,9) & (-12,-10,-8,-7,-6,1) \end{bmatrix}$$

$$R_2 \rightarrow R_2 - \tilde{a}_{h21}R_1 \text{ , where } \tilde{a}_{h21} \neq \tilde{0}$$
$$R_3 \rightarrow R_3 - \tilde{a}_{h31}R_1 \text{ , where } \tilde{a}_{h31} \neq \tilde{0}$$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,2,4,6) & (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) \\ (-7,-4,-1,1,4,7) & (-11,-6,-1,2,6,10) & (-3,-1,1,3,6,12) & (-24,-16,-8,-4,0,4) \\ (-10,-6,-2,2,6,10) & (-19,-11,0,5,10,15) & (-10,-6,-2,2,6,10) & (-39,-28,-17,-10,-6,4) \end{bmatrix}$$

$$R_3 \rightarrow \frac{R_3}{\tilde{a}_{h34}} \text{ , where } \tilde{a}_{h34} \neq \tilde{0}$$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,2,4,6) & (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) \\ (-7,-4,-1,1,4,7) & (-11,-6,-1,2,6,10) & (-3,-1,1,3,6,12) & (-24,-16,-8,-4,0,4) \\ (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-19}{16},\frac{-11}{16},0,\frac{5}{16},\frac{10}{16},\frac{15}{16}) & (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-39}{16},\frac{-28}{16},\frac{-17}{16},\frac{-10}{16},\frac{-6}{16},\frac{4}{16}) \end{bmatrix}$$

$$R_1 \rightarrow R_1 - \tilde{a}_{h14}R_3 \text{ , where } \tilde{a}_{h14} \neq \tilde{0}$$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,2,4,6) & (-2,-1,0,1,2,6) & (-10,-6,-2,2,6,10) \\ (-7,-4,-1,1,4,7) & (-11,-6,-1,2,6,10) & (-3,-1,1,3,6,12) & (-24,-16,-8,-4,0,4) \\ (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-19}{16},\frac{-11}{16},0,\frac{5}{16},\frac{10}{16},\frac{15}{16}) & (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-39}{16},\frac{-28}{16},\frac{-17}{16},\frac{-10}{16},\frac{-6}{16},\frac{4}{16}) \end{bmatrix}$$

$$R_2 \to R_2 - \tilde{a}_{h24} R_3 \text{ , where } \tilde{a}_{h24} \neq \tilde{0}$$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,2,4,6) & (-2,-1,0,1,2,6) & (-10,-6,-2,2,6,10) \\ (-7,-4,-1,1,4,7) & (-11,-6,-1,2,6,10) & (-3,-1,1,3,6,12) & (-28,-16,-4,4,16,28) \\ (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-19}{16},\frac{-11}{16},0,\frac{5}{16},\frac{10}{16},\frac{15}{16}) & (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-39}{16},\frac{-28}{16},\frac{-17}{16},\frac{-10}{16},\frac{-6}{16},\frac{4}{16}) \end{bmatrix}$$

$$R_2 \to \frac{R_2}{\tilde{a}_{h23}} \text{ , where } \tilde{a}_{h23} \neq \tilde{0}$$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,2,4,6) & (-2,-1,0,1,2,6) & (-10,-6,-2,2,6,10) \\ (\frac{-7}{3},\frac{-4}{3},\frac{-1}{3},\frac{1}{3},\frac{4}{3},\frac{7}{3}) & (\frac{-11}{3},\frac{-6}{3},\frac{-1}{3},\frac{2}{3},\frac{6}{3},\frac{10}{3}) & (\frac{-3}{3},\frac{-1}{3},\frac{1}{3},\frac{3}{3},\frac{6}{3},\frac{12}{3}) & (\frac{-28}{3},\frac{-16}{3},\frac{-4}{3},\frac{4}{3},\frac{16}{3},\frac{28}{3}) \\ (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-19}{16},\frac{-11}{16},0,\frac{5}{16},\frac{10}{16},\frac{15}{16}) & (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-39}{16},\frac{-28}{16},\frac{-17}{16},\frac{-10}{16},\frac{-6}{16},\frac{4}{16}) \end{bmatrix}$$

$$R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} (\frac{-13}{3},\frac{-7}{3},\frac{-1}{3},\frac{4}{3},\frac{10}{3},\frac{25}{3}) & (\frac{-13}{3},\frac{-6}{3},\frac{1}{3},\frac{7}{3},\frac{18}{3},\frac{29}{3}) & (-6,-3,-1,\frac{2}{3},\frac{7}{3},7) & (\frac{-58}{3},\frac{-34}{3},\frac{-10}{3},\frac{10}{3},\frac{34}{3},\frac{58}{3}) \\ (\frac{-7}{3},\frac{-4}{3},\frac{-1}{3},\frac{1}{3},\frac{4}{3},\frac{7}{3}) & (\frac{-11}{3},\frac{-6}{3},\frac{-1}{3},\frac{2}{3},\frac{6}{3},\frac{10}{3}) & (-1,\frac{-1}{3},\frac{1}{3},\frac{3}{3},\frac{6}{3},\frac{12}{3}) & (\frac{-28}{3},\frac{-16}{3},\frac{-4}{3},\frac{4}{3},\frac{16}{3},\frac{28}{3}) \\ (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-19}{16},\frac{-11}{16},0,\frac{5}{16},\frac{10}{16},\frac{15}{16}) & (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-39}{16},\frac{-28}{16},\frac{-17}{16},\frac{-10}{16},\frac{-6}{16},\frac{4}{16}) \end{bmatrix}$$

$$c = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathbb{C}(\hat{A}) = \{c_1, c_2, c_3, c_4\}$ *and* $\mathbb{R}(\hat{A}) = \{r_1, r_2, r_3\}$

clearly c_1, c_3 *and* c_4 *are linearly independent,*

$$\text{The basis of } \mathbb{C}(\hat{A}) = \begin{Bmatrix} (-2,-1,0,1,2,6) & (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) \\ (-1,0,1,2,4,6) & (-2,-1,0,1,2,6) & (-6,-4,-2,-1,0,1) \\ (-1,0,1,3,6,9) & (-1,0,1,3,6,9) & (-12,-10,-8,-7,-6,1) \end{Bmatrix}$$

