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An Exact Method for a Class of Third Order Nonlinear Differential Equations

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ABSTRACT. In this work, an existing method for solving certain classes of nonlinear second order ordinary differential equations is extended to nonlinear third order ordinary differential equations. The method simply expresses the solution of the nonlinear differential equation in terms of solutions of an equivalent fourth order linear differential equation.

1 Introduction

Nonlinear differential equations arise naturally in many areas of science and engineering. Various methods have been investigated to solve these equations numerically or analytically. The class of equations considered in [1] is reducible to third order linear differential equation with constant or variable coefficients. The method consists in constructing a linear third order equation with three linearly independent solutions which is then reduced to an equivalent nonlinear second order differential equation. In this note the method is extended to third order nonlinear differential equation by matching it with an equivalent fourth order linear differential equation.

2 Analysis

Consider the following fourth order linear differential equation

$$v'''' + p(s)v''' + q(s)v'' + r(s)v' + w(s)v = 0$$

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Let v_1, v_2, v_3 and v_4 be four linearly independent solutions. Let

$$u(s) = c_1 v_1(s) + c_2 v_2(s) + c_3 v_3(s).$$

Clearly $u(s)$ satisfies the given DE:

$$u'''' + p(s)u''' + q(s)u'' + r(s)u' + w(s)u = 0 \tag{1}$$

Define

$$x = \frac{a(s)u' + b(s)u'' + c(s)u'''}{u} = V(s) \tag{2}$$

We have,

$$V'(s) = a\left[\frac{u''}{u} - \left(\frac{u'}{u}\right)^2\right] + b\left[\frac{u'''}{u} - \frac{u'u''}{u^2}\right] + c\left[\frac{u''''}{u} - \frac{u'''u'}{u^2}\right] + \frac{a'u'}{u} + \frac{b'u''}{u} + \frac{c'u'''}{u} \tag{3}$$

From (2) it follows that,

$$\frac{u'''}{u} = \frac{1}{c}\Big(v - \frac{au'}{u} - \frac{bu''}{u}\Big) \tag{4}$$

From (1) and (4) it follows that,

$$\frac{u''''}{u} = -\frac{p}{c}\Big(v - \frac{au'}{u} - \frac{bu''}{u}\Big) - \frac{qu''}{u} - \frac{ru'}{u} - w \tag{5}$$

Replace $\frac{u''''}{u}, \frac{u'''}{u}$ by their above expressions in (3). We get,

$$\begin{aligned} V'(s) =& a\Big(\frac{u''}{u} - \Big(\frac{u'}{u}\Big)^2\Big) + \frac{b}{c}\Big(v - \frac{au'}{u} - \frac{bu''}{u}\Big) \\ & - \frac{bu'u''}{u^2} - p\Big(v - \frac{au'}{u} - \frac{bu''}{u}\Big) \\ & - qc\frac{u''}{u} - \frac{rcu'}{u} - cw - \frac{u'}{u}\Big(v - \frac{au'}{u} - \frac{bu''}{u}\Big) \\ & + \frac{a'u'}{u} + \frac{b'u''}{u} + \frac{c'}{c}\Big(v - \frac{au'}{u} - \frac{bu''}{u}\Big) \\ =& \frac{bv}{c} - pv - cw + \Big(\frac{-ab}{c} + pa - v - rc + a'\Big)\frac{u'}{u} \\ & - \frac{ac'}{c} + \Big(a - \frac{b^2}{c} + pb - qc\Big)\frac{u''}{u} + b' - \frac{bc'}{c} \end{aligned}$$

which can be written as,

$$V'(s) = GV - cw + [H - V]\frac{u'}{u} + K\frac{u''}{u}; \tag{6}$$

where

$$G = \frac{b}{c} - p + \frac{c'}{c}$$

$$H = \frac{-ab}{c} + pa - rc + a' - \frac{ac'}{c}$$

$$K = a - \frac{b^2}{c} + pb - qc + b' - \frac{bc'}{c}$$

From (6), we have

$$\frac{u''}{u} = \frac{1}{k}\Big(V' - GV + cw - (H - V)\frac{u'}{u}\Big) \tag{7}$$

Differentiate (6) to get

$$V''(s) = G'V + GV' - c'w - cw' + (H' - V')\frac{u'}{u} + (H - V)\Big(\frac{u''}{u} - \Big(\frac{u'}{u}\Big)^2\Big)$$
$$+ k'\frac{u''}{u} + K\Big(\frac{u'''}{u} - \frac{u''u'}{u^2}\Big)$$