$$\text{The basis of } \mathbb{R}(\hat{A}) = \begin{Bmatrix} (\frac{-13}{3},\frac{-7}{3},\frac{-1}{3},\frac{4}{3},\frac{10}{3},\frac{25}{3}) & (\frac{-13}{3},\frac{-6}{3},\frac{1}{3},\frac{7}{3},\frac{18}{3},\frac{29}{3}) & (-6,-3,-1,\frac{2}{3},\frac{7}{3},7) & (\frac{-58}{3},\frac{-34}{3},\frac{-10}{3},\frac{10}{3},\frac{34}{3},\frac{58}{3}) \\ (\frac{-7}{3},\frac{-4}{3},\frac{-1}{3},\frac{1}{3},\frac{4}{3},\frac{7}{3}) & (\frac{-11}{3},\frac{-6}{3},\frac{-1}{3},\frac{2}{3},\frac{6}{3},\frac{10}{3}) & (-1,\frac{-1}{3},\frac{1}{3},\frac{3}{3},\frac{6}{3},\frac{12}{3}) & (\frac{-28}{3},\frac{-16}{3},\frac{-4}{3},\frac{4}{3},\frac{16}{3},\frac{28}{3}) \\ (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-19}{16},\frac{-11}{16},0,\frac{5}{16},\frac{10}{16},\frac{15}{16}) & (\frac{-10}{16},\frac{-6}{16},\frac{-2}{16},\frac{2}{16},\frac{6}{16},\frac{10}{16}) & (\frac{-39}{16},\frac{-28}{16},\frac{-17}{16},\frac{-10}{16},\frac{-6}{16},\frac{4}{16}) \end{Bmatrix}$$

$dim(\mathbb{C}(\hat{A})) = \rho_c(\hat{A}) = 3$

$dim(\mathbb{R}(\hat{A})) = \rho_r(\hat{A}) = 3$

$\therefore$ *The common value of row rank and column rank is fuzzy rank,*

(i,e)., $\rho_r(\hat{A}) = \rho_c(\hat{A}) = \rho_f(\hat{A}) = 3.$

Proposition 4.10 *Let* $\hat{A} \in \mathbb{F}_{mn}$ *with* $\rho_r(\hat{A}) = r$. *Then there exist matrices* $\hat{B} \in \mathbb{F}_{mr}$ *and* $\hat{C} \in \mathbb{F}_{rn}$ *such that* $\rho_r(\hat{A}) = \rho_r(\hat{C}) = r$ *and* $\hat{A} = \hat{B}\hat{C}$. *Since* $\rho_r(\hat{A}) = r$, *therefore row space of* $\hat{A}$, $\mathbb{R}(\hat{A})$ *has r linearly independent row vectors of* $\hat{A}$. *Now , if* $\mathbb{R}(\hat{A})$ *is generated by the r linearly independent rows of an* $r \times n$ *matrix* $\hat{C}$, *then* $\rho_r(\hat{C}) = r$ *and there exist* $m \times r$ *matrix* $\hat{B}$, *such that* $\hat{A} = \hat{B}\hat{C}$.

Thus $\hat{A} = \hat{B}\hat{C}$ *and* $\rho_r(\hat{A}) = \rho_r(\hat{C}) = r$.

Proposition 4.11 *Let* $\hat{A} \in \mathbb{F}_{mn}$ *with* $\rho_c(\hat{A}) = s$. *Then there exist matrices* $\hat{B} \in \mathbb{F}_{ms}$ *and* $\hat{C} \in \mathbb{F}_{sn}$ *such that* $\rho_r(\hat{A}) = \rho_r(\hat{B}) = s$ *and* $\hat{A} = \hat{B}\hat{C}$. *Since* $\rho_c(\hat{A}) = r$, *therefore column space of* $\hat{A}$, $\mathbb{C}(\hat{A})$ *has s linearly independent row vectors of* $\hat{A}$. *Now , if* $\mathbb{C}(\hat{A})$ *is generated by the s linearly independent columns of an* $s \times m$ *matrix* $\hat{B}$, *then* $\rho_r(\hat{B}) = s$ *and there exist* $n \times s$ *matrix* $\hat{C}$, *such that* $\hat{A} = \hat{B}\hat{C}$.

Thus $\hat{A} = \hat{B}\hat{C}$ *and* $\rho_r(\hat{A}) = \rho_r(\hat{C}) = r$.

Proposition 4.12 *Let $\hat{A} \in \mathbb{F}_{mn}$ with $\rho_f(\hat{A}) = \rho_r(\hat{A}) = \rho_c(\hat{A}) = r$. Then there exist matrices $\hat{B} \in \mathbb{F}_{ms}$ and $\hat{C} \in \mathbb{F}_{sn}$ such that $\hat{A} = \hat{B}\hat{C}$ with $\rho_f(\hat{A}) = \rho_r(\hat{A}) = \rho_c(\hat{A}) = r$. This proof is obtained by combining preposition 4.11 and 4.12.*

For these last three prepositions, we see that the matrix $\hat{A}$ can be decomposed into two matrices$\hat{B}$ and $\hat{C}$. (i,e), $\hat{A} = \hat{B}\hat{C}$. This decomposition is called, row rank factorization (preposition 4.11), column rank factorization (preposition 4.12) and is called only fuzzy rank factorization (preposition 4.13).

Example 4.13 *For the Hexagonal fuzzy matrix $\hat{A}$ of example 4.10 we see that $\hat{A} = \hat{B}\hat{C}$ where,*

$\hat{B} =$

$$\begin{pmatrix} (\frac{-13}{3}, \frac{-7}{3}, \frac{-1}{3}, \frac{4}{3}, \frac{10}{3}, \frac{25}{3}) & (\frac{-13}{3}, \frac{-6}{3}, \frac{1}{3}, \frac{7}{3}, \frac{18}{3}, \frac{29}{3}) & (-6, -3, -1, \frac{2}{3}, \frac{7}{3}, 7) & (\frac{-58}{3}, \frac{-34}{3}, \frac{-10}{3}, \frac{10}{3}, \frac{34}{3}, \frac{58}{3}) \\ (\frac{-7}{3}, \frac{-4}{3}, \frac{-1}{3}, \frac{1}{3}, \frac{4}{3}, \frac{7}{3}) & (\frac{-11}{3}, \frac{-6}{3}, \frac{-1}{3}, \frac{2}{3}, \frac{6}{3}, \frac{10}{3}) & (-1, \frac{-1}{3}, \frac{1}{3}, \frac{3}{3}, \frac{6}{3}, \frac{12}{3}) & (\frac{-28}{3}, \frac{-16}{3}, \frac{-4}{3}, \frac{4}{3}, \frac{16}{3}, \frac{28}{3}) \\ (\frac{-10}{16}, \frac{-6}{16}, \frac{-2}{16}, \frac{2}{16}, \frac{6}{16}, \frac{10}{16}) & (\frac{-19}{16}, \frac{-11}{16}, 0, \frac{5}{16}, \frac{10}{16}, \frac{15}{16}) & (\frac{-10}{16}, \frac{-6}{16}, \frac{-2}{16}, \frac{2}{16}, \frac{6}{16}, \frac{10}{16}) & (\frac{-39}{16}, \frac{-28}{16}, \frac{-17}{16}, \frac{-10}{16}, \frac{-6}{16}, \frac{4}{16}) \end{pmatrix} \in \mathbb{F}_{3\times 4}$$