From (4) and (7) we get

$$V''(s) = G'V + GV' - c'w - cw' + (H' - V')\frac{u'}{u} + (H - V)\Big(\frac{1}{k}\Big(V' - GV + cw - (H - V)\frac{u'}{u}\Big) - \Big(\frac{u'}{u}\Big)^2\Big)$$
$$+ \frac{k'}{k}\Big(V' - GV + cw - (H - V)\frac{u'}{u}\Big) + \frac{K}{c}\Big(v - a\frac{u'}{u} - b\frac{u''}{u}\Big) - K\Big(\frac{u'}{u}\frac{u''}{u}\Big)$$

From (7) again, we get

$$V''(s) = G'V + GV' - c'w - cw' + (H' - V')\frac{u'}{u} + (H - V)\Big(\frac{1}{k}\Big(V' - GV + cw - (H - V)\frac{u'}{u}\Big) - \Big(\frac{u'}{u}\Big)^2\Big)$$
$$+ \frac{k'}{k}\Big(V' - GV + cw - (H - V)\frac{u'}{u}\Big) + \frac{K}{c}\Big(v - a\frac{u'}{u} - b\frac{1}{k}\Big(V' - GV + cw - (H - V)\frac{u'}{u}\Big)\Big)$$
$$- K\Big(\frac{u'}{u}\frac{1}{k}\Big(V' - GV + cw - (H - V)\frac{u'}{u}\Big)\Big)$$

After simplifying and regrouping terms, we get

$$V''(s) = \frac{u'}{u}\Big(H' - V' - \frac{1}{K}(H - V)^2 - \frac{K'}{K}(H - V)\frac{aK}{c}$$
$$+ \frac{b}{c}(H - V) - V' + GV - cw\Big) + G'V + GV' - c'w - cw' + \frac{1}{K}(H - V)(V' - GV + cw)$$
$$+ \frac{K'}{K}(V' - GV + cw) + \frac{KV}{c} - \frac{b}{c}(V' - GV + cw)$$

For convenience of notation and to avoid lengthy obvious calculations rewrite it as,

$$V''(s) = \frac{u'}{u}M + Z \tag{8}$$

where $M = H' - V' - \frac{1}{K}(H - V)^2 - \frac{K'}{K}(H - V)\frac{aK}{c} + \frac{b}{c}(H - V) - V' + GV - cw$

and

$$Z = G'V + GV' - c'w - cw' + \frac{1}{K}(H - V)(V' - GV + cw) + \frac{K'}{K}(V' - GV + cw) + \frac{KV}{c}$$
$$- \frac{b}{c}(V' - GV + cw)$$

From (8) it follows that

$$\frac{u'}{u} = \frac{1}{M}(V'' - Z) \tag{9}$$

which is clearly a function of V, V' and V''. By differentiating (8), we get,

$$V'''(s) = \Big(\frac{u''}{u} - \Big(\frac{u'}{u}\Big)^2\Big)M + \frac{u'}{u}M' + Z' \tag{10}$$

From (7) and (9) we have,

$$\frac{u''}{u} = \frac{1}{k}\Big(V' - GV + cw - (H - V)\frac{1}{M}(V'' - Z)\Big) \tag{11}$$

Replace $\frac{u'}{u}$ and $\frac{u''}{u}$ by their respective expressions from (9) and (11) in (10). We get the following third order non-linear differential equation.

$$V''' = \frac{1}{k}\Big(V' - GV + cw - (H-V)\frac{1}{M}(V'' - Z)\Big)M - \Big(\frac{1}{M}(V'' - Z)\Big)^2 M + \frac{1}{M}(V'' - Z)M' + Z' \tag{12}$$

The next step is to rewrite the above differential equation (12) in terms of x, x', x'' and x'''.

From (2) we have $x = V(s)$. This implies that $s = V^{-1}(x)$.