$$\hat{C} = \begin{pmatrix} (-2,-1,0,1,2,6) & (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) \\ (-1,0,1,2,4,6) & (-2,-1,0,1,2,6) & (-6,-4,-2,-1,0,1) \\ (-1,0,1,3,6,9) & (-1,0,1,3,6,9) & (-12,-10,-8,-7,-6,1) \end{pmatrix} \in \mathbb{F}_{3\times 3}$$

Hence, we see that $\rho_f(\hat{A}) = \rho_r(\hat{B}) = \rho_c(\hat{C}) = 3$

Therefore, the decomposition $\hat{A} = \hat{B}\hat{C}$ is a column and row rank factorization and also the both cases are same rank then the fuzzy rank factorization is same.

Definition 4.14 *Let $\hat{A}$ and $\hat{B}$ be the two HFMs such that $\hat{A}\hat{B}$ is defined by $\rho(\hat{A}\hat{B}) \leq min\{\rho(\hat{A}), \rho(\hat{B})\}$.*

Let $\hat{A}\hat{B}$ be $m \times n$ HFMs then $\rho(\hat{A} + \hat{B}) \leq \rho(\hat{A}) + \rho(\hat{B})$. Let $\hat{A} = \hat{X}\hat{Y}$, $\hat{B} = \hat{U}\hat{V}$ be rank factorization of $\hat{A}\hat{B}$ then,

$$\hat{A} + \hat{B} = \hat{X}\hat{Y} + \hat{U}\hat{V} = \begin{bmatrix} \hat{X} & \hat{Y} \end{bmatrix} \begin{bmatrix} \hat{Y} \\ \hat{V} \end{bmatrix}$$

By the definition,

$$\rho(\hat{A} + \hat{B}) \leq \rho[\hat{X}, \hat{U}]$$

Let $x_{h1}, x_{h2}, \ldots, x_{hp}$ and $x_{h1}, x_{h2}, \ldots, x_{hq}$ be the bases of $\mathfrak{B}(\hat{X}), \mathfrak{B}(\hat{X})$ respectively, any vector in the column space $[\hat{X}, \hat{U}]$ can be expressed as a linear combination of these $p + q$ vectors.

Thus,

$$\begin{aligned} \rho[\hat{X}, \hat{U}] &\leq \rho(\hat{X}) + \rho(\hat{U}) \\ &\leq \rho(\hat{A}) + \rho(\hat{B}) \end{aligned}$$

$\therefore \rho(\hat{A} + \hat{B}) \leq \rho(\hat{A}) + \rho(\hat{B})$.

Definition 4.15 *Let $\hat{A} = (\tilde{a}_{hij})_{m\times n} \in \mathbb{F}_{mn}$, then $\hat{A}$ is called row full rank if $\rho_r(\hat{A}) = m$ and $\hat{A}$ is called column full rank if $\rho_c(\hat{A}) = n$ and let $\hat{A} \in \mathbb{F}_{mn}$ then $\hat{A}$ is said to be fuzzy full rank if $\rho_f(\hat{A}) = \rho_r(\hat{A}) = \rho_c(\hat{A}) = n$.*

Example 4.16 *Consider a Hexagonal fuzzy matrix*

$$\hat{A} = \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) & (-1,1,4,6,8,12) \\ (-1,0,1,2,4,6) & (2,4,6,8,10,12) & (0,8,10,12,14,16) \\ (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) & (1,4,5,7,9,10) \end{bmatrix} \text{ into RREF.}$$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) & (-1,1,4,6,8,12) \\ (-1,0,1,2,4,6) & (2,4,6,8,10,12) & (0,8,10,12,14,16) \\ (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) & (1,4,5,7,9,10) \end{bmatrix}$$

$R_1 \to \frac{R_1}{\tilde{a}_{h11}}$, *where* $\tilde{a}_{h11} \neq \tilde{0}$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) & (-1,1,4,6,8,12) \\ (-1,0,1,2,4,6) & (2,4,6,8,10,12) & (0,8,10,12,14,16) \\ (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) & (1,4,5,7,9,10) \end{bmatrix}$$

$R_2 \to R_2 - \tilde{a}_{h21} R_1$, *where* $\tilde{a}_{h21} \neq \tilde{0}$

$R_3 \to R_3 - \tilde{a}_{h31} R_1$, *where* $\tilde{a}_{h31} \neq \tilde{0}$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) & (-1,1,4,6,8,12) \\ (-7,-4,-1,1,4,7) & (-16,-8,0,5,10,15) & (-30,-12,0,7,14,21) \\ (-8,-3,-1,1,3,8) & (-19,-6,-2,3,9,15) & (-29,-6,0,7,14,20) \end{bmatrix}$$

$R_1 \to R_1 - \tilde{a}_{h12} R_2$, *where* $\tilde{a}_{h12} \neq \tilde{0}$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-10,-6,-2,2,6,10) & (-1,1,4,6,8,12) \\ (-7,-4,-1,1,4,7) & (-16,-8,0,5,10,15) & (-30,-12,0,7,14,21) \\ (-8,-3,-1,1,3,8) & (-19,-6,-2,3,9,15) & (-29,-6,0,7,14,20) \end{bmatrix}$$

$R_1 \to R_1 - \tilde{a}_{h13} R_3$, *where* $\tilde{a}_{h13} \neq \tilde{0}$

$$\sim \begin{bmatrix} (-2,-1,0,1,2,6) & (-10,-6,-2,2,6,10) & (-13,-7,-2,2,7,13) \\ (-7,-4,-1,1,4,7) & (-16,-8,0,5,10,15) & (-30,-12,0,7,14,21) \\ (-8,-3,-1,1,3,8) & (-19,-6,-2,3,9,15) & (-29,-6,0,7,14,20) \end{bmatrix}$$

$$c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbb{C}(\hat{A}) = \{c_1, c_2, c_3\} and \mathbb{R}(\hat{A}) = \{r_1, r_2, r_3\}$

clearly c_1, c_2 *and* c_3 *are linearly independent,*

The basis of $\mathbb{C}(\hat{A}) = \begin{Bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) & (-1,1,4,6,8,12) \\ (-1,0,1,2,4,6) & (2,4,6,8,10,12) & (0,8,10,12,14,16) \\ (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) & (1,4,5,7,9,10) \end{Bmatrix}$

The basis of

$$\mathbb{R}(\hat{A}) = \begin{Bmatrix} (-2,-1,0,1,2,6) & (-10,-6,-2,2,6,10) & (-13,-7,-2,2,7,13) \\ (-7,-4,-1,1,4,7) & (-16,-8,0,5,10,15) & (-30,-12,0,7,14,21) \\ (-8,-3,-1,1,3,8) & (-19,-6,-2,3,9,15) & (-29,-6,0,7,14,20) \end{Bmatrix}$$

$dim(\mathbb{C}(\hat{A})) = \rho_c(\hat{A}) = 3$

$dim(\mathbb{R}(\hat{A})) = \rho_r(\hat{A}) = 3$

$\therefore$ *The common value of row rank and column rank is fuzzy rank,*

(i,e).,$\rho_r(\hat{A}) = \rho_c(\hat{A}) = \rho_f(\hat{A}) = 3.$

5 Cross Vector and Schein Rank

Cross vector play an important role in fuzzy as well as Hexagonal fuzzy matrix theory to evaluate the rank of matrices.

Definition 5.1 *For vectors u,v the cross vector (u,v) is the HFM* $\hat{A} = (\tilde{a}_{hij}) = u^t v$ *such that* $\tilde{a}_{hij} = u_{hi} v_{hi}$*, where* $\tilde{a}_{hij}, u_{hi}, v_{hi} \in \mathbb{F}$ *in fuzzy algebra.*

Example 5.2 *We consider two vectors* $u = [(-2,-1,0,1,2,6),(-1,0,1,2,4,6),(-1,0,1,3,6,9)]$ *and* $v = [(-1,0,1,3,6,9),(1,3,5,7,9,11),(1,2,3,4,6,8)]$*, then the cross vector of these two vectors u and v is,*

$$u^t v = \begin{pmatrix} (-2,-1,0,1,2,6) \\ (-1,0,1,2,4,6) \\ (-1,0,1,3,6,9) \end{pmatrix} \begin{pmatrix} (-1,0,1,3,6,9) & (1,3,5,7,9,11) & (1,2,3,4,6,8) \end{pmatrix}$$

$$= \begin{pmatrix} (-6,-3,0,3,6,18) & (-12,-6,0,6,12,36) & (-8,-4,0,4,8,24) \\ (-3,0,3,6,12,18) & (-6,0,6,12,24,36) & (-4,0,4,8,16,24) \\ (-3,0,3,9,18,27) & (-6,0,6,18,36,63) & (-4,0,4,12,24,30) \end{pmatrix}$$

Here we see that the row vectors of the cross vectors are linear combination of the vector $[(-1,0,1,3,6,9), (1,3,5,7,9,11),(1,2,3,4,6,8)]$ *and the column vectors are the linear combination of the vector* $[(-2,-1,0,1,2,6), (-1,0,1,2,4,6),(-1,0,1,3,6,9)]^t$*, which shows that the row vectors and column vectors of the cross vector are linearly dependent and row rank= column rank= fuzzy rank=1.*

Proposition 5.3 *A non-zero Hexagonal fuzzy matrix is a cross vector if and only if it has a row rank as well as column rank 1 over the fuzzy algebra* $\mathbb{F}$*. The condition is necessary. Let* $\hat{A}$ *be the Hexagonal fuzzy matrix which is a cross vector then A can be written as* $\hat{A} = u^t v$*, where u and v are vectors over the fuzzy algebra* $\mathbb{F}$*.*
This implies that every rows of the matrix $\hat{A}$ *can expressed as the linear combination of only one vector* u^t*. Thus row rank of* $\hat{A}$ *=column rank of* $\hat{A}$*=1.*
The condition is sufficient,
Let the row rank as well as column rank of the matrix $\hat{A}$ *is 1. Then there exist only one linearly independent column vectors* u^t *and only one linearly independent row vector v, such that* $\hat{A}$ *can be expressed as* $\hat{A} = u^t v$*.*

Proposition 5.4 *Let* $\hat{A} \in \mathbb{F}_{mn}$*, then the fuzzy rank* $\rho_f(\hat{A})$ *satisfy the following properties,*

1. $\rho_f(\hat{A}) \le min(\rho_r(\hat{A}), \rho_r(\hat{A}))$.
2. $\rho_f(\hat{P}\hat{A}\hat{Q}) \le \rho_f(\hat{A})$*, if the HFM* $\hat{P}\hat{A}\hat{Q}$ *is defined.*
3. $\rho_f(\hat{A})$ *is the smallest size of a set s of vectors such that* $\mathbb{R}(\hat{A}) \subseteq < s >$.
4. $\rho_f(\hat{A})$ *is the least number of matrices (of rank 1) whose sum is* $\hat{A}$.

1. Let us consider $\rho_r(\hat{A}) = r$*, then by preposition 4.11 there exist matrices* $\hat{B} \in \mathbb{F}_{mr}$ *and* $\hat{C} \in \mathbb{F}_{rn}$ *such that* $\rho_r(\hat{A}) = \rho_r(\hat{C}) = r$ *and* $\hat{A} = \hat{B}\hat{C}$*.*
Therefore, by definition of fuzzy rank $\rho_f(\hat{A}) \le \rho_r(\hat{A})$*. Also the preposition 4.12, using the column rank factorization we similarly get* $\rho_f(\hat{A}) \le \rho_c(\hat{C})$*.*

Therefore $\rho_f(\hat{A}) \leq min(\rho_r(\hat{A}), \rho_c(\hat{A}))$.

2. *we take* $\rho_f(\hat{A}) = t$, *then there exist a least integer t with* $\hat{B} \in \mathbb{F}_{mt}$ *and* $\hat{C} \in \mathbb{F}_{tn}$ *such that* $\hat{A} = \hat{B}\hat{C}$, *which is the fuzzy rank factorization of* $\hat{A}$.

Since $\hat{P}\hat{A}\hat{Q}$ *is defined, therefore we consider that* $\hat{P} \in \mathbb{F}_{pm}$ *and* $\hat{Q} \in \mathbb{F}_{nq}$.