Define a new independent variable t by another implicit relation of the form

$$\frac{ds}{dx} = f(x)\dot{x}^n = f(x)\Big(\frac{dx}{dt}\Big)^n \tag{13}$$

From (2), it follows that

$$V'(s) = \frac{dx}{ds} (or) \frac{ds}{dx} = \frac{1}{V'(s)} \tag{14}$$

Differentiate (14) with respect to x. We get

$$\frac{d^2s}{dx^2} = -\frac{V''(s)}{\Big(V'(s)\Big)^2}\frac{ds}{dx}$$

$$(or)$$

$$-\frac{\frac{d^2s}{dx^2}}{\Big(\frac{ds}{dx}\Big)^2} = V''(s)\frac{ds}{dx} \tag{15}$$

Differentiate (13) with respect to x. We get

$$\begin{aligned}\frac{d^2s}{dx^2} &= f'(x)\dot{x}^n + nf(x)\dot{x}^{n-1}\frac{dt}{dx}\ddot{x} \\ &= f'(x)\dot{x}^n + nf(x)\dot{x}^{n-2}\ddot{x}\end{aligned} \tag{16}$$

From (2), $V = x$.

Therefore, and from (13)

$$V' = \frac{dx}{ds} = \frac{1}{f(x)\dot{x}^n} \tag{17}$$

From (15), and (17), it follows that

$$V'' = \frac{-\Big(n\ddot{x} + \frac{f'}{f}\dot{x}^2\Big)}{f^2\dot{x}^{2n+2}} \tag{18}$$

By Differentiating (18), we get a formula for V'''. Finally, replace V, V', V'' and V''' by their above expressions in the differential equation (12), we get a nonlinear differential equation in x, x', x'' and x''' for which the solution is exact. We next illustrate with a numerical example.

3 Example

Consider the differential equation

$$\dddot{x} + 4kx\ddot{x} + 6k^2\dot{x}x^2 + k^3x^4 + 3k\dot{x}^2 = 0 \tag{19}$$

which is a Chazy class of equation [2,3,4,5].

To solve it using the proposed technique, we start with the differential equation

$$v'''(s) = 0 \tag{20}$$

whose four linearly independent solutions are;

v_1 =constant, $v_2 = s, v_3 = s^2$, and $v_4 = s^3$.

Let

$$u(s) = c_1v_1 + c_2v_2 + c_3v_3 \tag{21}$$

Clearly, $u''''(s) = 0$.

Let

$$x = \frac{1}{k}\frac{u'}{u} = V(s) \tag{22}$$

Considering the particular case of $f(x) = 1$ and $n = -1$ in (13), we have

$$\frac{ds}{dx} = \frac{dt}{dx} \tag{23}$$

i.e., $s = t + constant$. Now, (22) and (23) define the new variables. Proceeding along the lines outlined in the analysis we obtain

$$V'(s) = \frac{1}{k}\frac{u''}{u} - kV^2$$

Hence

$$\frac{u''}{u} = k(V' + kV^2).$$

Therefore,

$$\begin{aligned} V''(s) &= \frac{1}{k}\Big(\frac{u'''}{u} - \frac{u'u''}{u^2}\Big) - 2kVV' \\ &= \frac{1}{k}\Big(\frac{u'''}{u} - k^2V(V' + kV^2)\Big) - 2kVV' \\ &= \frac{1}{k}\frac{u'''}{u} - 3kVV' - k^2V^3 \end{aligned} \tag{24}$$

Therefore

$$\frac{u'''}{u} = k(V'' + 3kVV' + k^2V^3)$$

Note that

$$V'''(s) = \frac{1}{k}\Big(\frac{u''''u - u'''u'}{u^2} - \Big) - 3k(V')^2 - 3kVV'' - 3k^2V^2V'$$

By replacing $\frac{u'''}{u}$ and $\frac{u'}{u}$ and noting that $u'''' = 0$, we obtain

$$V'''(s) = -4kVV'' - 6k^2\dot{x}x^2 + k^3x^4 + 3k(V')^2$$

$$or$$

$$\dddot{x} + 4kx\ddot{x} + 6k^2\dot{x}x^2 + k^3x^4 + 3k(\dot{x})^2 = 0$$

Since $u(s) = a + bs + cs^2$ and $x = \frac{1}{k}\frac{u'}{u}$ it follows that one family of infinitely many solutions of (19) is given by

$$x(t) = \frac{b + 2c(t+e)}{k(a + b(t+e) + c(t+e)^2)}$$

4 Concluding Remarks

In this note, an exact method to solve particular classes of third order nonlinear differential equations is presented. It is a simple extension of the second order nonlinear case. The main advantage is that it provides exact solutions which is not the case for most methods of nonlinear differential equations.

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