Hence $\hat{P}\hat{A}\hat{Q} = \hat{P}(\hat{A}\hat{B})\hat{Q} = (\hat{P}\hat{B})(\hat{C}\hat{Q}) = \hat{V}\hat{W}$.

Where $\hat{V} = \hat{P}\hat{B} \in \mathbb{F}_{pt}$ *and* $\hat{W} = \hat{C}\hat{Q} \in \mathbb{F}_{qt}$. *Thus we have a decomposition of* $\hat{P}\hat{A}\hat{Q}$. *Hence by definition of fuzzy rank we get* $\rho_f(\hat{P}\hat{A}\hat{Q}) \leq t = \rho_f(\hat{A})$.

3. *Since* $\rho_f(\hat{A}) = t$, *then by fuzzy rank factorization we get* $\hat{A} = \hat{B}\hat{C}$ *with* $\hat{B} \in \mathbb{F}_{mt}$ *and* $\hat{C} \in \mathbb{F}_{tn}$, *where t is a least positive integer. Then the result we get* $\mathbb{R}(\hat{A}) = \mathbb{R}(\hat{A}\hat{B}) = \mathbb{R}(\hat{C})$.

Let $\mathbb{R}(\hat{C}) =$ *sł, where s is the smallest spanning set of* $\mathbb{R}(\hat{C})$.

Hence $\mathbb{R}(\hat{A}) \subseteq$ *sł.*

4. *Let s be the least number of matrices with rank 1 such that ,*

$$\hat{A} = \hat{A}_1 + \hat{A}_2 + \cdots + \hat{A}_s$$

Here each $\hat{A}_i^s$ *are of rank 1, therefore by preposition 5.3 each* $\hat{A}_i^s$ *are cross vectors. So there exist vectors* $u_i \in \mathbb{V}_m$ *and* $v_i \in \mathbb{V}_n$ *such that* $\hat{A} = u_i^t v_i$ *for each* $i \in \{1, 2, \ldots, n\}$. *Let*$\hat{U} \in \mathbb{F}_{ms}$ *and* $\hat{V} \in \mathbb{F}_{sn}$ *be defined such that* $i^t h$ *column of* $\hat{U}$ *be* u_i^t *and* $i^t h$ *row of* $\hat{V}$ *be* v_i. *Then* $\hat{A} = \hat{U}\hat{V}$ *is a fuzzy rank factorization and hence* $\rho_f(\hat{A}) = s$.

Example 5.5 *Let us consider* $u_1 = [(-2,-1,0,1,2,6),(-1,0,1,2,4,6)] \in \mathbb{V}_2$, $u_2 = [(-1,0,1,3,6,9),(1,2,3,4,6,8)] \in \mathbb{V}_2$, $v_1 = [(-1,0,1,3,6,9),(-1,1,6,8,10,12),(-2,-1,0,1,2,6)] \in \mathbb{V}_3$
and $v_2 = [(1,2,3,4,6,8),(2,4,6,8,10,12),(-1,0,1,2,4,6)] \in \mathbb{V}_3$. *Now the cross vectors*

$$\hat{A}_1 = u_1^t v_1$$
$$= \begin{pmatrix} (-2,-1,0,1,2,6) \\ (-1,0,1,2,4,6) \end{pmatrix} \begin{pmatrix} (-1,0,1,3,6,9) & (-1,1,6,8,10,12) & (-2,-1,0,1,2,6) \end{pmatrix}$$
$$= \begin{pmatrix} (-6,-3,0,3,6,18) & (-12,-6,0,6,12,36) & (-2,-1,0,1,2,6) \\ (-3,0,3,6,12,18) & (-6,0,6,12,24,36) & (-1,0,1,2,4,6) \end{pmatrix}$$

$$\hat{A}_2 = u_2^t v_2$$
$$= \begin{pmatrix} (-1,0,1,3,6,9) \\ (1,2,3,4,6,8) \end{pmatrix} \begin{pmatrix} (1,2,3,4,6,8) & (2,4,6,8,10,12) & (-1,0,1,2,4,6) \end{pmatrix}$$
$$= \begin{pmatrix} (-4,0,4,12,24,36) & (-7,0,7,21,42,63) & (-2,0,2,6,12,18) \\ (4,8,12,16,24,32) & (7,14,21,28,42,56) & (2,4,6,8,12,16) \end{pmatrix}$$

$$\hat{A} = \hat{A}_1 + \hat{A}_2$$
$$= \begin{pmatrix} (-6,-3,0,3,6,18) & (-12,-6,0,6,12,36) & (-2,-1,0,1,2,6) \\ (-3,0,3,6,12,18) & (-6,0,6,12,24,36) & (-1,0,1,2,4,6) \end{pmatrix} + \begin{pmatrix} (-4,0,4,12,24,36) & (-7,0,7,21,42,63) & (-2,0,2,6,12,18) \\ (4,8,12,16,24,32) & (7,14,21,28,42,56) & (2,4,6,8,12,16) \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} (-10,-3,4,15,30,54) & (-19,-6,7,27,54,99) & (-4,-1,2,7,14,24) \\ (1,8,15,22,36,50) & (1,14,27,40,66,92) & (1,4,7,10,16,22) \end{pmatrix}$$

Also,

$$\hat{U} = \begin{pmatrix} (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) \\ (-1,0,1,2,4,6) & (1,2,3,4,6,8) \end{pmatrix}$$

$$\hat{V} = \begin{pmatrix} (-1,0,1,3,6,9) & (-1,1,6,8,10,12) & (-2,-1,0,1,2,6) \\ (1,2,3,4,6,8) & (2,4,6,8,10,12) & (-1,0,1,2,4,6) \end{pmatrix}$$

$$\hat{U}\hat{V} = \begin{pmatrix} (-10,-3,4,15,30,54) & (-19,-6,7,27,54,99) & (-4,-1,2,7,14,24) \\ (1,8,15,22,36,50) & (1,14,27,40,66,92) & (1,4,7,10,16,22) \end{pmatrix} = \hat{A}$$

$\therefore \hat{U}\hat{V} = \hat{A}$.

Corollary 5.6 *Let $\hat{A} \in \mathbb{F}_{mn}$ and $\hat{B} \in$, the following results are true,*

(i). $\rho_f(\hat{A}\hat{B}) \leq min(\rho_r(\hat{A}), \rho_r(\hat{B}))$

(ii). $\rho_f(\hat{A}\hat{B}) \leq min(\rho_c(\hat{A}), \rho_c(\hat{B}))$

(iii). $\rho_f(\hat{A}\hat{B}) \leq min(\rho_r(\hat{A}), \rho_r(\hat{B}), \rho_c(\hat{A}), \rho_c(\hat{B}))$

(i). Let us consider $\rho_r(\hat{A}) = r$, then by preposition 4.11 there exist matrices $\hat{B} \in \mathbb{F}_{mr}$ and $\hat{C} \in \mathbb{F}_{rn}$ such that $\rho_r(\hat{A}) = \rho_r(\hat{C})$ and $\hat{A} = \hat{B}\hat{C}$.

Therefore, by definition of fuzzy rank $\rho_f(\hat{A}\hat{B})$

we say that $\rho_f(\hat{A}\hat{B}) \leq \rho_f(\hat{A}) < \rho_r(\hat{A})$,

similarly $\rho_f(\hat{A}\hat{B}) \leq \rho_f(\hat{B}) < \rho_r(\hat{B})$,

Thus $\rho_f(\hat{A}\hat{B}) \leq min\{\rho_r(\hat{A}), \rho_r(\hat{B})\}$

(ii). In similar way we also prove that $\rho_f(\hat{A}\hat{B}) \leq min\{(\rho_c(\hat{A}), \rho_c(\hat{B})\}$

(iii). Using (i) and (ii) we concluded that $\rho_f(\hat{A}\hat{B}) \leq min\{\rho_r(\hat{A}), \rho_r(\hat{B}, \rho_c(\hat{A}), \rho_c(\hat{B})\}$.

Corollary 5.7 *Let $\hat{A} \in \mathbb{F}_{mn}$ then $\rho_f(\hat{A}\hat{A}^T) \leq min\{\rho_r(\hat{A}), \rho_c(\hat{A})\}$ and $\rho_f(\hat{A}^T\hat{A}) \leq min\{\rho_r(\hat{A}), \rho_c(\hat{A})\}$. This proof is straightforward to proof (iii) of corollary 5.6*

Definition 5.8 *The schein rank $\rho_s(\hat{A})$ of a Hexagonal fuzzy matrix $\hat{A}$ is the least number of matrices of rank 1 whose sum is $\hat{A}$.*

Example 5.9 *The schein ranks of the Hexagonal fuzzy matrix*

$$\hat{A} = \begin{bmatrix} (-2,-1,0,1,2,6) & (0,0,0,0,0,0) & (-2,-1,0,1,2,6) \\ (-2,-1,0,1,2,6) & (-2,-1,0,1,2,6) & (-1,0,1,2,4,6) \\ (0,0,0,0,0,0) & (-2,-1,0,1,2,6) & (-2,-1,0,1,2,6) \end{bmatrix}$$ *are 3,because*

$\hat{A} = \hat{A}_1 + \hat{A}_2 + \hat{A}_3$.

Where,

$$\hat{A}_1 = \begin{pmatrix} (-2,-1,0,1,2,6) \\ (-2,-1,0,1,2,6) \\ (0,0,0,0,0,0) \end{pmatrix} \begin{pmatrix} (-2,-1,0,1,2,6) & (0,0,0,0,0,0) & (0,0,0,0,0,0) \end{pmatrix}$$

$$= \begin{pmatrix} (-2,-1,0,1,2,6) & (0,0,0,0,0,0) & (0,0,0,0,0,0) \\ (-2,-1,0,1,2,6) & (0,0,0,0,0,0) & (0,0,0,0,0,0) \\ (0,0,0,0,0,0) & (0,0,0,0,0,0) & (0,0,0,0,0,0) \end{pmatrix}$$

$$\hat{A}_2 = \begin{pmatrix} (0,0,0,0,0,0) \\ (-2,-1,0,1,2,6) \\ (-2,-1,0,1,2,6) \end{pmatrix} \begin{pmatrix} (0,0,0,0,0,0) & (-2,-1,0,1,2,6) & (0,0,0,0,0,0) \end{pmatrix}$$

$$= \begin{pmatrix} (0,0,0,0,0,0) & (0,0,0,0,0,0) & (0,0,0,0,0,0) \\ (0,0,0,0,0,0) & (-2,-1,0,1,2,6) & (0,0,0,0,0,0) \\ (0,0,0,0,0,0) & (-2,-1,0,1,2,6) & (0,0,0,0,0,0) \end{pmatrix}$$

$$\hat{A}_3 = \begin{pmatrix} (-2,-1,0,1,2,6) \\ (-1,0,1,2,4,6) \\ (-2,-1,0,1,2,6) \end{pmatrix} \begin{pmatrix} (0,0,0,0,0,0) & (0,0,0,0,0,0) & (-2,-1,0,1,2,6) \end{pmatrix}$$

$$= \begin{pmatrix} (0,0,0,0,0,0) & (0,0,0,0,0,0) & (-2,-1,0,1,2,6) \\ (0,0,0,0,0,0) & (0,0,0,0,0,0) & (-1,0,1,2,4,6) \\ (0,0,0,0,0,0) & (0,0,0,0,0,0) & (-2,-1,0,1,2,6) \end{pmatrix}$$

With all $\hat{A}_i^s$ are of rank 1.

6 Scalar Multiplication

Scalar multiplication of matrices is also very important in matrix theory as well as in linear algebra.

Definition 6.1 *Let $\hat{A} = (\tilde{a}_{hij})$ be the Hexagonal fuzzy matrix. (i,e) $\hat{A} \in \mathbb{F}_{mn}$. Then for any scalar $k \in \mathbb{F}$, the multiplication of $\hat{A}$ is defined by $k\hat{A}$ and is defined by*

$$k\hat{A} = \begin{cases} (k\tilde{a}_{ij1}, k\tilde{a}_{ij2}, k\tilde{a}_{ij3}, k\tilde{a}_{ij4}, k\tilde{a}_{ij5}, k\tilde{a}_{ij6}) & if k > 0 \\ (k\tilde{a}_{ij6}, k\tilde{a}_{ij5}, k\tilde{a}_{ij4}, k\tilde{a}_{ij3}, k\tilde{a}_{ij2}, k\tilde{a}_{ij1}) & if k < 0 \end{cases}$$

Example 6.2 *For the matrix* $\hat{A} = \begin{bmatrix} (-2,-1,0,1,2,6) & (-1,0,1,3,6,9) \\ (-1,0,1,2,4,6) & (-1,1,6,8,10,12) \end{bmatrix}$ *and for* $k > 0$ *and* $k < 0$.

If $k = 2$ *then* $\hat{A}$ *becomes,*

$\hat{A} = \begin{bmatrix} (-4,-2,0,2,4,12) & (-2,0,2,6,12,18) \\ (-2,0,2,4,8,12) & (-2,2,12,16,20,24) \end{bmatrix}$ *and,*

$k = -1$*then* $\hat{A}$ *becomes,*

$\hat{A} = \begin{bmatrix} (-6,-2,-1,0,1,2) & (-9,-6,-3,-1,0,1) \\ (-6,-4,-2,-1,0,1) & (-12,-10,-8,-6,-1,1) \end{bmatrix}$

Let $\hat{A} \in \mathbb{F}_{mn}$ and if $k\hat{A}$ denotes the scalar multiplication of $\hat{A}$ with k then,

(i). $\mathbb{R}(k\hat{A}) \subseteq \mathbb{R}(\hat{A})$

(ii). $\mathbb{C}(k\hat{A}) \subseteq \mathbb{R}(\hat{A})$

(i). Let us consider $\hat{A} = (\tilde{a}_{hij}) \in \mathbb{F}_{mn}$, where $\tilde{a}_{hij} \in \mathbb{F}$. Then $k\hat{A} = (k\tilde{a}_{hij}) \in \mathbb{F}_{mn}$.

Therefore $\mathbb{R}(\hat{A}) \subseteq < s >$, where,

$S = \{(\tilde{a}_{h11}, \tilde{a}_{h12}, \dots, \tilde{a}_{h1n}), (\tilde{a}_{h21}, \tilde{a}_{h22}, \dots, \tilde{a}_{h2n}), \dots, (\tilde{a}_{hm1}, \tilde{a}_{hm2}, \dots, \tilde{a}_{hmn})\}$

(i,e), $\mathbb{R}(\hat{A})$ can be written as

$$\mathbb{R}(\hat{A}) = \sum_{i=1}^{m} x_{hi}(\tilde{a}_{hi1}, \tilde{a}_{hi2}, \dots, \tilde{a}_{hin})$$

and similarly

$$\mathbb{R}(k\hat{A}) = \sum_{i=1}^{m} kx_{hi}(\tilde{a}_{hi1}, \tilde{a}_{hi2}, \ldots, \tilde{a}_{hin})$$

Let α be any element of $\mathbb{R}(k\hat{A})$ then $\alpha \in \mathbb{R}(k\hat{A})$. (i.e), α can be written as
$\alpha = \sum_{i=1}^{m} kx_{hi}(\tilde{a}_{hi1}, \tilde{a}_{hi2}, \ldots, \tilde{a}_{hin})$ where $x_{hi} \in \mathbb{F}$,
$= \sum_{i=1}^{m} y_{hi}(\tilde{a}_{hi1}, \tilde{a}_{hi2}, \ldots, \tilde{a}_{hin})$ where $y_{hi} = kx_{hi} \in \mathbb{F}$.
Hence $\alpha \in \mathbb{R}(k\hat{A})$
Therefore $\mathbb{R}(k\hat{A}) \subseteq \mathbb{R}(\hat{A})$.
(ii).Similarly, by using the transpose of a matrix we get the same result for column space.
(i,e),$\mathbb{C}(k\hat{A}) \subseteq \mathbb{R}(\hat{A})$. Let $\hat{A} \in \mathbb{F}_{mn}$ and $k \in \mathbb{F}$ then,

(i). $\rho_r(k\hat{A}) \leq \rho_r(\hat{A})$

(ii). $\rho_c(k\hat{A}) \leq \rho_c(\hat{A})$

(iii). $\rho_f(k\hat{A}) \leq \rho_r(\hat{A})$

Let $\rho_r(\hat{A}) = p$, then $p \leq m$. Therefore, $\mathbb{R}(\hat{A})$ has p linearly independent vectors, say $\alpha_1, \alpha_2, \ldots, \alpha_p$. Then $\mathbb{R}(\hat{A}) = \sum_{i=1}^{p} x_{hi}\alpha_i$ and α_i cannot be expressed as the linear combination of order, (i,e), $\alpha_1, \alpha_2, \ldots, \alpha_(i-1), \alpha_(i+1), \ldots, \alpha_p$.
Let $\beta_1, \beta_2, \ldots, \beta_q$ be another row vectors of $\hat{A+}$ which can be expressed as the linear combinations of $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $p + q = m$, then any β_j can be expressed as,
$\beta_j = \sum_{i=1}^{p} \tilde{x}_{hij}\alpha_i$ with $j = 1, 2, .., q, \tilde{x}_{hij} \in \mathbb{F}$. So,
$k\beta_j = \sum_{i=1}^{p} k\tilde{x}_{hij}\alpha_i = \sum_{i=1}^{p} \tilde{x}_{hij}(k\alpha_i)$.
(i,e) $k\beta_j^s$ are expressed as linear combination of $k\alpha_i$ for $j = \{1, 2, \ldots, q\}$ and $i = \{1, 2, \ldots, p\}$.
(i,e) if $k\alpha_1, k\alpha_2, \ldots, k\alpha_p$ are linearly independent, then $\rho_r(k\hat{A}) = \rho_r(\hat{A}) = p$.
Here there is three cases arises,

case(i)

If $k > 0$, then
$k\tilde{a}_{hij} = \tilde{a}_{hij}$ for all i,j. For this case $k\alpha_i = \alpha_i$.
Therefore $\rho_r(k\hat{A}) = \alpha_r(\hat{A}) = p$.

case(ii)

If $k = 0$, then
$k\tilde{a}_{hij} = \hat{O}$.
Therefore, $k\hat{A}$ being a zero matrix for this case $\rho_r(k\hat{A}) = 0 < \rho_r(\hat{A}) = p$.

case(iii)

If $k < 0$, then
$k\tilde{a}_{hij} = \tilde{a}_{hij}$ for all i,j. For this case $k\alpha_i = -\alpha_i$. Then,
Therefore $\rho_r(k\hat{A}) = \rho_r(\hat{A}) = p$ and if one of $k\alpha_i$ expressed as linear combination of others then,

$$\rho_r(k\hat{A}) = \rho_r(\hat{A}) = p$$

Combining the above three cases we concluded that,
$\rho_r(k\hat{A}) \leq \rho_r(\hat{A})$.

(ii).Similarly, we can prove $\rho_c(k\hat{A}) \leq \rho_c(\hat{A})$.

(iii).Let $\rho_f(\hat{A}) = t$, then $\hat{A}$ can be written as the sum of t matrices of rank 1 say$\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_t$.

Then, $\hat{A} = \hat{A}_1 + \hat{A}_2 + \cdots + \hat{A}_t$ or $k\hat{A} = k\hat{A}_1 + k\hat{A}_2 + \cdots + k\hat{A}_t$.

Since $\hat{A}_i^{(}, s)$ are all cross vectors therefore $k\hat{A}_i$ are also cross vectors $i = 1, 2, \ldots, t$. Now if one of $k\hat{A}_i$ can be expressed as the sum of other

(i,e),$k\hat{A} = k\hat{A}_1 + k\hat{A}_2 + \cdots + k\hat{A}_(i-1) + k\hat{A}_(i+1) + \cdots + k\hat{A}_i$, for this case $\rho_f(k\hat{A}) = t$. The process is repeated until no such cross vector can be removed.

(i,e) $\rho_f(k\hat{A}) = \rho_f(\hat{A})$

If the $k\hat{A}$ become zero matrix then, $\rho_f(k\hat{A}) = 0 < \rho_f(\hat{A})$. Combining these cases then we have,

$$\rho_f(k\hat{A}) \leq \rho_f(\hat{A}).$$

Example 6.3 *Let us consider the Hexagonal fuzzy matrix*
$\hat{A} = \begin{pmatrix} (-10,-3,4,15,30,54) & (-19,-6,7,27,54,99) & (-4,-1,2,7,14,24) \\ (1,8,15,22,36,50) & (1,14,27,40,66,92) & (1,4,7,10,16,22) \end{pmatrix}$ *with* $\rho_r(\hat{A}) = 2$.
Also let $k = 1$ *then,*
$k\hat{A} = \begin{pmatrix} (-10,-3,4,15,30,54) & (-19,-6,7,27,54,99) & (-4,-1,2,7,14,24) \\ (1,8,15,22,36,50) & (1,14,27,40,66,92) & (1,4,7,10,16,22) \end{pmatrix}$ *then using RREF* $k\hat{A}$ *is reduced in the row rank and column rank is* $\rho_r(k\hat{A}) = \rho_c(k\hat{A}) = 2$.
Also we see that,
$k\hat{A} = \begin{pmatrix} (-2,-1,0,1,2,6) \\ (-1,0,1,2,4,6) \end{pmatrix} \begin{pmatrix} (-1,0,1,3,6,9) & (-1,1,6,8,10,12) & (-2,-1,0,1,2,6) \end{pmatrix} +$
$\begin{pmatrix} (-1,0,1,3,6,9) \\ (1,2,3,4,6,8) \end{pmatrix} \begin{pmatrix} (1,2,3,4,6,8) & (2,4,6,8,10,12) & (-1,0,1,2,4,6) \end{pmatrix}$
If $k = 0$ *then the* $\rho_r(k\hat{A}) < \rho_r(\hat{A})$ *as well as* $\rho_c(k\hat{A}) < \rho_c(\hat{A})$.
$\therefore \rho_f(k\hat{A}) = 2 < \rho_f(\hat{A})$.

7 Conclusion

In this paper, we find out the different types of ranks of Hexagonal fuzzy matrices such as fuzzy rank, fuzzy full rank, cross vector and schein rank. The evaluation of rank in a topics of Hexagonal fuzzy matrices in fuzzy algebra F have been attempted. Our future investigation of study may be in the domain of fuzzy linear space of fuzzy linear system under fuzzy rank method.

References

[1] K.S.Abdukhalikov, M.S.Tulenbaev and U.U.Umirbarv, *on fuzzy bases of vector space*, Fuzzy sets and systems, vol.63,(1994),201-206.

[2] K.S.Abdukhalikov, *The dual of fuzzy subspace*, Fuzzy sets and systems, vol.82,(1996),375-381.

[3] D.Dubois and H.Prade, *Operations on fuzzy number*, International journal of systems, vol.9(6),(1976), 613-626.

[4] H.Hedayati, *On properties of fuzzy subspace of vector space*, Ratio Math, vol.19,(2009),1-10.

[5] H.N.Ismail and N.N.Morsi, *Fuzzy rank of fuzzy matrix*, Fuzzy sets and Systems, vol.41(2),(1991),243-249.

[6] S.Kaguei and A.Ohsato, *The rank of a fuzzy matrix and its evaluation*, Fuzzy sets and systems, vol.38(3),(1990),355-364.

[7] K.H.Kim and F.W.Roush, *Generalized fuzzy matrices*, Fuzzy sets and systems, vol.4,(1980),293-315.

[8] P.Lubczonok, *Fuzzy vector space*, Fuzzy sets and systems, vol.38,(1990),329-343.

[9] M.Manivizhi, K.Latha, K.Tripurasundari,*Rank of a Type-2 Triangular fuzzy matrix*, Imperial journal of interdisciplinary Research, vol.3(2),(2017),538-545.

[10] S.Mondal and M.Pal,*Interval- valued fuzzy vector space*, Annl.pure.appl.math, vol.2(1),(2012),86-95.

[11] S.Mondal and M.Pal,*Rank of Interval valued fuzzy matrices*, Afrika Mathematika, vol.27(2),(2016), 97-114.

[12] J.N.Mordenson, *Bases of fuzzy vector space*, Information Science, vol.67,(1993),87-92.

[13] M.Pal, *Interval valued fuzzy matrices with interval valued rows and columns*, Fuzzy Information and Engineering, vol.7,(2015), 335-368.

[14] M.Pal,*Fuzzy matrices with fuzzy rows and columns*, Journal of intelligent and fuzzy system, vol.30(1), (2016),561-573.

[15] F.G.Shi and C.E.Huang, *Fuzzy bases and fuzzy dimension of fuzzy vector space*, Math. Communication, vol.15(2),(2010),303-310.

[16] A.K.Shyamal and M.Pal, *Two new operators on fuzzy matrices*, J.Appl.Math and Computing, vol.15,(2004), 91-107.

[17] A.K.Shyamal and M.Pal,*Triangular fuzzy matrices*,Iranian journal of fuzzy systems, vol.4(1),(2007),75-87.

[18] D.Stephen Dinagar and U.Hari Narayanan, *A note on Arithmetic operations of Hexagonal fuzzy number using -cut method*, Inter.J.App.fuzz.sets.and.Arti.Intelligence, vol.6,(2016),145-162.

[19] M.G.Thomson, *Convergence of power of a fuzzy matrix*, J.Math.Annl, vol.57,(1977), 476-480.

[20] Wang Hongxu and He Zhongxiong, *Solution of fuzzy matrix rank*, Fuzzy Mathematics, vol.4,(1984),35-44.

[21] Wang Houngxu and YI Chongxin,*The minimum row(column)space of fuzzy matrix*, BUSEFAL, vol.33, (1987).